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APPLIED MECHANICS

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VOL. II

STRENGTH OF MATERIALS

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PREFACE

This volume has been prepared primarily for the use of students in the Departments of Engineering in the Massachusetts Institute of Technology and is intended to cover the fundamentals of the subject, in so far as they may be required in subsequent work in structural and machine design and in the ordinary problems of engineering practice.

As preparation a student should have a knowledge of Differential and Integral Calculus, the principles of Statics and Dynamics and the methods of determining Centers of Gravity and Moments of Inertia of areas and solids.

Considerable attention has been given to the logical development of the subject and care has been taken to point out the limitations of the different theories; emphasis being laid on the divergence of the conditions met in practice from the ideal conditions, under which the theoretical formulas are deduced, and on modifications necessary, or advisable, when the formulas are used under ordinary working conditions.

The methods of the Calculus have been employed throughout the text, when these furnish the most direct method of analysis, and in the more difficult parts of the subject the derivation of equations has been given in detail, for the benefit of the student who may understand the principles employed but may not have acquired facility in the solution of the differential equations involved.

Graphical methods of solution and, particularly, graphical representations of results, such as diagrams of load intensities, shearing forces, bending moments, normal and shearing stresses, have been freely employed. A considerable number of problems, involving the application of each of the theories discussed, have been included in the text and solutions have been given in detail wherever it has appeared that these would be an aid to a clear understanding of the subject.

While the text is intended to include the material required for a fairly comprehensive knowledge of the subject, the chapters have been arranged in such a manner that the more difficult parts appear at the end; and hence, for a briefer course the latter parts of certain chapters and in some cases the entire chapter may be omitted without destroying the continuity in the presentation of the subject. For example, in a brief course parts of Chapters II, III, IV, V, VII, IX and X, and the whole of Chapters VI and XI to XIV, inclusive, may be omitted.

Finally, to our colleagues on the instructing staff of the Massachusetts Institute of Technology we wish to express our appreciation of the helpful suggestions for which we have been indebted to them during the preparation of this volume.

C. E. F. AND W. A. J.

December, 1918.

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APPLIED MECHANICS

STRENGTH OF MATERIALS.

CHAPTER I.

PHYSICAL PROPERTIES OF MATERIALS.

1. Strength of Materials. — The subject entitled Strength of Materials ordinarily includes the study of the strength and stability, the distribution of the internal forces and the deformations in the different parts of machines and structures.

The larger part of the underlying theory is based on the principles of Statics and certain laws, governing the elasticity of materials, which are determined by experiment. Where these are insufficient to provide a solution of a problem, purely empirical formulas, based on the results of experiments or the experience of practice, are used.

The subject may, therefore, be considered as comprising two parts: the mathematical, based on the principles of Statics as applied to rigid bodies; and the experimental, by which the laws governing the deformation of elastic bodies and the strength of different materials are determined. The elasticity and the strength of different materials vary to a wide extent and in many cases, particularly the different metals, these properties are very considerably affected by external conditions and by treatment in the process of manufacture. Hence it will be found that the mathematical and experimental parts of the subject are practically inseparable and that, in applying the formulas deduced by the theory, different constants must be obtained to suit each particular case as it arises. This makes practically all the derived formulas semi-empirical at least, and in the choice of the different constants required, the judgment and experience of the engineer must play an important part.

The larger part of this volume will necessarily deal with the theoretical side of the subject and we shall be obliged to restrict the illustrations of its application to a few of the more common materials which are used in engineering work. In order to obtain a thorough facility in the practical application of the formulas which are to be deduced, a comprehensive knowledge of the physical properties of materials is required. Within the limits of this volume, however, it will be necessary to confine ourselves to the definitions of those properties and the discussion of the methods of applying the experimental data obtained from the tests of a few materials, which may be taken as representative of the methods to be followed in further applications of the principles of the subject.

2. Properties of Materials. — Whenever a body of any material is subjected to the action of external forces, a certain amount of deformation occurs. This deformation may be permanent or, if the external forces are removed, the body may return partly, or wholly, to its original shape and dimensions.

Elasticity. — The material is said to be perfectly elastic if the deformation produced by a system of forces acting upon the body entirely disappears when the forces are removed.

Plasticity. — The material is said to be perfectly plastic if the entire deformation produced by a system of forces acting upon the body remains when the forces are removed.

Ductility. — A material is said to be ductile when it can be drawn out by *tension*, as in the process of drawing wire.

Malleability. — A material is said to be malleable when it can be flattened out under *compression*, as in the process of rolling plates. This property is similar to ductility, but different materials do not possess the two properties to the same degree. Both properties are evidently forms of plasticity.

Brittleness. — A material is brittle when it lacks toughness and tenacity. Such a material breaks easily under a sudden shock or blow.

Hardness. — This term is used in two senses: first to denote resistance to abrasion and second to denote resistance to indentation. The two kinds of hardness are more or less related, but do not differ in the same degree for all materials.

As a matter of fact, no solid material is thoroughly plastic, that is, absolutely devoid of elasticity, and no solid material is perfectly elastic, but the materials ordinarily used in engineering construction, such as iron, steel, wood, stone and concrete, within certain limits, possess the latter property to a very marked degree. When the recovery of form of a body which has been subjected to stress is only partial we say that the recovery is due to the elasticity which the material possesses and that the permanent deformation after the stress is removed is due to its plasticity. In the case of the materials mentioned above, however, unless the forces to which they are subjected exceed certain limits, the plasticity is so small that it is negligible and the materials may be considered to be perfectly elastic.

The amount of ductility and malleability and the degree of hardness and brittleness possessed by certain materials, the metals in particular, will be found to be affected to a very considerable extent by the treatment which they receive in the process of manufacture.

3. Stress. — We have defined stress (Vol. I, Art. 77) as the force exerted at a section between two contiguous bodies or two parts of the same body, the section in most cases being taken as a plane. A stress may be *simple* or *complex.* The *simple stresses* are: (a) tension, such as the stress on the right cross section *mn* (Fig. 1), of a straight rod subjected to a pull along its axis; (b) compression, such as the stress on the right cross section *mn* (Fig. 2) of a straight bar subjected to pressure along its axis; (c) shear, such as the stress on the section *mn* (Fig. 3) of a rivet joining two plates, which are subjected to a pull P, as shown.

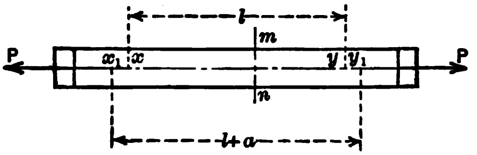


FIG. 1.

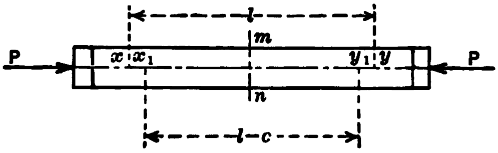


FIG. 2.

Tension and compression stresses are sometimes called *normal* or *direct* stresses.

A *complex stress*, or *oblique stress*, is the resultant of a *normal* and a *shear* stress and can always be analysed by resolving into the simple stresses which are its components. (Vol. I, Art. 78.)

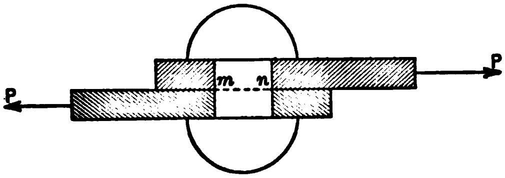


Fig. 3.

A stress may be *uniform*, or it may be *varying*. The *intensity of stress*, or *stress intensity*, in the case of a uniform stress is the stress exerted on a unit of area; and hence it will be equal to

$$\frac{P}{A},$$

where P is the stress on the area A. In the case of a varying stress, the stress intensity at any point is the quotient obtained by dividing the stress on a very small area at the point by the area, which at the limit is evidently equal to

$$\frac{dP}{dA}.$$

Units of Stress. — The units in which *stresses* are measured are the units of force: the *pound*, the *ton*, etc. The units in which *intensities of stress* are measured, or the *unit stresses*, as they are frequently called, are the *lb. per. sq. in.*, the *ton per sq. ft.*, etc.

Hereafter tensile and compressive stress intensities may be denoted by the symbols p_t and p_c, respectively, or simply by the letter p, and a shearing stress intensity will be denoted by the letter s.

In emphasizing the distinction between *stress* and *intensity of stress* it frequently adds to clearness to denote the former by the term *total stress*, although, if we adhere to the exact definition of the term stress as heretofore given, the use of the word total is unnecessary. It is also frequently true that a considerable repetition of terms is required if the term *stress intensity*, or *unit stress*, is used in every case where a strict rendering requires it. In such cases the term stress, simply, is used to denote intensity of stress, the context making it clear that the unit stress rather than the total stress is meant. The student should always note, wherever the term stress is used, whether the total stress or the stress intensity is meant.

4. Strain. — When a body is subjected to the action of forces it undergoes a certain amount of deformation. In general, any two points in the body undergo a relative displacement when the deformation takes place. The displacement may be such that: (*a*) The line joining the two points changes in length only, its direction remaining the same; (b) the line changes in direction only, its length remaining the same; (c) the line changes in both length and direction. If O and A are two points a small distance apart and ΔOA represents the change in the distance between them during the displacement, the limit of the ratio $\frac{\Delta OA}{OA}$ as OA approaches zero is the *strain* in the direction OA. In a few common cases where bodies are subjected to simple stresses it is very easy to define and measure the strains produced, but in cases where the stresses are *complex* the analysis of the strain becomes more complicated.

Tensile Strain. — A bar of uniform section and material which is subjected to a uniform tensile stress P (Fig. 1) may be taken as an illustration of a case in which the strain can be easily determined. All lines within the bar which are parallel to its axis will undergo a change in length, the increase in the length of each line being proportional to its length in the unstrained state.

If l is the distance between any two points on the axis of the bar before the stress is applied, this distance will be increased by a small amount a, when the bar is under stress. The ratio obtained by dividing the increase a by the original length l is called the *tensile strain* in the bar. We shall denote this strain by the symbol e_t; hence

$$e_t = \frac{a}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Since the ratio $\frac{a}{l}$ is the same, whatever the original distance between the points, the strain is *uniform* throughout the length of the bar. If the bar were not of uniform section and material, the strain would vary from point to point and the value of e_t at any point would evidently be represented by the expression

$$e_t = \frac{da}{dl}. \quad \cdots\cdots \quad (2)$$

Compressive Strain. — If a bar of uniform section and material is subjected to a uniform compressive stress under a load P, applied along the axis (Fig. 2), the distance l, between two points on the axis, will decrease by an amount c. In this case the ratio $\frac{c}{l}$ is called the *compressive strain* in the bar. As in the case of tension this ratio is the same whatever the distance l may be and the strain is *uniform*. We will denote this strain by the symbol

$$e_c = \frac{c}{l}. \quad \cdots\cdots \quad (3)$$

For a bar of non-uniform section or material the expression for the strain at any point would evidently be

$$e_c = \frac{dc}{dl}. \quad \cdots\cdots \quad (4)$$

Units of Strain.—It should be noted that tensile and compressive strains are *ratios* of the change in length of a line joining two points to its original length and hence the *units* in which the quantities are expressed are *numbers* only. When a bar is subjected to a uniform tensile, or a uniform compressive, stress, it is obvious that the strain produced is numerically equal to the extension, or contraction, per unit of length; but the magnitude of the strain is expressed in numbers and not in linear units.

Hereafter we shall frequently use the letter e, simply, to denote either a tensile or a compressive strain, the sign of e, where necessary, being taken plus for tension and minus for compression.

In both of the preceding cases the strains along the axis of the bar are accompanied by lateral strains in directions at right angles to the axis, the tensile stress producing a contraction and the compressive stress an extension in that direction. A discussion of the relations between these strains will follow later. (Art. 5.)

Simple Extension. — Both tensile and compressive strains may be designated by the term *extension*, the extension being positive in the case of a tensile strain and negative in the case of a compressive strain. In either of the cases represented by Figs. (1 and 2) the lateral strains accompanying the strain in the direction of the axis of the bar could be prevented by the application of forces at right angles to the axis uniformly distributed over the surface of the bar. If this were done the resulting strain in the direction of the axis would be known as a *simple extension.*

Shearing Strain. — We shall not be able to give a simple practical illustration of this case, as a deformation of this kind nearly always occurs either in combination with an elongation or a contraction, or in cases where the amount of the distortion varies from point to point through the body under strain. The following may be given, however, as an hypothetical illustration.

Suppose the opposite faces of a rectangular prism (Fig. 4) to be subjected to the equal and opposite uniform shearing stresses P parallel to the face *mnop*. Under the action of these stresses the prism will tend to distort into the shape mno_1p_1.

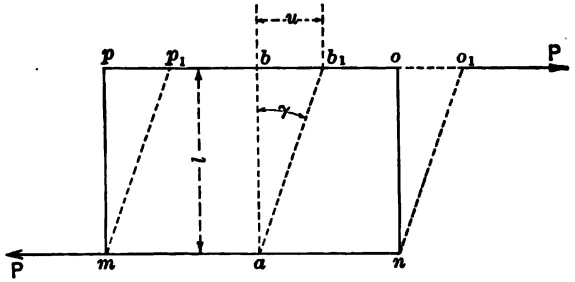


Fig. 4.

Other forces acting on the prism will be required to maintain equilibrium, but with these we need not be concerned. All points along the line *po* will be displaced the same amount along that line, and any line *ab*, parallel to *mp*, will be displaced to the position ab_1, parallel to mp_1.

A distortion such as this is known as a *simple shear.* It may be considered to be the result of the sliding of elementary layers parallel to the faces *mn* and *op* under the action of the shearing forces P.

Units of Strain. — The change in inclination γ *of the perpendicular ab, expressed in radians, will be the numerical measure of the strain, and, since this is always small, the value of* γ may be expressed as

$$\gamma = \frac{u}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The prism *mnop* may be considered to be a portion of a body distorted in simple shear, and, if all portions are distorted the same amount, the shear may be said to be *uniform*. If the amount of the shear in different portions of the body varies, the value of γ will evidently be given by the expression

$$\gamma = \frac{du}{dl}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The foregoing description of a simple shear affords a definition of an elementary type of *shearing strain*. It should be noted that the volume of the prism is unchanged by the shear and that the shear is measured by the change in inclination of two lines originally at right angles.

5. Change of Volume under Stress. — As stated in Art. (4), a tensile strain is accompanied by contraction and a compressive strain by extension in all directions at right angles.

For example, a bar of elastic material, subjected to a uniform tensile stress, undergoes a uniform contraction in all directions at right angles to the axis of the bar. If this contraction is expressed in strain units, it will be found, in the case of a bar of iron or steel, to be between $\frac{1}{3}$ and $\frac{1}{4}$ of the strain in the direction of the axis. While this ratio between the two strains varies for different materials, and different grades of the same material, it is practically constant for all bars of the same grade of any one material. It is customarily called *Poisson's ratio* and may be denoted by the symbol $\frac{1}{m}$. Hence the lateral strain in all directions at right angles to the axis of a bar subjected to tension (Fig. 1) will be equal to

$$-\frac{e_t}{m},$$

the negative sign indicating that this strain is of the opposite character to the strain e_t in the direction of the axis.

Similarly, in the case of a bar subjected to a uniform compressive stress (Fig. 2), a lateral strain is produced in all directions at right angles to the axis, which will be equal to

$$-\frac{e_c}{m},$$

the value of m in this case being practically the same as when the bar is subjected to tension.

To determine the proportionate change in volume in a bar subjected to a tensile stress we may assume the bar to be of square section, the dimensions of which in the unstrained state are $a \times a$, the original length being l. The volume of the bar in the unstrained state will then be equal to

$$V = la^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If e_t is the tensile strain in the direction of the axis of the bar, its length after straining will be equal to

$$l + e_t l = l\,(1 + e_t) \text{ (Art. 4)}$$

and the dimension of the side of the square cross section will be

$$a - \frac{e_t}{m}a = a\left(1 - \frac{e_t}{m}\right).$$

The volume of the strained bar will, therefore, be equal to

$$V_1 = l\,(1 + e_t)\,a^2\left(1 - \frac{e_t}{m}\right)^2$$

$$= la^2\left[1 + e_t\left(\frac{m-2}{m}\right) - e_t^2\left(\frac{2\,m-1}{m^2}\right) + \frac{e_t^3}{m^2}\right]. \quad . \quad . \quad (2)$$

The change in volume due to the strain will be equal to

$$V_1 - V = la^2\left[e_t\left(\frac{m-2}{m}\right) - e_t^2\left(\frac{2\,m-1}{m^2}\right) + \frac{e_t^3}{m^2}\right]. \quad . \quad (3)$$

The strain e_t is such a small quantity that all terms containing higher powers of e_t than the first may be neglected without introducing an appreciable error; and an inspection of equation (3) shows that if the value of m is greater than 2, the volume is slightly increased when the bar is subjected to a tensile stress. Hence the density will be correspondingly less.

A similar analysis in the case of a bar subjected to compression would show a slight decrease in volume and increase in density due to the stress.

6. Elastic Limits. — As previously stated (Art. 2) the metals and other materials used in construction are very nearly elastic provided the forces to which they are subjected are not too great.

It will be found in the case of all the materials mentioned, however, that if loads sufficient to produce stresses above certain limits are applied, a permanent deformation, or *set*, as it is called, will remain after the load is removed.

The greatest stress intensity to which a body of any given material can be subjected without the occurrence of a permanent set when the load producing the stress is removed is called the elastic limit of the material. The definition will apply in the case of each of the simple stresses; thus we speak of the *elastic limit in tension*, the *elastic limit in compression* and the *elastic limit in shear.*

It is frequently the case that when a material undergoes a stress below the elastic limit for the first time a very slight permanent set will remain on the removal of the stress, but succeeding applications of the same stress produce no additional permanent deformation. Such a material can be considered to be perfectly elastic for all practical purposes, the slight permanent set, caused by the first application of stress, being probably due to overcoming certain *initial strains*, as they are called, which are set up during the process of manufacture.

In the case of most of the materials used in construction, the strains produced, when stresses below the elastic limit are applied, are proportional to the stress intensities. This law is commonly known as *Hooke's law.* It leads to a definition frequently given for the elastic limit, viz.: *The elastic limit of a material is the limit of stress intensity above which the intensity of stress ceases to be proportional to the strain.* While this definition is practically correct for some of the materials mentioned, it is not strictly true that the stress intensity is proportional to the strain for all elastic materials, and hence the first definition given for the elastic limit is a more correct one.

When a body is subjected to a stress exceeding the elastic limit the ratio between the stress and the strain is less than that between stresses and strains below the elastic limit. In other words, the strain increases more rapidly than the stress after the elastic limit is exceeded and, as a rule, the rate of increase in the strain becomes greater and greater in proportion to the increase in stress, until the breaking point of the material is reached.

The foregoing statement of the relations between stresses and strains is true in general for each of the simple stresses, except that, as we shall see later, a slight modification will be necessary in the case of certain materials, when subjected to compressive stresses.

7. Moduli of Elasticity. — For any elastic material which follows *Hooke's law* the ratio of the tensile stress intensity to the strain produced by that stress is called the *tensile modulus of* *elasticity*. We shall denote its value by the symbol E_t. Hence, following the notation previously adopted,

$$E_t = \frac{p_t}{e_t}. \quad \ldots \ldots \ldots \quad (1)$$

Similarly, the ratio of the compressive stress intensity to the strain produced is called the *compressive modulus of elasticity* and will be denoted by the symbol

$$E_c = \frac{p_c}{e_c}. \quad \ldots \ldots \ldots \quad (2)$$

The metals and most of the other materials with which we have to deal follow the law of proportionality of stress and strain quite closely, if not exactly, below the limits of stress to which they are ordinarily subjected; and it is customary to assign values for the modulus of elasticity of each of these materials, even if the stress to strain ratio is not quite constant. In the cases where this ratio is not exactly constant the values usually given for the moduli of elasticity are the averages obtained for low limits of stress.

For all elastic materials the values of the tensile and compressive moduli of elasticity are practically the same and it is customary to denote both quantities by the letter E, hence

$$E = E_t = E_c. \quad \ldots \ldots \ldots \quad (3)$$

This quantity is also commonly known as *Young's modulus*, after Dr. Thomas Young, who was the first to define it.

The ratio of the intensity of a shearing stress to the shearing strain for an elastic material, following *Hooke's law*, is called the *shearing modulus of elasticity*, or the *modulus of rigidity*. We shall denote it by the symbol G; hence, following the notation previously adopted,

$$G = \frac{s}{\gamma}. \quad \ldots \ldots \ldots \quad (4)$$

The modulus of rigidity in general has a different value from the tensile modulus of elasticity, the relation between the values in the case of steel, for example, being approximately

$$G = \frac{2}{5}E. \quad \ldots \ldots \ldots \quad (5)$$

Units of E and G. — The units in which the values of E and G are expressed are the same as the units of stress intensity, viz.: *lbs. per sq. in.*, *lbs. per sq. ft.*, etc. Since a strain is always expressed by a number, which is the same whatever system of linear units is used, it is evident that the units in which the ratio of stress intensity to strain is expressed must be the same as the unit stresses. This fact may possibly be made clearer by the following modification of the original definition given by Young for the quantity we call the modulus of elasticity, viz.:

The modulus of elasticity is the weight, which, if applied along the axis of a bar of a unit cross section, would produce an elongation equal to the original length of the bar, provided the ratio of stress to strain could remain constant during the distortion.

8. Relation of Tensile Stresses to Strains above the Elastic Limit. — If a straight bar of very ductile steel, or *soft steel*, as it is called, is subjected to an increasing load in tension, the tensile strain in the bar will increase in direct proportion to the load, very nearly, until the elastic limit is reached. After passing the elastic limit the rate of increase in the strain becomes greater than the rate of increase in the stress.

When the stress intensity has reached a value a little higher than the elastic limit, a sudden yielding, or stretching, of the bar takes place without any increase in load; in fact, this yielding may continue at a stress slightly below that at which it begins and the bar will elongate a very considerable amount before any further increase in the load is made.

Yield Point. — The stress intensity at which this action occurs is called the *yield point* of the material in tension. At this point there is apparently a partial breaking down of the structure of the metal which results, as will be shown later, in a marked change in some of its properties. While very fine measurements are necessary to determine the stretch in the bar at lower loads, the elongation at this point is easily discernible by the eye.

After passing the yield point an increase in stress is required to produce a further increase in strain, but the proportion between the increase in strain and the corresponding increase in stress becomes greater as the load is increased.

For loads below the yield point the diminution in the cross section of the bar, due to the lateral strain (Art. 5), can be determined only by very accurate measurement, but above this stress it is easily discernible. The ratio between the lateral and the longitudinal strain is not the same as for stresses below the elastic limit (Art. 5), but increases as the load increases.

After the load reaches the maximum required to break the bar, the stretching continues to a very considerable extent until, just before fracture, a local reduction in the cross section takes place as indicated at A (Fig. 5). As this reduction increases, the load required to continue the stretching becomes less until at the point of fracture the total stress is considerably less than the maximum required to break the bar.

Stress-Strain Diagram. — If we plot the results of a tensile test, like the one just described, with the ordinates representing stress intensities and the abscissæ the corresponding strains, the smooth line drawn through the points thus obtained is called a *stress-strain* diagram. Such a diagram for a bar of soft steel tested in tension is shown by the curve marked C (Fig. 6).

In computing stress intensities it is customary to divide the total stress by the original area of the cross section of the bar before undergoing strain, and hence above the yield point the stress-strain diagram C does not show the actual stress intensities, but only the arbitrary values obtained on the assumption that the area of the cross section remains unchanged.

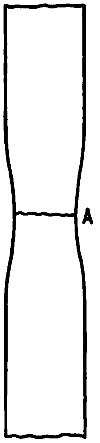


Fig. 5.

Breaking Strength. — The maximum stress intensity, computed in the above manner, which the bar sustains before breaking, is called the *breaking strength*, or *ultimate strength*, of the material in tension, or, simply, its *tensile strength*. An inspection of the plot C (Fig. 6) will show that the breaking strength of the soft steel bar is 50,000 lbs. per sq. in. and that the yield point is between 33,000 and 34,000 lbs. per sq. in.

The curve marked c (Fig. 6) is obtained by plotting the actual stress intensities as ordinates. For stresses below the yield point this line practically coincides with the curve C but diverges more and more from it as the breaking point is approached. Such a plot, when compared with the stress-strain diagram C, shows the difference between the stress intensity, figured in the usual way, and the actual stress intensity, but is of no practical value.

Elastic Limit. — In order to indicate the elastic limit clearly it is necessary to plot the strains to a larger scale than that used in Fig. 6. The diagram marked C (Fig. 7) is made by plotting the strains in the soft steel to a scale 25 times as large as that used in

Fig. 6. From this diagram it may be seen that the elastic limit of the steel is between 24,000 and 25,000 lbs. per sq. in. It is evidently impossible to represent any strain beyond the yield point within the limits of the plot.

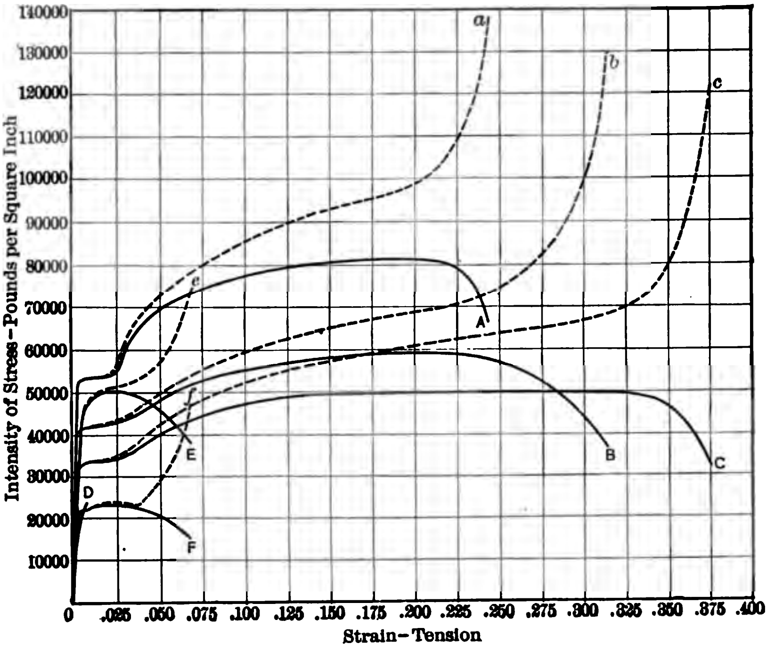


FIG. 6.

Modulus of Elasticity. — A computation of the ratio of stress intensity to strain for any point below the elastic limit will give 30,000,000 lbs. per sq. in. as the value of the modulus of elasticity of the material.

Similar diagrams for two other grades of steel, for cast iron, copper and aluminum are shown in Fig. 6. The plots marked with the large letters are made with the stress intensities computed in the ordinary way and those marked with the small letters represent the actual stress intensities. In Fig. 7, the same diagrams are plotted with strains to the larger scale.

The curves marked *B* and *b* are plotted from the results of a test on a bar of what is usually called "medium" steel and those marked *A* and *a* from a test on a bar of a still harder grade of "machinery" steel, so called.

The elastic limits, yield points and breaking strengths may easily be determined from an inspection of the plots, and it will also be seen that the values of the moduli of elasticity of the three grades of steel are the same.

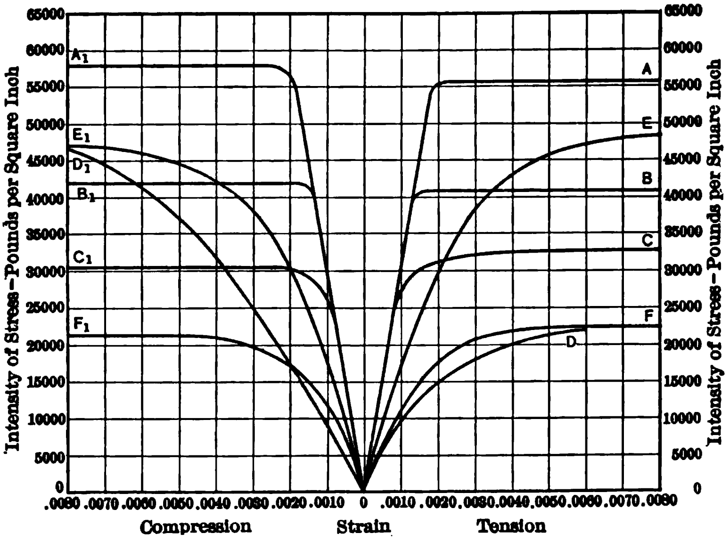


FIG. 7.

The curves marked *D* are plotted from the results of a tensile test on a bar of cast iron of ordinary grade. An inspection of these plots shows that cast iron does not follow Hooke's law exactly, even for low stresses, the stress-strain diagram being a curve throughout. There is no marked elastic limit and no yield point, and the reduction in the area of the cross section is so slight that the curve *D*, for which the stress intensities are figured from the original area of the section, gives very nearly the actual stress intensities also. Although the ratio of stress to strain is not constant it is customary to assign a value for the modulus of elasticity of cast iron, calculated from the values of the stress intensity and strain at a low load; the value in the case illustrated being about 11,000,000 lbs. per sq. in.

The curves marked *E* and *e* represent the results of a tensile test on a piece of copper trolley wire and those marked *F* and *f* the results of a tensile test on a piece of rolled aluminum rod. An inspection of the diagrams (Fig. 7) shows that while the material in these specimens did not follow Hooke's law exactly, the ratio of stresses to strains for low loads was more nearly constant than in the case of cast iron. The elastic limits are not definitely marked; in fact, there are none if the second definition of the elastic limit (Art. 6) is strictly adhered to. An inspection of the plots (Fig. 6) shows that there is no yield point similar to that obtained on a steel specimen, no marked yielding occurring until the breaking point is nearly reached. As in the case of cast iron the moduli of elasticity may be computed from the stress intensity to strain ratios at low loads, the value for the copper trolley wire being very nearly 18,000,000 lbs. per sq. in. and that for the aluminum 12,000,000 lbs. per sq. in.

The foregoing results have been cited to illustrate the variation in the physical properties of a few of the metals. If other materials are investigated a much wider variation will be found. Other tests might be cited to show that the same materials when subjected to different treatment show marked variations from the results given.

Load-Deformation Diagrams. — In ordinary experimental work it is not customary to plot the stress-strain diagrams as illustrated, but instead to use the values of the total tensile load on the bar as ordinates and the total elongations in a given length, called the *gaged length* as abscissæ. A diagram plotted in this way is called a *load-deformation* diagram and, if the scales are properly chosen, it will be very similar to the stress-strain diagram in form.

9. Measure of Ductility. — In addition to determining the physical properties of a material as given in Art. 8 it is customary to measure its ductility when subjected to tension in one, or both, of two ways.

First Method: By determining the total amount a certain gaged length of a test bar elongates before fracture occurs. The result is expressed as a percentage ratio between the increase in the gaged length and its original value before the bar is subjected to strain and is commonly called the *percentage* of *ultimate elongation.* The values obtained from the tests on the three grades of steel (Art. 8) were, respectively, for A, 24.4 per cent; for B, 31.4 per cent and for C, 37.5 per cent. The value obtained for cast iron was very small, viz., for D, 0.6 per cent. The values obtained from the tests on the copper and aluminum were, respectively, for *E*, 6.5 per cent and for *F*, 6.8 per cent. These values may be easily verified from the diagrams (Figs. 6 and 7). The gage length in each case was 8 inches.

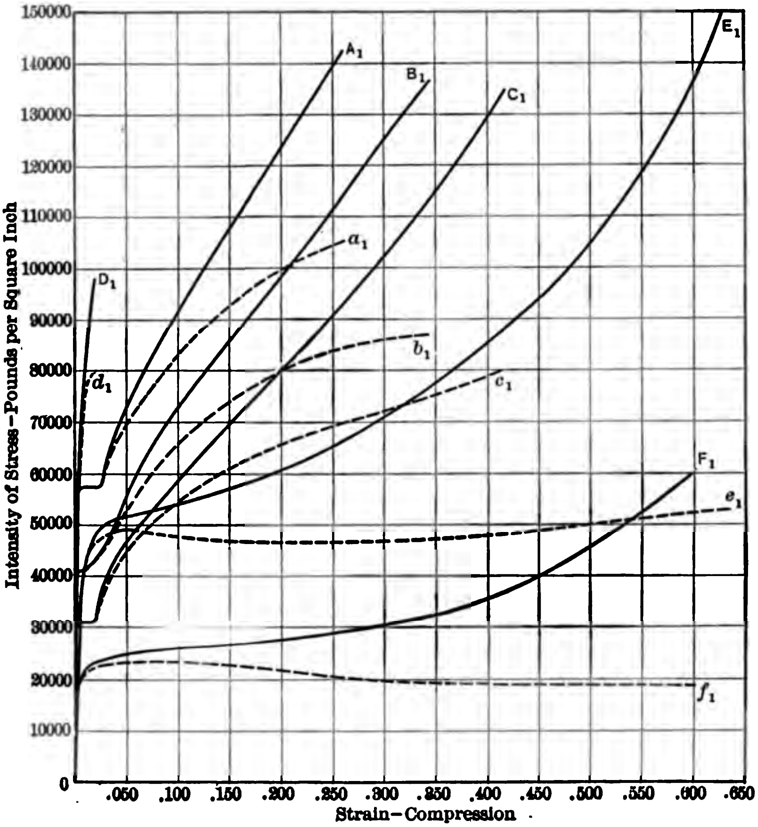


FIG. 8.

Second Method: By dividing the difference in the area of the smallest cross section at the time of fracture and the original area before the stress is applied by the original area and expressing the ratio in per cent. The result is called the *percentage of reduction of the area of the cross section.* Designating the materials by the letters on the plots (Figs. 6 and 7), the values obtained from the tests quoted were, respectively: machinery steel, *A*, 51.4 per cent;

medium steel, B, 66.4 per cent; soft steel, C, 73.3 per cent; cast iron, D, 0.6 per cent; copper trolley wire, E, 52.3 per cent; rolled aluminum rod, F, 67.8 per cent.

10. Effect of Shape of Test Piece on Tensile Properties. — The most common form of test piece for determining the properties of a material in tension is circular in section and of uniform diameter, either throughout its entire length (Fig. 9 *a*) or throughout the middle portion of its length (Figs. 9 *b* and 9 *c*).

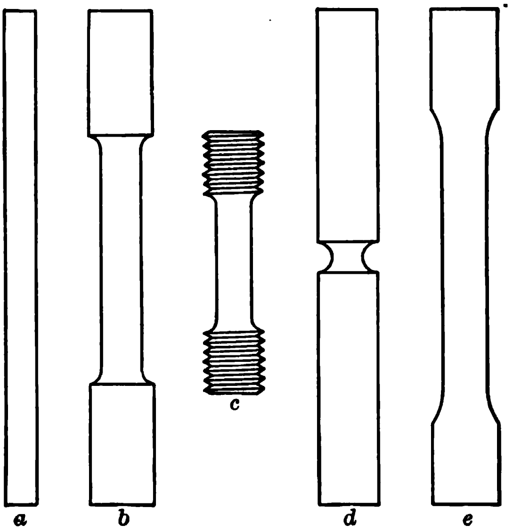


Fig. 9.

The straight cylindrical piece (Fig. 9 *a*) is suitable for a ductile material but in the case of a less ductile material there is a liability of breaking the piece at the points at which it is held by the grips of the testing machine. For the latter material the test piece (Fig. 9 *b*) is better suited, where it is evident that the break must occur in the straight portion between the large ends to which the grips are applied. For brittle material, like cast iron, the form Fig. 9 *c* is used. This is a modification of Fig. 9 *b*, the ends being threaded to fit suitable holders, which should be made with spherical bearings to ensure a correct alignment of the pull on the test piece.

In determining the tensile properties of any material it will be found that, provided the length of the straight portion of the test piece is four diameters or more, the values obtained for the elastic properties and for the breaking strength of the material are not affected to any appreciable extent by varying the size of the piece. It will be found, however, that the measurements of ductility will be affected by varying either the diameter or the length of the test piece. Both the percentage of the reduction of area and the ultimate elongation will decrease as the diameter of the piece is increased, and for any given diameter the percentage of the ultimate elongation will diminish as the gaged length of the piece is increased.

It is, therefore, apparent that comparative values of the tensile properties of different materials, particularly of their ductility, can be obtained only by using test pieces of the same dimensions.

The test piece of the form (Fig. 9 *d*) is given as an example of a form which gives incorrect results. It is evident that the fracture must occur at the section at the bottom of the groove and that it would be impossible to measure the elastic properties or the ductility of the material in the ordinary way. Moreover, the breaking strength obtained from such a specimen will be from 3 to 30 per cent higher than the value obtained from the cylindrical test piece of a length of four diameters or over, the amount of the variation depending on the ductility of the material.

In determining the properties of sheet metal, flat plates and certain rolled sections it is customary to use a test piece of rectangular cross section of the form (Fig. 9 *e*), cut to certain standard dimensions, the thickness of the piece being uniform throughout.

11. Relation of Compressive Stresses to Strains above the Elastic Limit. — A cube, a short cylinder, or a prism, of a brittle material when subjected to an increasing compressive stress will finally fail by breaking down along certain lines of cleavage, making angles in the neighborhood of 30 degrees with the direction of the line of pressure, as indicated by the sketch (Fig. 10 *a*).

A similar piece of a ductile material when subjected to the same treatment will squeeze out into a form similar to that illustrated (Fig. 10 *b*). After a certain pressure is reached, longitudinal cracks similar to those indicated in the sketch will appear at the circumference, but the piece will continue to flatten out and sustain a continually increasing load.

Hence a brittle material will have a definite *breaking strength* in compression, but in the case of the ductile material no breaking or ultimate load is reached. Under lower pressures, however, both the brittle and the ductile materials exhibit properties very similar to those shown when they are subjected to tension and the elastic limits and yield points in compression agree very closely with those shown in tension.

The plots A_1, etc. (Fig. 7), show the compressive stress-strain diagrams for short cylinders cut from the same pieces from which were taken the tensile test pieces for which the diagrams A, etc. (Fig. 7), were made. In each case the letter on the compression diagram is the same as that for the same material on the tension diagram with the subscript added.

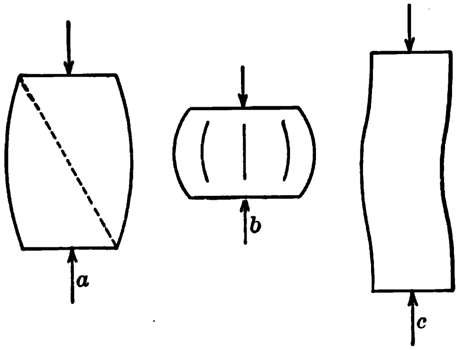


FIG. 10.

A comparison of the diagrams A_1, B_1, C_1 with the diagrams A, B, C will show that for each of the three grades of steel the moduli of elasticity, the elastic limits and the yield points are very nearly the same in compression as in tension. A comparison of the diagrams E_1, F_1 and E, F will show that for the stresses within the limits of the plots the properties of the copper and aluminum are very nearly the same in both compression and tension. The diagram D_1 for cast iron shows that there is not nearly as rapid a decrease in the stress-strain ratio in compression as there is in tension, that the breaking strength in compression is much higher than in tension and that, as in tension, there is no definite elastic limit and no yield point.

The plots shown (Fig. 8) represent the compressive stress-strain diagrams for the same materials with the strains measured to a small enough scale to enable the stress-strain relations above the yield points of the ductile materials to be shown. As in the case of the tensile tests, the stress intensities were computed by dividing the total load on the test piece by the area of the original cross section. The plots of these intensities are shown by the full lines A_1, etc., and may be compared with the corresponding tensile stress intensities to a similar scale (Fig. 6). The dotted lines a_1, etc. (Fig. 8), represent the actual stress intensities computed by dividing the load by the area of the greatest cross section at the time the load was sustained. The plots are of interest as they show the relation between the nominal stress intensity computed from the original area and the actual stress intensity on the greatest cross section, but otherwise are of no practical value.

With the exception of the cast iron shown by the plots D_1 and d_1 no breaking or ultimate strengths were obtained and the diagrams might have been continued indefinitely beyond the limits shown.

The breaking strength of the cast iron in compression was 98,000 lbs. per sq. in., nearly 4.5 times the breaking strength in tension (Fig. 7).

If cylinders, or prisms, of moderate length were used for test pieces, instead of the short pieces from which the foregoing results were obtained, it would be found that the stress-strain diagrams below the elastic limits for the ductile materials would be nearly identical with those shown; but that, at stresses very little greater than the yield points, the test pieces would buckle in the manner indicated (Fig. 10 *c*), after which the loads which they sustained would gradually diminish. Hence, with such test pieces, maximum values of stress intensities, or ultimate strengths in compression, would be obtained. Even a brittle material like cast iron when tested in this manner would fail by buckling with an ultimate strength much less than that obtained with the short test piece.

If the test pieces were made extremely long in comparison with the dimensions of the cross sections they will be found to bend and fail to sustain an increase in load before the stress intensity has reached the elastic limit of the material.

Hence the ultimate strength of a material in compression depends on the relation between the length and the cross section of the piece which is subjected to pressure. A more complete discussion of this relation will follow later under the theory of columns.

12. Stress-Strain Relations in Shear. — It is impracticable to determine the stress-strain relations in shear by direct measurements as in the case of tensile or compressive stresses. An indirect method of doing this will be discussed later under the theory of torsion.

To determine the ultimate or breaking strength in shear, a piece, such as a rivet or a bolt joining two plates together, may be broken in shear along the cross section between the plates by applying a pull in the manner indicated in Fig. 3, means being used at the same time to prevent the bending of the plates due to the lines of pull on the two not coinciding. The ultimate strength of such a piece in shear, or the *shearing strength* as it is usually called, is computed by dividing the greatest load which the piece sustains before failure by the area of its original cross section. The shearing strengths of other materials may be determined in a similar manner.

When the material is in the form of a flat plate, another method of determining its shearing strength is to find the pressure necessary to force a hole through the plate by means of a punch and die and to divide this pressure by the original area of the section of the metal cut through. Owing to the fact that the amount of distortion before failure is generally less in this case than when a bolt or rivet is broken in shear, the value of the shearing strength of a material determined in this way is, as a rule, higher than the value obtained by testing a bolt or rivet.

13. Strains and Stresses due to Changes in Temperature. — All materials undergo small deformations with changes in temperature unless restrained by the action of external forces. When free from stress the amount of the deformation, or strain, due to a change in temperature depends not only on the material but also, to a slight extent, on the treatment it has received during the process of manufacture.

Practically all the materials with which we have to deal expand equally in all directions under an increase in temperature and the ratio of the increase in any linear dimension of a body, under a change in temperature of one degree, to the original value of the dimension is called the coefficient of linear expansion of the material of which the body is composed.

In the case of a straight rod the coefficient of linear expansion would evidently be equal to the change in length of a unit length of the rod under a change in temperature of one degree. In other words, *the coefficient of linear expansion is the extension due to a unit change in temperature,* and its value for any material will depend on the scale of temperature used.

If a body is constrained so that it is not free to expand or contract, stresses will be set up by a change in temperature. If a straight rod, for example, is rigidly held at the ends, a compressive stress will be set up by an increase and a tensile stress by a decrease in its temperature.

The intensity of stress on any cross section of the rod due to a temperature change would be the same as that corresponding to the strain which would be induced by the same change in temperature if the rod were free.

14. Effect of a Stress above the Elastic Limit. — If a bar of steel is subjected to a tensile stress beyond the elastic limit and the stress is then gradually removed and measurements of strain are taken meanwhile, the stress-strain diagram will be very nearly a straight line and when the stress is entirely removed a certain *permanent strain,* or *set* (Art. 6), will remain. The difference between this permanent strain and the total strain in the bar before the load was removed may be called the *elastic strain* at that load and the ratio between this elastic strain and the corresponding stress intensity will be approximately the same as the stress-strain ratio before the elastic limit was exceeded.

If, immediately after the stress has been reduced to zero, the load on the bar is again gradually increased and measurements of strains are made, no definite elastic limit will be found until the load previously applied to the bar has been very nearly reached.

The diagram for a bar of soft steel which has been tested in the above manner is shown by the plot *A* (Fig. 11). The original elastic limit for this steel was 27,500 lbs. per sq. in. After increasing the stress to 35,500 lbs. per sq. in. the load was gradually reduced to zero and then increased immediately. On the second increase of load the elastic limit was found to be about 35,000 lbs. per sq. in. On the diagram the line *bc* for the decreasing load and the line *cd* for the increasing load very nearly coincide and differ very little from a straight line parallel to *oa*, the stress-strain line below the elastic limit. The permanent set is represented by the abscissa *oc*. After reaching the second elastic limit the load was still further increased, the stress-strain diagram following very nearly the same line that it would if no reduction in the load had been made, until a stress of about 46,000 lbs. per sq. in. was reached. The load was again reduced to zero and immediately increased, with a result very similar to that obtained before, the elastic limit this time being a little over 45,000 lbs. per sq. in. The test was then completed by carrying the load to the breaking point.

The diagram marked *B* (Fig. 11) shows the results of a similar test on a bar of medium steel. In this case the elastic limits obtained after decreasing the stress to zero, first from 44,500 lbs. per sq. in. and then from 57,500 lbs. per sq. in., were slightly higher than the stresses previously sustained by the bar. A further increase in the values of these special elastic limits would be obtained if, after the load had been removed, the bar had been allowed a period of rest before the load was again increased. Such a period of rest would result also in a decrease in the amount of ductility remaining in the bar below the normal, which would differ little, if any, from that represented by the ultimate strain on the plot. In other words, a period of rest after the *overstrain,*

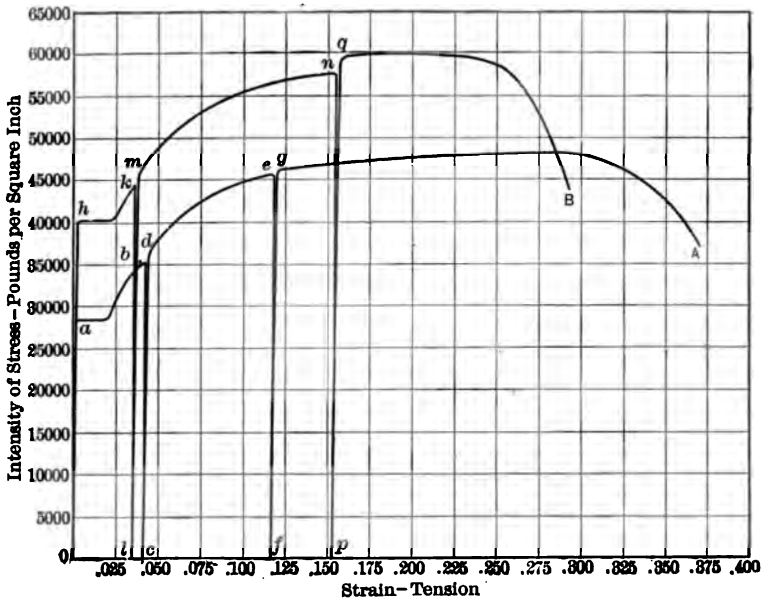


FIG. 11.

as it is frequently called, will further increase the elastic limit and reduce the ductility of the material.

If the lines *bc, cd, ef, fg* on diagram *A* and the lines *kl, lm, np, pq* on diagram *B* were plotted with the strains to a much larger scale they would be found to be slightly curved instead of straight; that is, the stress intensities are not exactly proportional to the strains below the new elastic limits produced by overstraining. The proportion varies so little, however, that the term elastic limit is used to denote the stress intensity at which a distinct change occurs.

Hysteresis of Strain.—The time which elapses during the decrease or increase of the load on the bar will also affect the strains obtained. If the stress is quickly decreased and then held constant the strain will continue to decrease for a considerable time after. The reverse is true if the stress is suddenly increased to a value below the elastic limit, the strain in this case increasing for a considerable time under a constant stress until a state of equilibrium is finally reached. This change in strain under a constant stress is called the *hysteresis of strain.* It may be measured by the difference between the initial strain at a given stress and its final value after the strain has become constant. For stresses below the original elastic limit the hysteresis is barely discernible, but it becomes more distinct after a piece has been subjected to a stress beyond the elastic limit and the stress is then removed.

In the foregoing discussion the results of tensile tests on steel have been cited as illustrations. Similar results would be obtained with steel under compression and with other ductile materials in a greater or less degree.

15. Resilience. — In a general sense resilience may be defined as the ability, which a body under strain possesses, of returning to its original dimensions when the cause of the strain is removed.

In mechanics the term is used to denote the *amount of work* which a body under strain is capable of doing in returning to its unstrained state; that is, the *resilience is the potential energy due to the strain.* A distinction must be made between this quantity and the work done in producing deformation, which will be equal to the resilience only when the body is composed of a perfectly elastic material.

If we refer to Fig. 12, in which the first part of the stress-strain diagram *B* (Fig. 11) is reproduced with the strains to a larger scale, it will be evident that the work done in producing the stress intensity *a* on the cross section of the bar will be equal to the shaded area *oag* multiplied by the proper factor. Since the diagram is plotted to represent unit stresses and strains this factor would be equal to the product of the scales of the stress intensities and strains multiplied by another factor depending on the area of the cross section and the length of the bar. In this case the work done in producing the strain is very nearly equal to the resilience at the stress intensity *a*, since practically no permanent set would remain after the removal of the load, and the area *oag* would represent the work done by the bar during the recovery of its original length.

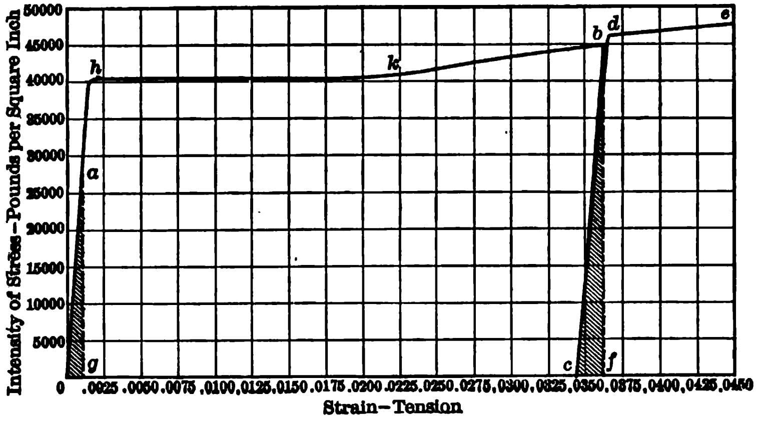


FIG. 12.

If, however, a load sufficient to produce a stress intensity *b*, above the elastic limit, were applied, the work done in producing the strain would be equal to the area *ohkbfo*, multiplied by the same factor as before. In this case the resilience at the stress intensity *b* would be equal to the area *cbf*, multiplied by the same factor; *bc* being the stress-strain line under the decreasing load. It is evident that when the stress intensity is above the yield point a large part of the work done is used up in producing permanent strain, a very small portion remaining in the form of strain energy. Similar results would be obtained with a steel bar subjected to compression and to a varying degree with other materials subjected to tension or compression.

In the light of the foregoing discussion the expression representing the resilience of a bar of any material which follows Hooke's law, under a uniform tensile stress, may be deduced as follows: Let A = the area of the cross section, l = the length of the bar, E = the modulus of elasticity and e = the strain when the stress intensity is equal to p. The total stress on the cross section of the bar will then be equal to $P = pA$ and the total elongation in the bar to $a = el$ and from the form of the shaded portion oag of the diagram (Fig. 12), it is evident that the resilience will be equal to

$$R = \frac{P}{2} \times a = \frac{pA}{2} \times el. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But

$$e = \frac{p}{E};$$

hence

$$R = \frac{P^2 l}{2AE} = \frac{1}{2}\frac{p^2}{E} Al = \frac{1}{2}\frac{p^2}{E} V, \quad . \quad . \quad . \quad . \quad (2)$$

where $V = Al$, the volume of the bar.

In the case of a bar subjected to uniform compressive stress, the expression for the resilience at any stress intensity p will evidently be that given by equation (2); and, if the moduli of elasticity in tension and compression are equal, the values of the resilience in tension and compression will be the same.

Modulus of Resilience. — If V = unity, equation (2) becomes

$$R_0 = \frac{1}{2}\frac{p^2}{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The quantity R_0 is called the *modulus of resilience* of the material at the stress intensity p. It is evident that the resilience of a bar of any dimensions subjected to a uniform tensile or compressive stress of intensity p will be equal to

$$R = R_0 V, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where R_0 = the modulus of resilience of the material at the stress intensity p.

Proof Resilience. — When the stress intensity p is equal to the elastic limit of the material the value of R is sometimes called the *proof resilience* of the bar.

When the stress-strain lines for decreasing stresses are approximately parallel, as those shown in Fig. 11, it is evident that the resilience under a direct stress above the elastic limit will be approximately represented by the same equation (2) as when the stress intensity is equal to or less than the elastic limit.

The term resilience is not restricted in its application to cases of direct stress only, but is used to denote the strain energy due to shear and also that due to more complex stresses, such as are produced by bending and torsion. The deduction of the expressions for its value in such cases must bè deferred until later.

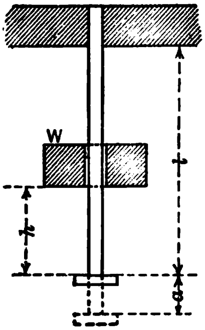


FIG. 13.

16. Effect of an Impact Producing a Direct Stress. — A tensile stress due to an impact may be produced by allowing a sliding weight to fall against a stop on the lower end of a bar which is rigidly held in a vertical position by a support at its upper end, as shown in Fig. 13. If the bar is perfectly elastic it will act like a spring, stretching a maximum amount a when the blow is struck and then oscillating slightly until equilibrium is restored, the final elongation being the same as would be produced by a gradual application of the weight.

If the bar is of uniform section and material, and l = its original length, A = the area of the cross section, p = the maximum stress intensity produced and h = the distance through which the weight falls before striking the stop, we may write

$$W(h + a) = \frac{1}{2}pAa = \frac{1}{2}\frac{p^2Al}{E}. \quad . \quad . \quad . \quad . \quad (1)$$

Solving for p we obtain

$$p = \frac{2W}{A}\left(\frac{h + a}{a}\right) = \sqrt{\frac{2WE(h + a)}{Al}}. \quad . \quad . \quad . \quad (2)$$

If we eliminate a, by substituting its value $a = \frac{pl}{E}$ in equation (2), and reduce we obtain

$$p = \frac{W}{A} + \sqrt{\frac{2WhE}{Al} + \frac{W^2}{A^2}} = \frac{W}{A}\left[1 + \sqrt{\frac{2AhE}{Wl} + 1}\right]. \quad (3)$$

The above equations are based on the assumption that all of the work done by the force of gravity acting on the falling weight is used up in stretching the bar. Since neither the weight nor the stop can be absolutely rigid it is evident that a part of the energy of the blow would be absorbed in compressing the weight and the stop and hence the actual value of p would always be somewhat less than that given by equation (2) or (3).

The effect of a weight falling on a bar of elastic material in a manner to produce compression would evidently be the same as in the case of tension.

When the force of the blow is great enough to produce a stress above the elastic limit it is evident that the above equations no longer hold true. If the stress produced is above the yield point, nearly all of the energy of the blow is used up in producing permanent distortion; and the greatest stress intensity will be very much less than that given by equation (2).

17. Suddenly Applied Load. — If in the preceding case the weight W were suspended so as to be in contact with the stop (Fig. 13), without exerting any pressure upon it, and then were allowed to fall we would have an illustration of what is usually called a suddenly applied load in tension; that is, a load applied instantly, but without any blow, or shock. Substituting $h = 0$ in equation (2) (Art. 16), we obtain for the value of the maximum intensity of stress produced by a load applied in this manner,

$$p = \frac{2\,W}{A}, \quad \ldots \ldots \ldots \quad (1)$$

which is evidently double the intensity which would be produced, if the load W were gradually applied to the bar.

The same relation would exist between the maximum stress intensity due to a suddenly applied load and a gradually applied load in compression.

As in the case of the stress due to impact (Art. 16), if the maximum stress intensity p is above the elastic limit the relation expressed by equation (1) no longer holds true; and above the yield point the value of p due to a suddenly applied load will be very much less than that given by the equation.

18. Effect of the Rate of Application of the Load on the Maximum Stress Intensity. — In Art. 17 we have deduced the relation between the maximum direct stress intensity produced by a suddenly applied and a gradually applied load. In many cases in practice the load to which a piece is subjected is not applied instantaneously, but at a rate sufficient to produce a maximum stress intensity greater than would be obtained if it were gradually applied. In the light of the preceding discussion the maximum stress intensity produced by such a load would vary with the rate of application, but would never be greater than in the case of a suddenly applied load.

A load, rapidly applied in the above manner, or applied so as to produce a shock (Art. 16), is frequently called a *dynamic load*, in distinction from a stationary or *static load*.

On account of the uncertainty which usually exists in practice in regard to the exact rate of application of a dynamic load, the determination of the maximum stress intensity produced must be largely by an estimate based on the results of experience.

When the maximum stress intensity produced by a dynamic load exceeds the yield point the difference between it and the stress intensity due to the same static load is, as has already been stated (Arts. 16–17), not nearly so great as when the elastic limit is not exceeded. From an inspection of the stress-strain diagrams for ductile steel (Figs. 11–12), it would appear that when the maximum stress intensity produced by a rapidly applied load, which was not of the nature of a blow, exceeded the yield point it would be very little greater than that produced by a static load of the same magnitude.

Another factor, not previously considered, enters into the case, however; namely, the rate at which a strain above the yield point, or, as is commonly stated, the rate at which the flow of the material above this point takes place. Thus, it would be found in the case of the steel for which the diagrams are shown (Figs. 11 and 12), that, if the load were very rapidly increased, the stress-strain line above the elastic limit would be higher than that shown, since time enough would not be allowed for the complete distortion due to each increment of load to take place. The result would be that all of the stresses corresponding to given amounts of distortion, including the ultimate strength, would be higher than those obtained with a gradually increasing load.

The results obtained with other materials would be similar, to a greater or less degree, to those obtained with steel; and also, the results obtained under compressive stresses would be affected similarly to those obtained under tensile stresses.

It is evident from the above that in order to determine correctly the properties of a material under a tension or compression stress the rate of application of the load must be slow enough to allow the normal amount of distortion due to each increment of load to take place.

19. Effect of Varying Stresses. — It is a well known fact that when a piece is subjected to a varying stress, such as would be produced by a continually changing, or by an intermittent load, failure will occur when the maximum intensity of the varying stress is considerably below the breaking strength, determined in the usual way (Art. 8), provided the fluctuation in the stress is repeated a sufficient number of times. The first thorough investigation of the influence of varying stresses in producing fracture, and on other properties of iron and steel, was made by Wohler, between 1860 and 1870. Since then a large amount of experimenting along the same line on iron and steel and other materials has been done by other investigators.

It is not within the province of this text to give in detail the results of these investigations, but merely to state in a general way some of the conclusions which may be drawn from them. These conclusions follow:

When a piece is subjected to a continually fluctuating stress in tension, which varies from zero to a given maximum intensity, it will finally break after a certain number of repetitions of the variation in stress, if the maximum stress intensity is above the elastic limit of the material, determined in the usual way (Art. 6). The number of repetitions of the variation in stress required to produce fracture will increase as the maximum intensity of the stress is decreased to a value at which an indefinite number of repetitions fails to produce fracture.

When a piece is subjected to a varying stress which fluctuates between a maximum and a minimum intensity, the latter being greater than zero, the value of the maximum intensity at which the piece will endure an indefinite number of repetitions of the variation, without fracture, is greater than where the variation each time is between a maximum and zero. As the range of variation decreases the value of this maximum increases, until it becomes equal to the ultimate strength in tension when the variation in stress equals zero.

The foregoing conclusions will apply to a limited extent in the case of a varying compressive stress, but, since short pieces of a ductile material have no ultimate strength in compression and pieces of moderate length fail when the stress intensity is at, or near, the yield point (Art. 11), the effect of a varying stress on the point of failure is not so marked as in the case of a varying stress in tension.

When a piece is subjected to a load which varies in such a manner that the stress continually fluctuates from a maximum value in tension to a maximum value in compression, and vice versa, it will fail ultimately at a lower maximum stress than when subjected to varying stress in tension or compression alone. The effect of the continual reversal of stress is much more severe than that due to the variation of one kind of stress and, if the variation is repeated long enough, failure will frequently occur when the greater of the two maximum stress intensities is considerably below the elastic limit of the material, determined in the usual way.

Most of the experimental investigation of the effect of this kind of a varying stress has been made by bending pieces back and forth, either directly or by rotating, as a shaft, under a constant load. In the latter case, as will be shown later, the maximum intensities of the alternate tensile and compressive stresses are always the same.

While the effect of a continual reversal of stress by bending is probably not exactly the same as that due to subjecting a piece to uniform tensile and compressive stresses alternately, the results obtained by the former method are of much greater practical value than those that might be obtained by the latter, and our conclusions in regard to the effect of continually reversing stresses are based on the results of experiments with pieces subjected to repeated bending.

As in the case of a varying stress of one kind, it may be said that the number of repetitions of the variation required to produce fracture increases as the maximum stress intensity is decreased, until a stress is reached at which an indefinite number of repetitions fails to produce fracture; also, if the alternate maximum intensities of the tensile and compressive stresses are not the same, the greatest stress at which a piece will stand an indefinite number of repetitions is higher than when the two maximum intensities are equal.

Fatigue. — The cause of failure under varying stresses when the greatest stress intensity is considerably below the *ultimate strength* of the material has been ascribed to the *fatigue of the material* under repeated or varying stress. Investigations have been made to determine the nature of this fatigue, that is, the change in the structure of the material under repeated stress, but the results of these will not be discussed here.

The conditions under which a varying stress is applied, as well as the nature and the amount of the variation, have a considerable effect upon the stress intensity at which failure occurs. Thus, the rate of the fluctuation of the stress will have an influence, the maximum stress intensity at failure being somewhat lower for the same amount of variation when the rate is rapid than when it is slow. If there is a sudden change in the size or shape of the cross section of the piece the maximum stress intensity at failure will be considerably lower than when the section is uniform throughout.

Certain formulas of an empirical nature have been proposed to represent the value of the maximum stress intensity at failure under different conditions of varying stress, in terms of the ultimate strength of the material and the range of the variation of the stress. On account of the number of different conditions which have an influence on the stress at failure, such formulas are of little practical value and, as experimental data sufficient to completely verify them is lacking also, they will not be quoted here.

As previously stated, the object of the foregoing discussion has been, not to quote values which might be suitable to use in certain specified cases, but to emphasize the fact that under a varying or repeated stress a piece will fail at a much lower stress intensity than its ultimate strength under a static load; and that the nature and the amount of the fluctuation in stress, as well as the conditions under which it occurs, will have a decided influence on the maximum stress intensity which a piece will stand an indefinite number of repetitions.

20. Working Strength. — Factor of Safety. — The *working strength* is the maximum intensity of stress to which a member of a machine or structure may be subjected without exceeding the limits of safety. The term is applied to any of the simple stresses, tension, compression or shear.

The *factor of safety* is the ratio between the breaking strength under a static load and the value assigned for the working strength. Its value depends on the conditions under which a piece is subjected to stress and may be as low as 2 or 3, or as high as 20 or more.

The *working load* is the load required to produce a maximum stress intensity in a member equal to the working strength. The term is used in the case of any of the three simple stresses and, also, as we shall see later, is applied to the load required to produce a maximum stress intensity equal to the working strength in a piece subjected to a complex stress.

The *working resilience* is the resilience of a piece under its working load.

The determination of the proper value of the factor of safety is one of the most important questions with which the engineer has to deal. While it is comparatively easy, by assigning small values to the working strength, to make any piece strong enough to bear any load to which it may be subjected, economy in the use of material and in the cost of a structure demand that the working strength shall be as high as proper considerations of safety will admit.

Hence, in the light of the discussion in this chapter, the value of the working strength for a given member will depend on whether the piece is to be subjected to a constant or a varying load and, if the latter, on the amount and rapidity of the variation, on the liability to shock, and, to a certain extent, on the form of the member.

A satisfactory answer to the question as to what value to use, in any particular case, must be based on a thorough knowledge of the properties of the material of which the member is composed and the results of long experience with its use under different conditions in practice. While, in a large number of cases with which the engineer has to deal, such data has been compiled in forms which require only a brief reference in order to obtain proper values for the working strength under different conditions, new problems are constantly arising, in the solution of which the engineer is thrown largely upon his own judgment and resources.

21. Problems — Physical Properties of Materials.

Problem 1.

From a tensile test of a steel bar, 1″ diam., the following data were obtained:

Maximum load	64,000 lbs.
Load at elastic limit	38,000 lbs.
Elongation in a length of 8″ between loads of 2000 lbs. and 30,000 lbs.	0.0096″
Total elongation in 8″ at fracture	1.70″
Average diameter of smallest section at fracture . . .	0.72″

Find the breaking strength, the elastic limit, the modulus of elasticity, the percentage of ultimate elongation in 8″, the percentage of the reduction of area.

Given a factor of safety of 5, find the working load, the working strength, the working elongation and resilience in a section 5 ft. in length.

Solution.—

(a) Breaking strength $= \frac{\text{breaking load}}{\text{original section}} = \frac{64{,}000}{0.7854} = 81{,}500$ lbs. per sq. in.

(b) Elastic limit $= \frac{\text{load at elastic limit}}{\text{original section}} = \frac{38{,}000}{0.7854} = 48{,}400$ lbs. per sq. in.

(c) Modulus of elasticity $= \frac{\text{stress intensity}}{\text{strain}}$.

The elongation in 8″, produced by increasing the load from 2000 lbs. to 30,000 lbs., is 0.0096″, hence

$$E = \frac{Pl}{Aa} = \frac{28{,}000 \times 8}{0.7854 \times 0.0096} = 29{,}700{,}000 \text{ lbs. per sq. in.}$$

(d) Percentage elongation.

The percentage elongation in 8″ will be equal to

$$\frac{1.7}{8.0} \times 100 = 21.3 \text{ per cent.}$$

(e) Percentage reduction of area:

Area of 1.00″ circle = 0.7854 sq. in. = original section
Area of 0.72″ circle = 0.4072 sq. in. = fractured section

Difference = 0.3782 sq. in. = reduction of area.

Hence, the percentage reduction of area will be equal to

$$\frac{0.3782}{0.7854} \times 100 = 48.2 \text{ per cent.}$$

(f) Working load $= \frac{\text{breaking load}}{\text{factor of safety}} = \frac{64{,}000}{5} = 12{,}800$ lbs.

(g) Working strength $= \frac{\text{working load}}{\text{original section}} = \frac{12{,}800}{0.7854} = 16{,}300$ lbs. per sq. in.

or,

Working strength $= \frac{\text{breaking strength}}{\text{factor of safety}} = \frac{81{,}500}{5} = 16{,}300$ lbs. per sq. in.

(h) Working elongation = working strain × length.

$$\text{Working strain} = \frac{\text{working strength}}{\text{modulus of elasticity}} = \frac{16{,}300}{29{,}700{,}000} = 0.000549,$$

hence,

$$\text{Working elongation} = 0.000549 \times 60 = 0.0329 \text{ in.}$$

It is evident, in this case, that the working elongation may also be found by direct proportion.

(i) Resilience.

Since the load is gradually applied, the resilience produced by the working load will be equal to

$$R = \frac{\text{working load}}{2} \times \text{working elongation} = \frac{12{,}800}{2} \times 0.0329 = 211 \text{ in. lbs.}$$

Problem 2.

From a tension test of a steel bar 2″ diam., the following data were obtained:

Maximum load	248,000 lbs.
Load at elastic limit	160,000 lbs.
Elongation in a length of 8″ between loads of 4000 lbs. and 136,000 lbs.	0.0112 in.
Total elongation in 8″ at fracture	1.76 ins.
Average diameter of smallest section at fracture	1.60 ins.

Find the breaking strength, the elastic limit, the modulus of elasticity, the percentage elongation in 8″, the percentage reduction of area.

If the factor of safety equals 5, find the working load, the working elongation, and the working resilience for a length of 10 ft.

What suddenly applied load will produce a stress in the bar equal to the working strength?

Problem 3.

Find the working load in tension for a steel eye bar, 20 ft. long, of rectangular cross section ¾″ × 5″. Find the resilience of the bar at this load. Assume working strength of steel in tension = 15,000 lbs. per sq. in. and E = 29,000,000 lbs. per sq. in.

Problem 4.

Find the safe axial load for a cast-iron column 15 ft. long, of the cross section shown (Fig. 14): also find the total amount the column will shorten under the load.

Assume the breaking strength of a cast-iron column = 30,000 lbs. per sq. in., factor of safety = 6, E = 14,000,000 lbs. per sq. in.

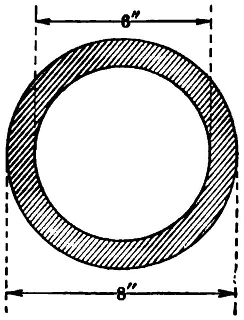


Fig. 14.

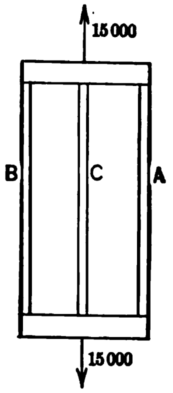


Fig. 15.

Problem 5.

A straight steel bar, 1½″ diameter and 10 ft. long is held rigidly at the ends so that its length remains constant during a change of temperature from 80° F.

to 40° F. If the stress in the bar is zero at 80° find the total pull exerted when the temperature is 40°.

Assume the coefficient of linear expansion of steel = 0.0000067 per degree Fahrenheit and E = 28,000,000 lbs. per sq. in.

Problem 6.

A weight of 40 lbs. is hung on a steel wire, 0.05″ diam. and 20 ft. long. Find the elongation and resilience of the wire under this load. If the temperature falls 10° F. after the weight is hung on the wire find the total change in length. Assume E = 30,000,000 lbs. per sq. in. and the coefficient of linear expansion = 0.0000062 per degree Fahrenheit.

Problem 7.

Two bars of steel A and B, each ¾″ diam., and a bar of copper C, 1″ diam., are fastened together as shown (Fig. 15) and subjected to a total load of 15,000 lbs. The bars are each 30″ long. Find the intensity of the tensile stress and the total elongation in each bar, assuming that the bars elongate equally.

Assume E = 28,000,000 lbs. per sq. in. for steel and E = 15,000,000 lbs. per sq. in. for copper.

Problem 8.

Find the elongation in a straight bar of steel 50 ft. long due to its own weight when suspended in a vertical position from its upper end. Assume E = 28,000,000 lbs. per sq. in. and the weight of steel = 0.28 lb. per cu. in.

Solution.— Assume the origin at the lower end of the bar (Fig. 16) and let l = the length of the bar (ins.); A = area of cross section (sq. ins.); w = the weight of the material per cubic inch; E = the modulus of elasticity of the material; p = the intensity of stress on any cross section; a = total elongation of the bar.

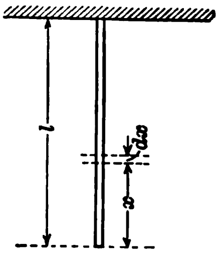


FIG. 16.

Then, the intensity of the stress at any section, at a distance x from the lower end, will be

$$p = \frac{Awx}{A} = wx,$$

and the corresponding strain

$$e = \frac{wx}{E}.$$

The elongation in a length dx will be

$$da = \frac{wx}{E}\,dx,$$

and, for the entire rod,

$$a = \frac{w}{E}\int_0^l x\,dx = \frac{wl^2}{2E}.$$

Substituting the values of w, l and E, in this equation, we obtain

$$\text{Total elongation} = a = \frac{28 \times 50 \times 50 \times 12 \times 12}{100 \times 2 \times 28{,}000{,}000} = 0.0018 \text{ in.}$$

Note. — The problem might be solved by finding the average strain and multiplying it by the total length of the rod, as follows:

The average stress intensity will be

$$\frac{28 \times 50 \times 12}{100 \times 2} = 84 \text{ lbs. per sq. in.}$$

The average strain will be

$$\frac{84}{28{,}000{,}000} = \frac{3}{1{,}000{,}000}.$$

Hence, the total elongation in 50 ft. is equal to

$$\frac{3 \times 50 \times 12}{1{,}000{,}000} = 0.0018 \text{ in.}$$

Problem 9.

A straight steel bar 10 ft. long, of rectangular cross section, varying uniformly from 1″ × 2″ at one end to 1″ × 4″ at the other, is subjected to an axial load of 25,000 lbs. Find the total elongation of the bar, assuming $E = 30{,}000{,}000$ lbs. per sq. in. Find the resilience of the bar at this load.

Problem 10.

A round bar of steel, 5 ft. long, which is 2″ diameter for a length of 3 ft. at the center and 1″ for a length of 1 ft. at each end, is subjected to an axial load of 12,000 lbs. Find the resilience of the bar at this load. Assume $E = 28{,}000{,}000$ lbs. per sq. in.

Compare this value with that for a bar of steel, 5 ft. long and 1″ diameter throughout, when subjected to the same load.

Problem 11.

A steel tube 0.5″ thick is fitted loosely on a solid cast-iron cylinder, 4″ diameter. The length of both tube and cylinder is 25″. If the whole is subjected to an axial load in compression of 100,000 lbs., find the intensity of the compressive stress in the steel and in the cast iron. Assume that the strains in both cast iron and steel in the direction of the axis of the cylinder are equal and that $E = 30{,}000{,}000$ lbs. per sq. in. for steel and $E = 14{,}000{,}000$ lbs. per sq. in. for cast iron.

Problem 12.

A solid cylinder of concrete, 8″ diameter and 24″ long, is subjected to an axial load in compression of 30,000 lbs. Find the maximum intensity of the compressive stress at any point in the middle portion of the cylinder.

Find the amount the cylinder is shortened if $E = 3{,}000{,}000$ lbs. per sq. in.

CHAPTER II.

ANALYSIS OF STRESS AND STRAIN.

§ 1. STRESS.

22. Analysis of Stress. — Stress has been defined (Art. 3) as the force which is exerted at a section of a body, the section being either plane or curved, and being taken either through the body itself, or forming the boundary surface with a contiguous body. A stress at any section of a body may usually be considered to be due to the action of external forces; but it may also be due to internal forces, such as magnetic, or gravitational, attractions between the particles of a body, and to the forces between the particles due to unequal expansion, or contraction, of its parts caused by temperature changes. Having defined the three simple stresses and shown that the stress on any plane section can always be resolved into normal and shearing components (Art. 3), we will now have occasion to use the term stress in a still broader sense, which is usually embodied in the expression *stress at a point.*

Whenever the term *intensity of stress at a point* is used, some plane passing through the point is always understood. Since an infinite number of planes can be passed through any given point and the stress intensity on each of these planes may be a different quantity, the stress at the point can be fully determined only when the resultant stress intensity on every plane passing through the point is known. For a complete determination of the stress throughout a given body the stress at every point in the body must be known.

It is convenient to consider two cases: (a) the case in which the resultant stresses on all planes passing through a point are parallel to a single plane; (b) the case in which they are not.

We shall confine our attention at present to the first case.

23. Plane Stress. — When the resultant stresses on all planes passing through a given point in a body are parallel to a single plane the body is said to be subjected to *plane stress* and the plane to which the stresses are parallel may be called the *plane of stress.*

In such a case the stress on any plane section, which is perpendicular to the plane of the stress, may in general be resolved into a normal and a shearing component which are both parallel to that plane.

We shall now deduce certain relations between the stress intensities on the different planes passing through a given point in a body subjected to plane stress and will show that the stress at any point is fully determined when the stress intensities on any two planes passing through the point and perpendicular to the plane of stress are known. In each case the Z plane is taken as the plane of stress and the X and Y planes are any two planes at right angles to each other, passing through any point O, which are perpendicular to the plane of stress.

We shall adopt the following notation:

n_x = the intensity of the normal stress component on the X plane at the point O,
s_x = the intensity of the shearing stress component on the X plane at the point O,
p_x = the intensity of the resultant stress component on the X plane at the point O,
n_y = the intensity of the normal stress component on the Y plane at the point O,
s_y = the intensity of the shearing stress component on the Y plane at the point O,
p_y = the intensity of the resultant stress component on the Y plane at the point O,
n_a = the intensity of the normal stress component on the A plane at the point O,
s_a = the intensity of the shearing stress component on the A plane at the point O,
p_a = the intensity of the resultant stress component on the A plane at the point O,

where the A plane is any plane through O, perpendicular to the Z plane, the normal to which makes an angle α with the axis OX and an angle β with the axis OY.

A normal component or a shearing component will be considered positive when the vector representing the stress intensity on the positive side of any plane is directed positively along the axis which is normal to the plane.

24. Relation of Intensities of Shearing Stress on the X and Y Planes. — Let the X and Y planes (Fig. 17) be any two planes at right angles passing through any point O of a body in equilibrium under plane stress parallel to the Z plane. Consider a small rectangular prism *Oabc*, three faces of which are bounded by the X, Y, Z planes. Assume that the dimensions *Oa* and *Oc* are so small that the stress intensities on opposite parallel faces may be considered equal and uniform and that the third dimension of the prism is unity. Such a prism, being a small portion of the entire body which is under stress, must be in equilibrium under the stresses acting on its six faces. Since the stresses on the opposite faces are equal and uniform, the normal components will balance; the shearing components on the faces *Oc* and *ab* will form a couple whose moment is equal to $-s_x\,(Oc)\,(Oa)$; and the shearing components on the faces *Oa* and *bc* will form a couple whose moment is $s_y\,(Oa)\,(Oc)$. To maintain equilibrium these couples must balance and hence

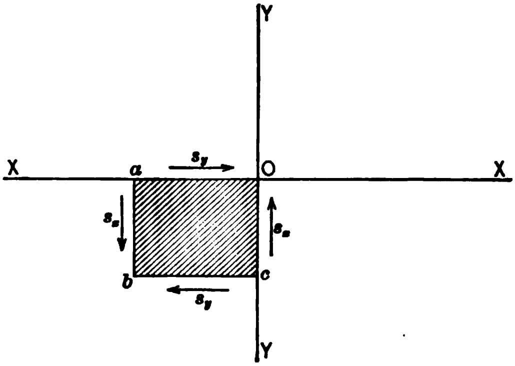


FIG. 17.

$$s_y\,(Oa)\,(Oc) - s_x\,(Oc)\,(Oa) = 0.$$

$$\therefore\ s_y = s_x.$$

Therefore, the intensities of the shearing components of the stresses on the X and Y planes, through any point of a body subjected to plane stress parallel to the Z plane, are equal.

25. Stress Intensities on Different Planes through a Point in a Bar Subjected to Uniform Tension. — Let Fig. (18) represent a bar subjected to a uniform tension P. Let OX and OY be a pair of rectangular coördinate axes through any point O, the axis OX coinciding with the central axis of the bar; A = the area of a cross section perpendicular to OX; B = the area of any oblique section whose normal OB makes an angle α with OX. Hence,

$$B = \frac{A}{\cos\alpha}.$$

Adopting the notation of Art. (23), we obtain for the intensity of stress at O on the X plane

$$n_x = \frac{P}{A},$$

and for the intensity of stress at O on the oblique plane, in the direction of the resultant P,

$$p_b = \frac{P}{B} = \frac{P}{A}\cos\alpha = n_x \cos\alpha. \quad \ldots \quad (1)$$

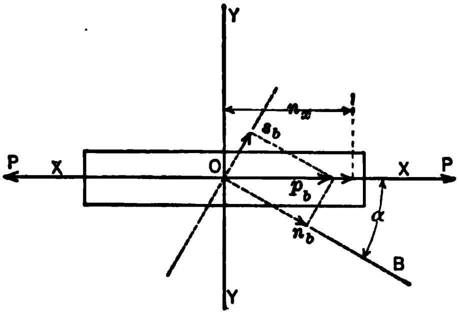


Fig. 18.

Resolving this resultant intensity into normal and shearing components on the oblique section, we have for the normal component

$$n_b = p_b \cos\alpha = n_x \cos^2\alpha \quad \ldots \ldots \quad (2)$$

and for the shearing component

$$s_b = p_b \sin\alpha = n_x \sin\alpha \cos\alpha = \frac{n_x}{2}\sin 2\alpha. \quad \ldots \quad (3)$$

Equations (1), (2) and (3) give the values for the resultant intensity of stress at the point O, on any oblique plane passing through O, and its normal and shearing components on that plane, in terms of the intensity of stress on the cross section and the angle between the planes.

Since the tension is uniform the same result might be obtained by resolving the resultant force P into two components, one

$$N = P \cos \alpha, \quad \ldots \ldots \quad (4)$$

normal to the oblique plane, and the other

$$T = P \sin \alpha, \quad \ldots \ldots \quad (5)$$

in the oblique plane. These components, when divided by the area of the oblique section, will give the same results as equations (2) and (3).

When

$$\alpha = 90°, \qquad n_b = 0 \qquad \text{and} \qquad s_b = 0,$$

there being evidently no stress on longitudinal sections through the bar.

26. Stress Intensities on Different Planes through a Point in any Body Subjected to Plane Stress. — Let OX, OY and OZ be three coördinate axes through any point O, the axis OZ being perpendicular to the plane of stress. Let OA be the normal to any plane through O, perpendicular to the Z plane, and let α and β be the angles between OA and OX and OY, respectively.

Assume that all the stress components on the X and Y planes are positive as shown (Fig. 19). Consider a small triangular prism Oab, bounded by the X, Y and Z planes, with the face ab parallel to the A plane. Let the dimensions Oa, Ob and ab be so small that the stresses on the faces of the prism may be considered uniform and the stress intensity on the face ab equal to that on the A plane. Let the length of the prism in the direction OZ equal unity. Since

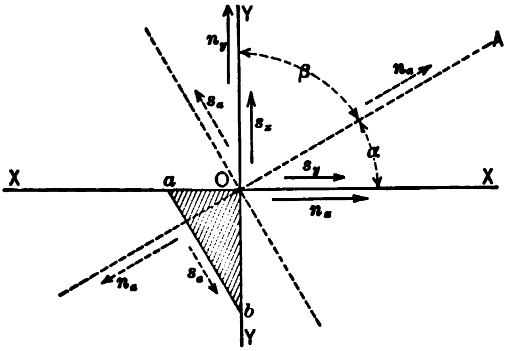


FIG. 19.

the intensities of the shearing stresses on the X and Y planes are equal (Art. 24), we may write

$$s_x = s_y = s_{xy},$$

where s_{xy} = the shearing intensity on either the X or the Y plane. It is evident that the intensity of the resultant stresses on the X and Y planes will be respectively equal to

$$p_x = \sqrt{n_x^2 + s_{xy}^2}$$

and

$$p_y = \sqrt{n_y^2 + s_{xy}^2}.$$

Since the prism Oab is in equilibrium under the stresses on its faces we may deduce the relations between n_a, s_a and p_a, the normal, shearing and resultant intensities of the stress on the face ab, and the stress intensities on the X and Y planes as follows:

Resolve all the stress components on the faces of the prism into components parallel and perpendicular to OA. Since the algebraic sum of the components in the two directions will equal zero,

$$n_a\,(ab) = n_x\,(Ob)\cos\alpha + n_y\,(Oa)\sin\alpha + s_x\,(Ob)\sin\alpha + s_y\,(Oa)\cos\alpha,$$

$$s_a\,(ab) = n_y\,(Oa)\cos\alpha - n_x\,(Ob)\sin\alpha - s_y\,(Oa)\sin\alpha + s_x\,(Ob)\cos\alpha.$$

Substituting $Oa = (ab)\sin\alpha$, $Ob = (ab)\cos\alpha$, $s_x = s_y = s_{xy}$, and eliminating (ab), we obtain

$$n_a = n_x\cos^2\alpha + n_y\sin^2\alpha + 2\,s_{xy}\sin\alpha\cos\alpha, \quad . \quad . \quad . \quad (1)$$

$$s_a = (n_y - n_x)\sin\alpha\cos\alpha + s_{xy}(\cos^2\alpha - \sin^2\alpha)$$

$$= \frac{n_y - n_x}{2}\sin 2\alpha + s_{xy}\cos 2\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The resultant stress intensity on (ab) will evidently be equal to

$$p_a = \sqrt{n_a^2 + s_a^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

from which equation its value in terms of the stress intensities on the X and Y planes might be obtained by substitution.

The value of p_a in terms of these stress intensities may be obtained more directly, however, by the addition of the stress components on the prism Oab, acting in the directions of the X and Y axes. Thus, if we resolve the resultant stress intensity on the face (ab) into a component E in the direction OX and a component F in the direction OY, we shall have

$$E\,(ab) = n_x\,(Ob) + s_y\,(Oa)$$

and

$$F\,(ab) = n_y\,(Oa) + s_x\,(Ob).$$

Eliminating (*ab*), (*Ob*) and (*Oa*), as before, we obtain

$$E = n_x \cos\alpha + s_y \sin\alpha, \quad . \; . \; . \; . \; . \; . \; . \tag{4}$$

$$F = n_y \sin\alpha + s_x \cos\alpha. \quad . \; . \; . \; . \; . \; . \; . \tag{5}$$

Therefore,

$$p_a = \sqrt{E^2 + F^2} = \sqrt{[(n_x \cos\alpha + s_y \sin\alpha)^2 + (n_y \sin\alpha + s_x \cos\alpha)^2]},$$

which easily reduces to

$$p_a = \sqrt{[n_x^2 \cos^2\alpha + n_y^2 \sin^2\alpha + 2\,s_{xy}(n_x + n_y)\sin\alpha\cos\alpha + s_{xy}^2]}. \tag{6}$$

The expressions (1), (2) and (6) give the intensities of the normal, shearing and resultant stresses on any plane, whose normal makes an angle α with the axis *OX*, in terms of the intensities of the normal and shearing stresses on the *X* and *Y* planes.

27. Planes on Which there is no Shear. — If we put equation (2) (Art. 26) equal to zero and solve for the value of $2\,\alpha$ we shall obtain

$$\frac{2\,s_{xy}}{n_x - n_y} = \frac{\sin 2\,\alpha}{\cos 2\,\alpha} = \tan 2\,\alpha. \quad . \; . \; . \; . \; . \; . \tag{1}$$

Since $\tan 2\,\alpha$ may have any value between $+\infty$ and $-\infty$, the solution of this equation will give real values for $2\,\alpha$ for all possible values of n_y, n_x and s_{xy}. Moreover, for every value of $\tan 2\,\alpha$ there are a series of values of $2\,\alpha$, differing by 180°, and hence in every case the solution of equation (1) will give two or more values of α, differing by 90°. Therefore, at every point of a body subjected to plane stress there are two planes passing through the point at right angles, perpendicular to the plane of stress, on which there is no shearing stress.

28. Principal Stresses. — Ellipse of Stress. — The planes of no shear (Art. 27) are called *principal planes of stress* and the stresses on these planes are called *principal stresses*. We shall now show that, at any given point in a body subjected to plane stress, the intensity of one of the principal stresses is greater than, and the intensity of the other less than, the resultant stress intensity at that point on any other plane passing through it.

In this case we will let the *X* and *Y* planes represent the principal planes of stress at the point *O* (Fig. 20). The expression for the resultant stress intensity on any other plane through *O* may then be obtained by substituting $s_{xy} = 0$ in equation (6) (Art. 26), which gives

$$p_a = (n_x^2 \cos^2\alpha + n_y^2 \sin^2\alpha)^{\frac{1}{2}}. \quad . \; . \; . \; . \; . \; . \tag{1}$$

It is evident from the construction (Fig. 20) that the vector *OM*, representing the intensity of the resultant stress p_a on the *A* plane, is a semi-diameter of an ellipse whose semi-major and semi-minor axes coincide with *OY* and *OX* and are equal respectively to n_y and n_x. That the point *M* is on an ellipse, constructed as indicated, may also be shown by the following simple proof. Let the coordinates of *M* be $x = Oa$ and $y = Ob$. It is then evident from equation (1) that

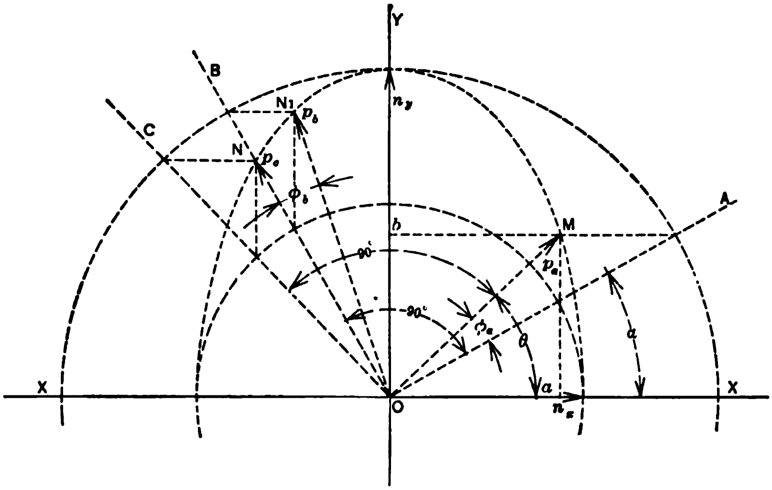


FIG. 20.

$$x = n_x \cos \alpha$$

and

$$y = n_y \sin \alpha.$$

Hence

$$\frac{x^2}{n_x^2} = \cos^2 \alpha$$

and

$$\frac{y^2}{n_y^2} = \sin^2 \alpha.$$

$$\therefore \quad \frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which is the equation of the ellipse shown in the figure.

Ellipse of Stress. — The ellipse just described is known as the *ellipse of stress*, and whenever a body is subjected to plane stress, the stress intensities on all planes through any point which are perpendicular to the plane of stress are represented by semidiameters of the ellipse of stress at that point.

It is therefore evident that the intensity of one of the principal stresses at any point is greater than, and of the other less than, the stress intensity on any other plane through the point.

29. Stress Components on any Plane in Terms of the Principal Stresses. — Let the X and Y planes be the *principal planes of stress* at any point O, the stress intensities on these planes being n_x and n_y, respectively. Let OA be the normal to any other plane through O, perpendicular to the Z plane. Substituting $s_{xy} = 0$ in equations (1), (2) and (6) (Art. 26), we obtain the expressions for the normal, shearing and resultant stress intensities on the A plane,

$$n_a = n_x \cos^2 \alpha + n_y \sin^2 \alpha, \quad \ldots \ldots \quad (1)$$

$$s_a = (n_y - n_x) \sin \alpha \cos \alpha = \frac{n_y - n_x}{2} \sin 2\alpha, \quad \ldots \quad (2)$$

and

$$p_a = (n_x^2 \cos^2 \alpha + n_y^2 \sin^2 \alpha)^{\frac{1}{2}}, \quad \ldots \ldots \quad (3)$$

p_a being represented by the vector OM (Fig. 20).

The angle θ, between OM and the axis OX, may be determined from the expressions

$$\cos \theta = \frac{Oa}{OM} = \frac{n_x \cos \alpha}{p_a}, \quad \ldots \ldots \quad (4)$$

$$\sin \theta = \frac{Ob}{OM} = \frac{n_y \sin \alpha}{p_a}. \quad \ldots \ldots \quad (5)$$

The angle $\phi_a = \theta - \alpha$, between OM and the normal OA, is called the *angle of obliquity* of the stress on the A plane. The magnitude of ϕ_a will change as α varies, its value being zero when $\alpha = 0°$, or $90°$, and a maximum for some intermediate value of α.

In a similar manner, by substituting $(90° + \alpha)$ for α, we obtain the expressions for the normal, shearing and resultant stress intensities on the B plane, at right angles to the A plane,

$$n_b = n_x \sin^2 \alpha + n_y \cos^2 \alpha, \quad \ldots \ldots \quad (6)$$

$$s_b = (n_x - n_y) \sin \alpha \cos \alpha, \quad \ldots \ldots \quad (7)$$

$$p_b = (n_x^2 \sin^2 \alpha + n_y^2 \cos^2 \alpha)^{\frac{1}{2}}, \quad \ldots \ldots \quad (8)$$

p_b being represented by the vector ON_1 (Fig. 20).

Adding (1) and (6) and reducing we have

$$n_a + n_b = n_x + n_y. \qquad (9)$$

That is, *the sum of the normal intensities of stress on all pairs of planes at right angles, passing through any point O and perpendicular to the plane of stress, is a constant.*

30. Equal Principal Stresses. — If the intensities of the principal stresses at any point in a body subjected to plane stress are equal in magnitude, the ellipse of stress becomes a circle; and it follows that the resultant stress intensities on all planes, passing through the point and perpendicular to the plane of stress, are equal. Two cases may arise: (a) When the principal stresses are both of the same sign; (b) when the principal stresses are of opposite signs.

(a) In this case $n_x = n_y$ and equations (1), (2), (3), (4) and (5) (Art. 29) reduce to

$$n_a = n_x, \qquad (1)$$

$$s_a = 0, \qquad (2)$$

$$p_a = n_x, \qquad (3)$$

$$\cos \theta = \cos \alpha, \qquad (4)$$

$$\sin \theta = \sin \alpha, \qquad (5)$$

respectively; which shows that *when the principal stresses at any point are equal and of the same sign, the stresses on all planes through that point, perpendicular to the plane of stress, are normal and of the same intensity.* This is evidently true when both principal stresses are negative as well as when both are positive.

(b) In this case $n_x = -n_y$ and equations (1), (2), (3), (4) and (5) (Art. 29) become

$$n_a = n_x (\cos^2 \alpha - \sin^2 \alpha) = n_x \cos 2\,\alpha, \qquad (6)$$

$$s_a = -2\, n_x \sin \alpha \cos \alpha = -n_x \sin 2\,\alpha, \qquad (7)$$

$$p_a = n_x, \qquad (8)$$

$$\cos \theta = \cos \alpha, \qquad (9)$$

$$\sin \theta = -\sin \alpha, \qquad (10)$$

respectively; which shows that *when the principal stresses at any point are equal in magnitude and of opposite sign, the resultant stress intensities on all planes through that point, perpendicular to the plane of stress, are equal, and the resultant stresses on all but principal planes are oblique.*

Equations (9) and (10) show that the angle, which the vector representing the resultant stress on the A plane makes with the axis

OX, is of the same magnitude and of the opposite sign to that which the normal OA makes with OX.

Equation (6) shows that when the angle between the normal to the plane and the axis OX is 45° the normal stress component is zero, the resultant stress on this plane being a shearing stress of the same intensity as the principal stresses. The intensity of this shearing stress is evidently a maximum (Art. 31).

31. Planes of Maximum Shear and of Maximum Obliquity.

Planes of greatest shear. — An inspection of equations (2) and (7) (Art. 29) will show that when α is equal to $\pm 45°$, $180° \pm 45°$, etc., the shearing stress intensities on the A and B planes will become equal to

$$s_a = \pm \frac{n_y - n_x}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and

$$s_b = \pm \frac{n_x - n_y}{2}; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

that is, the shearing stress intensities on the two planes, making angles of 45° with the planes of principal stress through a point, are greater than the shearing stress intensities on any other planes passing through the point.

If positive directions are assumed along the axes OA and OB and vectors, representing s_a and s_b, are laid off along the A and B planes, in accordance with the signs given by the solution of (1) and (2), the relation between the shearing stress intensities on any two planes at right angles (Art. 24) will be fulfilled in every case.

If we compare equations (1) and (6) (Art. 29), it will be evident that when $\alpha = 45°$, or $135°$,

$$n_a = n_b = \frac{n_x + n_y}{2},$$

that is, the normal components of the stresses on the planes of maximum shear are always equal and of a magnitude equal to the mean of the principal stresses.

Planes of greatest obliquity. — Referring to Fig. 20 we see that

$$\sin \phi = \frac{s_a}{p_a} = \frac{(n_y - n_x)\sin\alpha\cos\alpha}{(n_x^2\cos^2\alpha + n_y^2\sin^2\alpha)^{\frac{1}{2}}}; \quad . \; . \; . \quad (3)$$

also

$$\tan \phi = \frac{s_a}{n_a} = \frac{(n_y - n_x)\sin\alpha\cos\alpha}{n_x\cos^2\alpha + n_y\sin^2\alpha}, \quad . \; . \; . \; . \quad (4)$$

where n_x and n_y are principal stress intensities. To determine the value of α for which ϕ is a maximum we may differentiate (4), place the derivative equal to zero and substitute the value thus obtained in (3), as follows:

$$\frac{d}{d\alpha}(\tan\phi) = \frac{(n_y - n_x)(\cos^2\alpha - \sin^2\alpha)}{(n_x\cos^2\alpha + n_y\sin^2\alpha)}$$

$$- \frac{(n_y - n_x)\sin\alpha\cos\alpha}{(n_x\cos^2\alpha + n_y\sin^2\alpha)^2}\,2\,(n_y - n_x)\sin\alpha\cos\alpha = 0.$$

Transposing and reducing,

$$(\cos^2\alpha - \sin^2\alpha)(n_x\cos^2\alpha + n_y\sin^2\alpha) = 2\,(n_y - n_x)\sin^2\alpha\cos^2\alpha.$$

Substituting $\cos^2\alpha = 1 - \sin^2\alpha$ and solving,

$$\sin^2\alpha = \frac{n_x}{n_y + n_x}. \quad \text{(5)}$$

Hence $$\cos^2\alpha = 1 - \sin^2\alpha = \frac{n_y}{n_y + n_x}. \quad \text{(6)}$$

Substituting in (3) and letting ϕ_0 = the maximum value of ϕ,

$$\sin\phi_0 = \frac{(n_y - n_x)\left(\frac{n_x}{n_y + n_x}\right)^{\frac{1}{2}}\left(\frac{n_y}{n_y + n_x}\right)^{\frac{1}{2}}}{\left(\frac{n_x^2 n_y}{n_y + n_x} + \frac{n_y^2 n_x}{n_y + n_x}\right)^{\frac{1}{2}}} = \frac{n_y - n_x}{n_y + n_x}. \quad \text{(7)}$$

It is evident from equations (5) and (6) that the value of α for the plane of greatest obliquity of stress will not be 45° and hence the plane on which the obliquity of stress is a maximum is not the plane on which the shearing stress is a maximum.

It should be noted that the foregoing proof is restricted to the case where the principal stresses are of the same sign. If the principal stresses were of opposite signs the solution of equation (7) would give a value of $\sin\phi_0 > 1$.

When the principal stresses are of opposite sign the angle of greatest obliquity will be 90°, there always being in this case some plane on which the normal stress intensity is equal to zero. The angle α which the normal to the plane of greatest obliquity makes with OX can be found by putting the value of n_a (equation 1, Art. 29) equal to zero, and solving for α as follows:

$$n_a = n_x \cos^2\alpha + n_y \sin^2\alpha = 0,$$

$$\frac{\sin^2\alpha}{\cos^2\alpha} = -\frac{n_x}{n_y},$$

$$\frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \sqrt{-\frac{n_x}{n_y}}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which gives a real value for α whenever n_x and n_y are of opposite signs.

32. Principal Stresses in Terms of the Stresses on any Two Coördinate Planes at Right Angles to the Plane of Stress. — The magnitudes of the intensities of the principal stresses at any point O may be expressed in terms of the normal and shearing stress intensities at O on any two planes at right angles by eliminating the value of α between equations (1) and (2) (Art. 26), as follows:

For a principal plane of stress, equation (2) (Art. 26) may be written

$$(n_y - n_x)\sin\alpha\cos\alpha + s_{xy}(2\cos^2\alpha - 1) = 0,$$

from which

$$\cos^2\alpha = \frac{1}{2} + \frac{n_x - n_y}{2\,s_{xy}}\sin\alpha\cos\alpha; \quad . \quad . \quad . \quad . \quad (1)$$

also,

$$(n_y - n_x)\sin\alpha\cos\alpha + s_{xy}(1 - 2\sin^2\alpha) = 0,$$

from which

$$\sin^2\alpha = \frac{1}{2} - \frac{n_x - n_y}{2\,s_{xy}}\sin\alpha\cos\alpha. \quad . \quad . \quad . \quad . \quad (2)$$

Substituting the above values of $\cos^2\alpha$ and $\sin^2\alpha$ in equation (1) (Art. 26), we obtain

$$\begin{aligned} n_a &= \frac{n_x}{2} + \frac{n_x}{2\,s_{xy}}(n_x - n_y)\sin\alpha\cos\alpha + \frac{n_y}{2} \\ &\quad - \frac{n_y}{2\,s_{xy}}(n_x - n_y)\sin\alpha\cos\alpha + 2\,s_{xy}\sin\alpha\cos\alpha \\ &= \frac{n_x + n_y}{2} + \frac{(n_x - n_y)^2 + 4\,s_{xy}^2}{2\,s_{xy}}\sin\alpha\cos\alpha. \quad . \quad . \quad . \quad (3) \end{aligned}$$

From equation (1) (Art. 27),

$$\tan 2\alpha = \frac{2\,s_{xy}}{n_x - n_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and hence

$$\sin 2\alpha = \pm\frac{\tan 2\alpha}{\sqrt{1 + \tan^2 2\alpha}} = \pm\frac{2\,s_{xy}}{\sqrt{(n_x - n_y)^2 + 4\,s_{xy}^2}}.$$

Substituting in equation (3) and reducing, we obtain

$$n_a = \frac{n_x + n_y}{2} \pm \frac{1}{2}\sqrt{(n_x - n_y)^2 + 4\,s_{xy}^2}, \quad . \;\; . \;\; . \quad (5)$$

which gives the magnitude of both principal stress intensities, one value being obtained when the second term of the right-hand member is plus and the other when it is negative.

If we represent the principal stress intensities by n_1 and n_2, respectively, we have

$$n_1 = \frac{n_x + n_y}{2} + \frac{1}{2}\sqrt{(n_x - n_y)^2 + 4\,s_{xy}^2}, \quad . \;\; . \;\; . \quad (6)$$

$$n_2 = \frac{n_x + n_y}{2} - \frac{1}{2}\sqrt{(n_x - n_y)^2 + 4\,s_{xy}^2}. \quad . \;\; . \;\; . \quad (7)$$

If we let α_1 and α_2 represent the angles which the normals to the principal planes (differing by 90°) make with the X axis, their magnitudes may be obtained from equation (4).

If $n_y = 0$, that is, the normal component of the stress on one of the coördinate planes is zero, equations (4), (6) and (7) reduce to

$$\tan 2\,\alpha = \frac{2\,s_{xy}}{n_x}. \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (8)$$

$$n_1 = \frac{n_x}{2} + \frac{1}{2}\sqrt{n_x^2 + 4\,s_{xy}^2}. \quad . \;\; . \;\; . \;\; . \;\; . \quad (9)$$

$$n_2 = \frac{n_x}{2} - \frac{1}{2}\sqrt{n_x^2 + 4\,s_{xy}^2}. \quad . \;\; . \;\; . \;\; . \;\; . \quad (10)$$

The angles α_1 and α_2, which the normals to the principal planes make with the X axis, are obtained from equation (8).

If $n_y = 0$ and $n_x = 0$, that is, the stresses on the two coördinate planes are shearing stresses only, equations (4), (6) and (7) reduce to

$$\tan 2\,\alpha = \infty. \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (11)$$

$$n_1 = s_{xy}. \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (12)$$

$$n_2 = -\,s_{xy}. \quad . \;\; . \;\; . \;\; . \;\; . \;\; . \;\; . \quad (13)$$

From equation (11), 2 α will equal 90° or 270°, hence $\alpha_1 = 45°$ and $\alpha_2 = 135°$. The principal stresses are of equal magnitude and opposite sign (Art. 30).

Hence the principal stresses and therefore the state of stress at any point in a body under plane stress can be fully determined when the stress components on any two planes at right angles and perpendicular to the plane of stress are known. (Art. 29.)

33. Conjugate Stresses. — If the vector OM (Fig. 20) represents the resultant stress at the point O on the A plane, the resultant stress at the point O on a plane in the direction OM and perpendicular to the plane of stress will be represented by the vector ON. This may easily be shown by applying the conditions of equilibrium to a small prism at O whose base is a parallelogram $Omtn$ (Fig. 21) with the sides Om and On coinciding with OM and ON respectively and of such small dimensions that the stresses on the opposite faces may be considered equal and of uniform intensity. We will designate the plane in the direction OM as the C plane and let p_c equal the resultant stress intensity on this plane.

It is evident from the sketch (Fig. 21) that, since the stresses on the opposite faces of the prism $Omtn$ are equal and uniform, the resultant stresses on the faces On and mt must balance and those on the faces Om and nt must balance also. But to satisfy this condition the resultant stresses on the opposite faces Om and nt must act in the direction ON, otherwise they will form a couple. Since the intensity of the resultant stress on the C plane is represented by a semi-diameter of the ellipse of stress (Art. 28), it follows that ON (Fig. 20) represents the magnitude as well as the direction of the resultant stress p_c on the C plane.

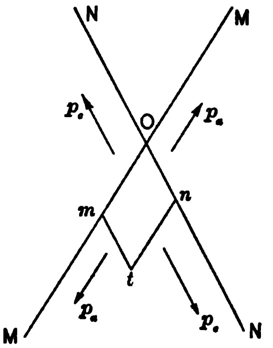


FIG. 21.

Stresses which bear the relation just described are known as *conjugate stresses:* and two planes so situated that the vector representing the resultant stress intensity on one plane lies in the other plane are known as *conjugate planes of stress.*

The angles of obliquity of any two conjugate stresses are evidently equal.

Principal stresses are conjugate stresses, where the planes are at right angles.

34. Resultant Stress on any Plane in Terms of the Principal Stresses. — This is a second solution of the theorem in Art. (29), based upon the results obtained in Art. (30).

Assume the X and Y planes to be the principal planes of stress at any point O, the principal stress intensities being n_x and n_y, respectively, and let OA be the normal to any other plane, passing through O perpendicular to the Z plane, and α = the angle between OA and OX (Fig. 22). Then evidently

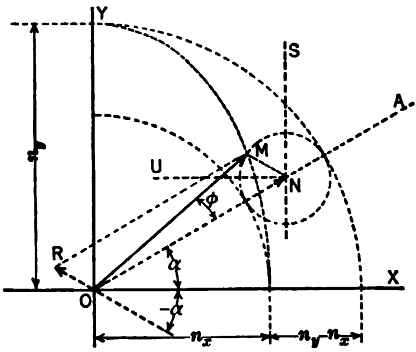


FIG. 22.

$$n_y = \frac{n_y + n_x}{2} + \frac{n_y - n_x}{2}$$

and

$$n_x = \frac{n_y + n_x}{2} - \frac{n_y - n_x}{2};$$

that is, the principal stresses can be resolved into two sets of *equal principal stresses* (Art. 30), one set having the same sign and equal in intensity to

$$\frac{n_y + n_x}{2}$$

and the other set having opposite signs and equal in intensity to

$$\frac{n_y - n_x}{2}.$$

The resultant stress intensities on the A plane can be found by determining, first, the component stress intensity on this plane due to the equal principal stress components of the same sign and, second, the component stress intensity due to the equal principal stress components of opposite sign and then finding their resultant. The first component will be equal to

$$p_{a_1} = \frac{n_y + n_x}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and will be normal to the A plane; that is, the angle between the vector p_{a_1} and OX will be equal to α. (Art. 30.) The second component will be equal to

$$p_{a_2} = \frac{n_y - n_x}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

and the angle between the vector p_{a_2} and OX will be equal to $-\alpha$ (Art. 30). If we lay off $ON = p_{a_1}$ and $OR = p_{a_2}$ (Fig. 22), it is evident that the resultant stress intensity on the A plane will be equal to

$$p_a = OM = [p_{a_1}^2 + p_{a_2}^2 - 2\, p_{a_1} p_{a_2} \cos 2\,\alpha]^{\frac{1}{2}}. \quad . \; . \quad (3)$$

Substituting the values of p_{a_1}, p_{a_2} and $\cos 2\,\alpha$ in equation (3) we obtain

$$p_a = \left[\left(\frac{n_y + n_x}{2}\right)^2 + \left(\frac{n_y - n_x}{2}\right)^2 - 2\left(\frac{n_y + n_x}{2}\right)\left(\frac{n_y - n_x}{2}\right)(\cos^2\alpha - \sin^2\alpha)\right]^{\frac{1}{2}},$$

which readily reduces to

$$p_a = (n_x^2 \cos^2\alpha + n_y^2 \sin^2\alpha)^{\frac{1}{2}}, \quad . \; . \; . \; . \; . \quad (4)$$

the value for p_a previously obtained in Art. (29). The point M is evidently a point on the ellipse of stress for the point O.

In making a graphical solution the point M may be located by laying off $ON = \frac{n_y + n_x}{2}$ along the normal OA, and constructing the angle $SNM = SNA$, the line SN being drawn parallel to OY, and then laying off $NM = OR = \frac{n_y - n_x}{2}$.

The angle of obliquity ϕ, of the stress p_a, increases with the value of α, from zero when $\alpha = 0°$, to a maximum; and then, as the value of α continues to increase, the value of ϕ decreases again to zero when $\alpha = 90°$.

The *maximum obliquity* is evidently obtained when the angle OMN is a right angle and, if we let ϕ_o = angle of greatest obliquity,

$$\sin \phi_o = \frac{n_y - n_x}{n_y + n_x} \text{ (Art. 31)}. \quad . \; . \; . \; . \; . \; . \quad (5)$$

35. Principal Stresses in Terms of the Stresses on any Two Planes Perpendicular to the Plane of Stress. — The following may be regarded as the converse of the proposition in Art. 34.

Let OA and OB be the normals to any two planes through the point O (Fig. 23), perpendicular to the plane of stress and let p_a and p_b equal the resultant stress intensities and ϕ_a and ϕ_b the angles of obliquity for the A and B planes, respectively. Let n_x and n_y equal the principal stress intensities at O, the X and Y planes being the principal planes of stress.

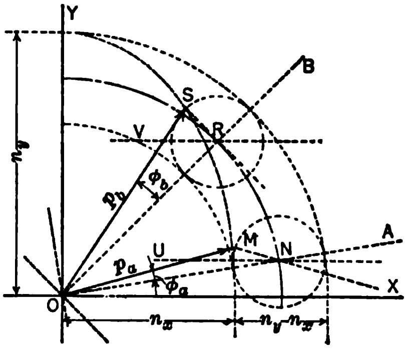


FIG. 23.

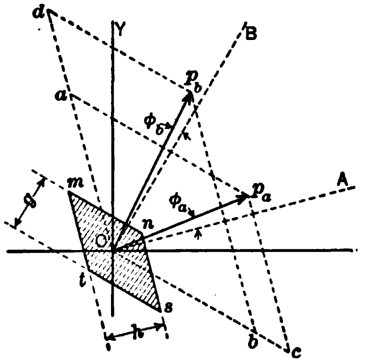


FIG. 24.

In an abstract problem it is evident that p_a, p_b, ϕ_a and ϕ_b cannot be taken at random but must be chosen in accord with the law of variation of the stress intensities on different planes through the point O.

Note.—To determine, whether or not the values chosen for p_a and p_b are consistent, we may proceed as follows: Let OA and OB (Fig. 24) be the normals to any two oblique planes A and B passing through the point O; p_a and ϕ_a be the magnitude and obliquity of the stress intensity on the A plane; p_b and ϕ_b be the magnitude and obliquity of the stress intensity on the B plane; *mnst* be the cross section of a prismatic particle of a unit length with its opposite faces parallel to the A and B planes as shown. Let the particle be taken so small that the stresses on the opposite faces may be assumed equal, opposite and uniform.

Resolve each of the stresses p_a and p_b into two components, parallel to the A and B planes, respectively. For the components of p_a, we have Oa, parallel to the A plane, and Oc parallel to the B plane.

For the components of p_b, we have Ob parallel to the B plane, and Od, parallel to the A plane.

Since the stresses on the opposite faces of the particle are uniform and of equal intensity the component $Oc \cdot ns$ of the stress on the face *ns* will be in equilibrium with the equal and opposite component on the face *mt*. Likewise the component $Od \cdot mn$ of the stress on the face *mn* will be in equilibrium with the equal and opposite component on the face *st*.

The remaining stress components will form two couples,

$$-Oa \cdot ns \cdot h$$

and

$$Ob \cdot mn \cdot g$$

which must be equal in magnitude if equilibrium is maintained. Hence,

$$Oa \cdot ns \cdot h = Ob \cdot mn \cdot g.$$

But

$$ns \cdot h = mn \cdot g.$$

Therefore,

$$Oa = Ob,$$

which is the condition of equilibrium of the stresses at a point on any two planes perpendicular to the plane of stress.

When the resultant stress intensities on the *A* and *B* planes are known, however, the values of n_x and n_y may be obtained as follows:

Imagine $p_a = OM$ (Fig. 23) to be resolved into the components

$$ON = \frac{n_y + n_x}{2}$$

and

$$NM = \frac{n_y - n_x}{2} \text{ (Art. 34);}$$

likewise, that $p_b = OS$ (Fig. 23) is resolved into the components

$$OR = \frac{n_y + n_x}{2}$$

and

$$RS = \frac{n_y - n_x}{2}$$

Then from the triangle *OMN*,

$$\left(\frac{n_y - n_x}{2}\right)^2 = p_a^2 + \left(\frac{n_y + n_x}{2}\right)^2 - 2\,p_a\left(\frac{n_y + n_x}{2}\right)\cos\phi_a, \quad (1)$$

and from the triangle *ORS*,

$$\left(\frac{n_y - n_x}{2}\right)^2 = p_b^2 + \left(\frac{n_y + n_x}{2}\right)^2 - 2\,p_b\left(\frac{n_y + n_x}{2}\right)\cos\phi_b. \quad (2)$$

Subtracting (2) from (1) and transposing,

$$p_a^2 - p_b^2 = \frac{n_y + n_x}{2}(2\,p_a\cos\phi_a - 2\,p_b\cos\phi_b), \quad . \; . \; . \quad (3)$$

from which we obtain

$$\frac{n_y + n_x}{2} = \frac{p_a^2 - p_b^2}{2\,(p_a\cos\phi_a - p_b\cos\phi_b)}. \quad . \; . \; . \quad (4)$$

The value of $\frac{n_y - n_x}{2}$ may now be obtained from either of the equations (1) or (2); and, from the equations

$$n_y = \frac{n_y + n_x}{2} + \frac{n_y - n_x}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$n_x = \frac{n_y + n_x}{2} - \frac{n_y - n_x}{2} \text{ (Art. 34),} \quad . \quad . \quad . \quad . \quad (6)$$

the magnitudes of the principal stresses can be found.

The directions of the principal planes of stress can be found graphically by constructing either of the triangles *ONM* or *ORS* and bisecting the angle *ONM*, or the angle *ORS*. Both of the bisecting lines *NU* and *RV* will be parallel to *OX*. An analytical solution for the angle *ONU* or *ORV* could evidently be made by solving the triangle for the magnitude of the angle *ONM* or for the magnitude of the angle *ORS* and then obtaining the angle *ONU* or the angle *ORV*.

When the *A and B planes are conjugate planes of stress,* $\phi_a = \phi_b$ (Art. 33), and, if we let the common angle of obliquity $\phi_a = \phi_b = \phi$, equation (4) will reduce to

$$\frac{n_y + n_x}{2} = \frac{p_a + p_b}{2 \cos \phi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Substituting this value of $\frac{n_y + n_x}{2}$ in equation (1) we obtain

$$\left(\frac{n_y - n_x}{2}\right)^2 = p_a^2 + \left(\frac{p_a + p_b}{2 \cos \phi}\right)^2 - 2\left(\frac{p_a + p_b}{2 \cos \phi}\right) p_a \cos \phi,$$

which easily reduces to

$$\frac{n_y - n_x}{2} = \sqrt{\left(\frac{p_a + p_b}{2 \cos \phi}\right)^2 - p_a p_b}. \quad . \quad . \quad . \quad . \quad (8)$$

When the *A and B planes are at right angles,* the shearing components s_a and s_b of the stress intensities on the two planes are equal and, if we let n_a and n_b equal the normal intensities of the stresses on the two planes and $s_a = s_b = s$, we shall have

$$p_a^2 = n_a^2 + s^2 \qquad \text{and} \qquad p_b^2 = n_b^2 + s^2;$$

also,

$$\cos \phi_a = \frac{n_a}{p_a} \qquad \text{and} \qquad \cos \phi_b = \frac{n_b}{p_b}.$$

Substituting these values in equation (4) we obtain

$$\frac{n_y + n_x}{2} = \frac{n_b^2 + s^2 - n_a^2 - s^2}{2\,(n_a - n_b)} = \frac{n_b + n_a}{2}; \quad . \; . \; . \quad (9)$$

and by substituting in equation (1) we obtain

$$\left(\frac{n_y - n_x}{2}\right)^2 = n_a^2 + s^2 + \left(\frac{n_b + n_a}{2}\right)^2 - 2\left(\frac{n_b + n_a}{2}\right)n_a,$$

which easily reduces to

$$\frac{n_y - n_x}{2} = \sqrt{\left(\frac{n_b - n_a}{2}\right)^2 + s^2}. \quad . \; . \; . \; . \quad (10)$$

It follows that the principal stresses and therefore the state of stress at any point in a body under plane stress can be fully determined when the stresses on any two planes passing through the point at right angles to the plane of stress are known (Art. 29).

36. Rankine's Graphical Solution. — A graphical solution, due to Rankine, affords a simple solution of the problem in Art. (35). It consists in constructing the triangles *ONM* and *ORS* (Fig. 23), by superimposing one over the other as follows:

Given the resultant stress intensities p_a and p_b on the *A* and *B* planes at any point *O* and the angles of obliquity ϕ_a and ϕ_b.

Draw a straight line *OC* (Fig. 25) to represent the normal to either plane and lay off *OM*, making the angle ϕ_a with *OC* to represent the magnitude of the stress intensity p_a and in a like manner lay off *OS* making the angle ϕ_b with *OC* to represent the magnitude of the stress intensity p_b. Draw *TN* perpendicular to and bisecting *MS*. The triangles *OMN* and *OSN* are evidently respectively equal to the triangles *OMN* and *ORS* (Fig. 23) and

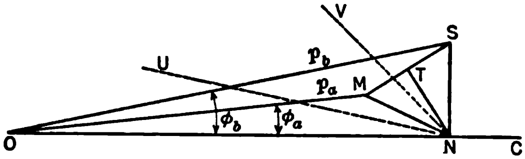


FIG. 25.

$$ON = \frac{n_y + n_x}{2}$$

and

$$MN = NS = \frac{n_y - n_x}{2}.$$

By drawing *NU*, bisecting the angle *ONM*, the angle *ONU* between the normals to the *A* and *X* planes is obtained.

In a similar manner by bisecting the angle *ONS* the angle between the *B* and the *X* planes is obtained.

37. Ratio of Conjugate Stresses. — When the normal components of the stresses on every plane passing through a point in a body under plane stress are of the same sign, that is, the normal components are either all tension or all compression, the ratio between the stress intensities on any pair of *conjugate planes of stress* through the point (Art. 33) may be expressed in terms of the common angle of obliquity of the conjugate stresses and the angle of greatest obliquity for any plane passing through the point as follows:

Let the *A* and *B* planes be any two conjugate planes of stress through a point *O* in a body under plane stress and p_a and p_b equal the respective intensities of the stresses on these planes, the common angle of obliquity being ϕ. Let ϕ_o = the greatest angle of obliquity of the stress on any plane passing through *O* and let n_y and n_x equal the principal stress intensities at *O*. Then

$$\sin \phi_o = \frac{n_y - n_x}{n_y + n_x} \text{ (Art. 31)}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Dividing equation (8) by equation (7) (Art. 35) we obtain

$$\frac{n_y - n_x}{n_y + n_x} = \frac{2 \cos \phi}{p_a + p_b} \sqrt{\left(\frac{p_a + p_b}{2 \cos \phi}\right)^2 - p_a p_b}. \quad . \quad . \quad . \quad (2)$$

Combining equations (1) and (2) and reducing

$$\sin \phi_o = \sqrt{1 - \frac{4\, p_a p_b \cos^2 \phi}{(p_a + p_b)^2}};$$

and hence

$$1 - \cos^2 \phi_o = 1 - \frac{4\, p_a p_b \cos^2 \phi}{(p_a + p_b)^2}$$

and

$$\frac{\cos^2 \phi}{\cos^2 \phi_o} = \frac{(p_a + p_b)^2}{4\, p_a p_b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By applying a principle of proportion, equation (3) can be transformed into

$$\frac{\cos^2 \phi - \cos^2 \phi_o}{\cos^2 \phi} = \frac{(p_a - p_b)^2}{(p_a + p_b)^2};$$

and hence

$$\frac{p_a + p_b}{p_a - p_b} = \frac{\cos\phi}{\sqrt{\cos^2\phi - \cos^2\phi_o}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again applying a principle of proportion

$$\frac{p_a}{p_b} = \frac{\cos\phi + \sqrt{\cos^2\phi - \cos^2\phi_o}}{\cos\phi - \sqrt{\cos^2\phi - \cos^2\phi_o}}, \quad . \quad . \quad . \quad . \quad (5)$$

where p_a must evidently be taken to represent the greater of the two conjugate stresses.

When the conjugate planes are at right angles, p_a and p_b are principal stresses and $\phi = 0$. In this case equation (5) reduces to

$$\frac{p_a}{p_b} = \frac{1 + \sin\phi_o}{1 - \sin\phi_o}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

a result which might be readily obtained by applying the principle of proportion to equation (1).

38. Problems. — Stresses at a Point.

Problem 1.

A steel bar 25 ft. long and 5 sq. in. cross section is subjected to a pull of 100,000 lbs. along its central axis. Find the magnitude of the resultant in tensity of stress and the normal and shearing components on an oblique plane making an angle of 30° with the cross section. Also find the resultant intensity and the normal and shearing components on a plane at right angles to the oblique plane. Compare the shearing components on the two oblique planes at right angles to each other and also the sum of the normal intensities on these planes with the normal intensity on the cross section.

Problem 2.

Following the notation of Art. (23), let $n_x = 16{,}000$ lbs. per sq. in., $n_y = 4000$ lbs. per sq. in. and $s_{xy} = 2000$ lbs. per sq. in. be the component stress intensities on two planes at right angles through the point O in a body under plane stress. Find the resultant intensity of stress at O on the A plane, whose normal OA makes an angle $\alpha = 30°$ with the axis OX; also find the intensities of the normal and shearing components of the stress on this plane.

First Solution. — The problem may be solved by substituting the known quantities in equations (1) and (2) (Art. 26) as follows:

Since $\alpha = 30°$, $\sin\alpha = \frac{1}{2}$, and $\cos\alpha = \frac{\sqrt{3}}{2}$, and hence from equation (1) we obtain for the normal component

$$n_a = 16{,}000 \times \frac{3}{4} + 4000 \times \frac{1}{4} + 2 \times 2000 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = 14{,}732 \text{ lbs. per sq. in.,}$$

the *positive sign* indicating *tension;* and from equation (2) we obtain for the shearing component

$$s_a = (4000 - 16{,}000) \times \frac{1}{2} \times \frac{\sqrt{3}}{2} + 2000 \times \left(\frac{3}{4} - \frac{1}{4}\right) = -4196 \text{ lbs. per sq. in.,}$$

the *negative sign* indicating that the shear is in the *opposite direction* to that shown in Fig. (19) (Art. 26).

The resultant stress intensity can be obtained by combining the normal and shearing intensities just found, giving

$$p_a = \sqrt{(14{,}732)^2 + (4196)^2} = 15{,}320 \text{ lbs. per sq. in.,}$$

or it can be found independently of the above solution by substituting in equation (6) (Art. 26) as follows:

$$\begin{aligned} p_a = \Big\{&(16{,}000)^2 \times \frac{3}{4} + (4000)^2 \times \frac{1}{4} + 2 \times 2000 \\ &\times (16{,}000 + 400) \times \frac{1}{2} \times \frac{\sqrt{3}}{2} + (2000)^2\Big\}^{\frac{1}{2}} \\ =\ &(234{,}640{,}000)^{\frac{1}{2}} = 15{,}320 \text{ lbs. per sq. in.} \end{aligned}$$

Second Solution. — The resultant stress intensity might also be found by determining the components E and F (equations (4) and (5) Art. 26), and combining as follows:

$$E = 16{,}000 \times \frac{\sqrt{3}}{2} + 2000 \times \frac{1}{2} = 14{,}856 \text{ lbs. per sq. in.,}$$

$$F = 4000 \times \frac{1}{2} + 2000 \times \frac{\sqrt{3}}{2} = 3732 \text{ lbs. per sq. in.,}$$

and hence

$$p_a = \sqrt{(14{,}856)^2 + (3732)^2} = 15{,}320 \text{ lbs. per sq. in.}$$

A sketch showing the intensities and directions of the components on a small particle, the faces of which are parallel to the X, Y and A planes, is shown in Fig. (26).

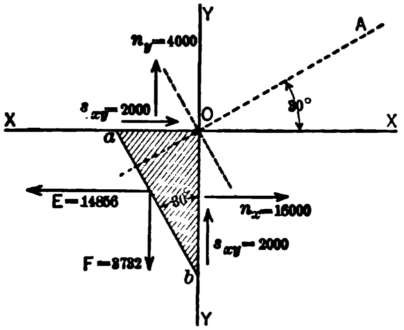


Fig. 26.

Such a sketch, or, in case the first solution is used, a sketch similar to Fig. (19), is a very considerable aid in avoiding confusion as to the directions of the unknown stress components.

Problem 3.

Given the intensities of the normal and shearing components of the stresses on two planes at right angles through a point O in a body subjected to plane stress, $n_x = 8000$ lbs. per sq. in., $n_y = 4000$ lbs. per sq. in. and $s_{xy} = 2000$ lbs. per sq. in., find the intensities of the principal stresses at O; also, the angles between the normals to the principal planes of stress and the axis OX.

Problem 4.

Solve Problem (3), assuming $n_x = 8000$ lbs. per sq. in., $n_y = 0$ and $s_{xy} = 2000$ lbs. per sq. in.

Problem 5.

Solve Problem (3), assuming $n_x = 8000$ lbs. per sq. in., $n_y = -4000$ lbs. per sq. in. and $s_{xy} = 2000$ lbs. per sq. in.

Problem 6.

Solve Problem (3), assuming $n_x = 8000$ lbs. per sq. in., $n_y = 0$ and $s_{xy} = -2000$ lbs. per sq. in.

Problem 7.

Given the principal stresses $n_y = 16{,}000$ lbs. per sq. in., $n_x = 8000$ lbs. per sq. in. at a point O, find the resultant intensity and its normal and shearing components on a plane passing through O whose normal makes an angle of 60° with the axis OX. Find the intensities of the shearing and normal stresses on the planes of maximum shear through O.

Problem 8.

Solve Problem (7), assuming $n_y = 16{,}000$ lbs. per sq. in. and $n_x = -8000$ lbs. per sq. in.

Problem 9.

Solve Problem (7), assuming $n_x = n_y = 8000$ lbs. per sq. in.

Problem 10.

Solve Problem (7), assuming $n_x = 8000$ lbs. per sq. in., $n_y = -8000$ lbs. per sq. in.

Problem 11.

Find the maximum obliquity of the stress in the case stated in Problem (7) and find the angles which the normals to the planes of greatest obliquity make with the axis OX. Find for this case the ratio of the conjugate stresses whose common angle of obliquity $\phi = 10°$.

Problem 12.

Find the maximum obliquity of the stress in the case stated in Problem (8) and find the angles which the normals to the planes of greatest obliquity make with the axis OX.

Problem 13.

Given two conjugate pressures, $p_a = -200$ lbs. per sq. in. and $p_b = -400$ lbs. per sq. in., having a common angle of obliquity $\phi = 20°$; determine the magnitudes of the principal stress intensities n_x and n_y: (a) graphically (Art. 36), (b) analytically (Art. 35).

Problem 14.

Determine graphically the directions of the normals OA and OB to the conjugate planes of stress in Problem (13) with respect to the normals OX and OY to the principal planes of stress.

§ 2. STRAIN.

39. Analysis of Strain. — The elementary types of strain have already been defined (Art. 4). For a more comprehensive definition we may state the following:

The *strain at any point* in a body is fully determined when the extensions of all lines radiating from the point are known, the term *extension* being taken to mean the ratio of the change in the length of a very short line to its original length, the change being either positive or negative.

A special case arises when a deformation is such that the extensions of all lines perpendicular to a given plane are zero. In such a case the body is said to be subjected to *plane strain*, and to determine fully the strain at any point it is necessary to determine the extensions in all directions parallel to the plane of the strain at that point.

40. Shearing Strain. — We will consider the distortion which takes place in a very small particle in a body which is subjected to plane strain. We will assume that before the deformation takes place the particle has the shape of a rectangular prism with the face *Oabc* (Fig. 27) parallel to the plane of the strain.

If this deformation of the particle is due to shear only, the face of the prism *Oabc* will be distorted into a parallelogram *Oa'b'c'*, of the same area as the original rectangle. The shearing strain will be measured by the change in the angle of inclination (Art. 4) of the two sides of the prism, *Oa* and *Oc*. This measure will be equal to the difference in circular measure between the angle *a'Oc'* and a right angle, and, since the strain is a very small quantity, if we assume the original dimensions of the particle to be dx and dy (Fig. 27), and let du and dv equal the displacements of the point b in the directions of the axes OX and OY, respectively, it will be sensibly equal to

$$\frac{du}{dy} + \frac{dv}{dx}.$$

Hence if the displacement of the point b relative to O is due to shear alone, the displacement may be considered equivalent to a simple shear (Art. 4) in the direction of the X plane equal to $\frac{dv}{dx}$ combined with a simple shear in the direction of the Y plane equal to $\frac{du}{dy}$ and, using a notation similar to that previously adopted, we will express the measure of the shear as

$$\gamma_{xy} = \frac{du}{dy} + \frac{dv}{dx}. \quad \dots \dots \quad (1)$$

The quantity γ_{xy} is called the *shearing strain* at O in the direction of the X and Y planes. As represented in Fig. (27) it is a positive quantity, but the displacement of the point b may be such that the angle $a'Oc'$ becomes greater than a right angle in which case the shearing strain is negative.

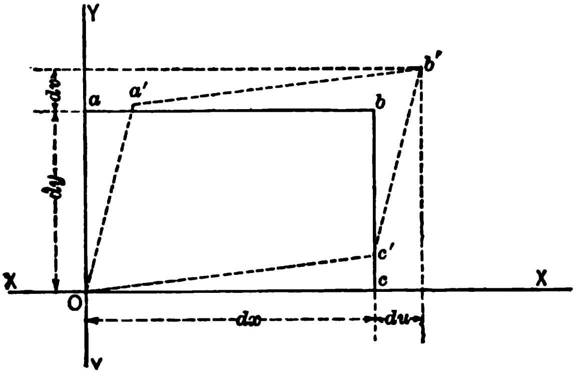


FIG. 27.

41. Simple Extension. — As another type, of plane strain, we may cite the case where the small rectangular prism $Oabc$ (Fig. 28) is distorted into a prism with a rectangular base $Oa'b'c'$. In this case the displacement of the point b relative to O is evidently due to two *simple* extensions (Art. 4); one in the direction OX, whose measure is $\frac{du}{dx}$, combined with one in the direction OY, whose measure is $\frac{dv}{dy}$.

Following our previous notation (Art. 4), we will denote the extension in the direction OX by the symbol

$$e_x = \frac{du}{dx}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and that in the direction OY by the symbol

$$e_y = \frac{dv}{dy}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It is evident that the distortion might be such that either e_x or e_y, or both together, could be negative. The increase of the volume, per unit of volume, due to the deformation may be called the *dilatation*, which, in the case of plane strain just cited, will be sensibly equal to

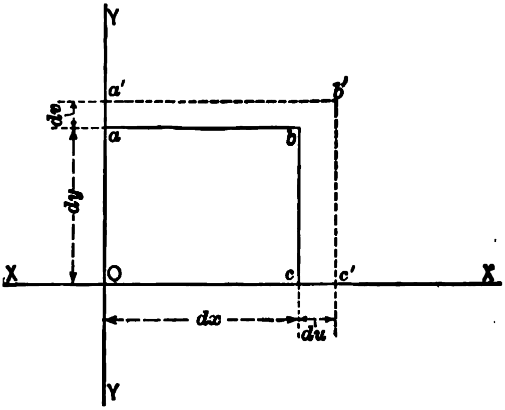


Fig. 28.

$$\Delta = \frac{du}{dx} + \frac{dv}{dy} = e_x + e_y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

42. Components of Strain. — We will now consider the general type of plane strain. Assume as before that any point b (Fig. 29) at a very small distance from any point O is displaced, relative to O, to the point b' during the deformation of the body, the line Ob changing in both length and direction. We will show that the change in the length of Ob can be expressed in terms of two simple extensions (Art. 41), along any two rectangular coördinate axes OX and OY through O and two simple shears (Art. 40) along the corresponding X and Y planes through O. Let α equal the angle between OA, which is the line Ob extended, and the axis OX. Consider a very small prism with sides parallel to OX and OY and with Ob as the diagonal of the base $Oabc$. In general the rectangle $Oabc$ will be distorted into a parallelogram $Oa'b'c'$ which has not the same area as the original rectangle.

Let dx and dy represent the original dimensions of the rectangle $Oabc$ and let ds the original length of the diagonal Ob.

Let $du + dx$ and $dv + dy$ represent the projections of Ob' on OX and OY respectively, and let e equal the unit extension, hereafter called the *extension* in the diagonal Ob.

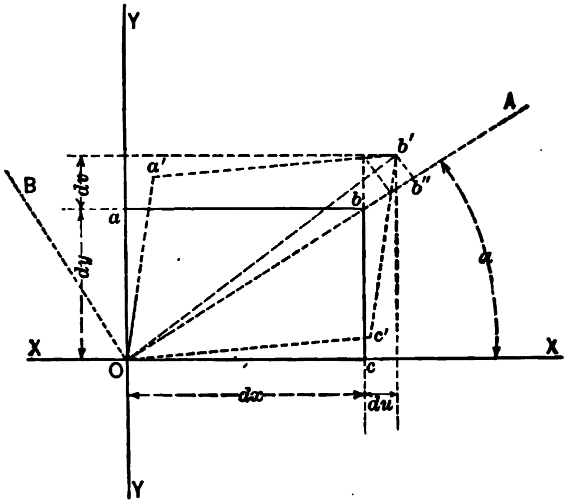


FIG. 29.

Let Ob'' equal the length of the projection of Ob' on OA. Since the angle bOb', measuring the change in direction of Ob during the deformation, is very small, it is evident, from the construction (Fig. 29), that the component bb'' of the displacement bb' is equal to

$$bb'' = dR = du \cos\alpha + dv \sin\alpha; \quad . \quad . \quad . \quad . \quad (1)$$

also, that the extension e is sensibly equal to

$$e = \frac{dR}{ds}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To determine the relation between the strain e and the strain components in the direction OX and OY, we note that, since the area of the parallelogram $Oa'b'c'$ is not in general the same as that of the rectangle *Oabc*, the deformation is the same as if the original rectangle were distorted by two simple extensions into a rectangle of the same area as *Oa'b'c'* and then distorted by shear into parallelogram shown. In other words the total displacement *du* is due partly to a simple extension of the prism *Oabc* in the direction *OX* and a simple shear in the direction of the *Y* plane. Expressed analytically,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Similarly the total displacement *dv* is due to simple extension in the direction *OY* combined with a simple shear along the *X* plane which is represented analytically by the expression

$$dv = \frac{\partial v}{\partial y} dy + \frac{\partial v}{dx} dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting the values of *du* and *dv* in (1) and combining with (2) we obtain

$$e = \frac{dR}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds}\cos\alpha + \frac{\partial u}{\partial y}\frac{dy}{ds}\cos\alpha + \frac{\partial v}{\partial y}\frac{dy}{ds}\sin\alpha + \frac{\partial v}{\partial x}\frac{dx}{ds}\sin\alpha,$$

which evidently reduces to

$$e = \frac{\partial u}{\partial x}\cos^2\alpha + \frac{\partial v}{\partial y}\sin^2\alpha + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\sin\alpha\cos\alpha, \quad . \quad (5)$$

where

$$\frac{\partial u}{\partial x} = e_x,$$

the component of the strain due to a simple extension in the direction *OX*,

$$\frac{\partial v}{\partial y} = e_y,$$

the component due to a simple extension in the direction *OY*, and

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy},$$

the component due to a shearing strain in the directions *OX* and *OY*.

Substituting these values in equation (5), the expression for the extension in any element making an angle α with *OX* reduces to

$$e = e_x\cos^2\alpha + e_y\sin^2\alpha + \gamma_{xy}\sin\alpha\cos\alpha. \quad . \quad . \quad . \quad (6)$$

The ratio of the component *b''b'* of the displacement of *b* (Fig. 29) to *Ot* will be sensibly equivalent to a simple shear on the *A* plane in the direction OB, at right angles to OA, the magnitude of which may be represented by the expression

$$\frac{b'b''}{Ob} = \frac{dQ}{ds}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where
$$dQ = dv \cos\alpha - du \sin\alpha. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Substituting the values of the total differentials (equations 3 and 4) and combining (8) with (7), we obtain

$$\begin{aligned} \frac{dQ}{ds} &= \frac{\partial v}{\partial y}\frac{dy}{ds}\cos\alpha + \frac{\partial v}{\partial x}\frac{dx}{ds}\cos\alpha - \frac{\partial u}{\partial x}\frac{dx}{ds}\sin\alpha - \frac{\partial u}{\partial y}\frac{dy}{ds}\sin\alpha \\ &= \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\alpha\cos\alpha + \frac{\partial v}{\partial x}\cos^2\alpha - \frac{\partial u}{\partial y}\sin^2\alpha. \quad . \quad . \quad (9) \end{aligned}$$

To determine completely the shearing strain we must obtain the expression for the component due to the simple shear on the B plane in the direction OA. For it is evident that a small element of the line OB taken perpendicular to OA before the strain is produced will in general change both in length and direction during the deformation.

We may, by the same method followed in deducing equation (9), determine the value of the angular displacement of OB due to a simple shear in the direction OA in terms of the angle α between OA and OX. The value will reduce to

$$\frac{dQ'}{ds'} = \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\alpha\cos\alpha + \frac{\partial u}{\partial y}\cos^2\alpha - \frac{\partial v}{\partial x}\sin^2\alpha. \qquad (10)$$

Adding (9) and (10) we obtain the value of the shearing strain in the directions OA and OB,

$$\begin{aligned} \gamma = \frac{dQ}{ds} + \frac{dQ'}{ds'} &= 2\left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\alpha\cos\alpha \\ &\quad + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)(\cos^2\alpha - \sin^2\alpha) \\ &= (e_y - e_x)\, 2\sin\alpha\cos\alpha + \gamma_{xy}(\cos^2\alpha - \sin^2\alpha) \\ &= (e_y - e_x)\sin 2\alpha + \gamma_{xy}\cos 2\alpha. \quad . \quad . \quad . \quad . \quad . \quad (11) \end{aligned}$$

43. Principal Strains. — If we differentiate equation (6) (Art. 42), we shall obtain

$$\frac{de}{d\alpha} = (e_y - e_x)\, 2\sin\alpha\cos\alpha + \gamma_{xy}(\cos^2\alpha - \sin\alpha). \quad . \quad . \quad (1)$$

Placing (1) equal to zero and reducing we obtain

$$\frac{\sin 2\alpha}{\cos 2\alpha} = \tan 2\alpha = \frac{\gamma_{xy}}{e_x - e_y}, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

which indicates that there are two values of α, differing by 90°, for one of which the extension e is a maximum and the other a minimum. A comparison of equation (1) with equation (11) (Art. 42) also shows that there is no shearing strain in the directions along which e is a maximum or a minimum.

Therefore, through any point O in a body subjected to plane strain there are two lines at right angles along which the extensions are greater or less than along any other lines through the point, the shearing strain along these directions being zero. These maximum and minimum extensions are known as the principal strains at the point O and the coördinate axes in these directions are the principal axes of the strain.

44. Strain Components in Terms of the Principal Strains. — If we let e_x and e_y equal the values of the principal strains in the directions OX and OY at any point, the expression for the extension in any direction OA, making an angle α with OX, may be found by substituting $\gamma_{xy} = 0$ in equation (6) (Art. 42), giving

$$e = e_x \cos^2 \alpha + e_y \sin^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, the expression for the shearing strain in the directions OA and OB at right angles may be obtained from equation (11) (Art. 42),

$$\gamma = (e_y - e_x)\, 2 \sin \alpha \cos \alpha = (e_y - e_x) \sin 2\alpha. \quad . \quad . \quad (2)$$

An inspection of equation (2) will show that when $\alpha = 45°$, the value of γ is a maximum, in which case

$$\gamma = e_y - e_x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the difference of the principal strains. Hence, the directions of maximum shearing strain make angles of 45° with the principal axes of strain and the magnitude of the maximum shearing strain is equal to the difference of the principal strains. If we substitute $90° + \alpha$ for α in equation (1) we shall obtain for the value of the extension in a direction at right angles to OA

$$e' = e_x \sin^2 \alpha + e_y \cos^2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Adding (1) and (4) and reducing, we have

$$e + e' = e_x + e_y = \Delta; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

that is, in any case of plane strain the sum of the extensions in any two directions at right angles through any point O in the plane of strain is a constant quantity, known as the *dilatation* at O (Art. 41).

It is evident that *the state of strain at any point in a body subjected to plane strain is fully determined when the principal strains at the point are known.*

45. Strain Accompanying Plane Stress. — The preceding discussion has been limited, for the sake of simplicity, to the type of strain known as plane strain. Such a strain would not in general be produced without the action of forces perpendicular to the plane of strain. When such forces are absent, as in the case of *plane stress*, the extensions and shear-strains at any point O along lines in the plane of the stress will be accompanied by an extension at O in a direction perpendicular to that plane (Art. 5). We will denote the extension in this direction by the symbol e_z.

The relations deduced in Arts. (39–44) will hold true, however, for the strain components at any point in directions parallel to the plane of stress, and the *dilatation* (Art. 44) will be represented by the expression

$$\Delta = e_x + e_y + e_z.$$

46. Relations between Stresses and Strains. — The relations between extensions and lateral strains and the relations between the simple stresses and strains have been discussed previously (Arts. 5 and 7).

In the general case of plane stress, the relations between the components of stress and strain at any point in a homogeneous isotropic body can be determined as follows: Assume three rectangular coördinate axes through any point O with the axes OX and OY in the plane of stress. Following the notation previously adopted, the normal stress on the X plane will cause an extension

$$\frac{n_x}{E},$$

in the direction OX, an extension

$$-\frac{n_x}{mE},$$

in the direction OY, and an extension

$$-\frac{n_x}{mE},$$

in the direction OZ.

Similarly the normal stress on the Y plane will cause extensions

$$\frac{n_y}{E}, \quad -\frac{n_y}{mE} \quad \text{and} \quad -\frac{n_y}{mE},$$

in the directions OY, OX and OZ respectively.

The resultant extension in the direction OX will therefore be equal to

$$e_x = \frac{n_x}{E} - \frac{n_y}{mE}, \quad \dots\dots\dots \quad (1)$$

and in the direction OY,

$$e_y = \frac{n_y}{E} - \frac{n_x}{mE}, \quad \dots\dots\dots \quad (2)$$

and in the direction OZ,

$$e_z = -\frac{n_x}{mE} - \frac{n_y}{mE}. \quad \dots\dots \quad (3)$$

The shearing stresses on the X and Y planes will produce a shearing strain in the directions OX and OY which will be equal to

$$\gamma_{xy} = \frac{s_{xy}}{G}. \quad \dots\dots\dots\dots \quad (4)$$

By solving equations (1) and (2) we readily obtain the following values of the normal stress intensities on the X and Y planes in terms of the extensions along OX and OY,

$$n_x = \frac{mE}{m^2 - 1}(me_x + e_y), \quad \dots\dots\dots \quad (5)$$

$$n_y = \frac{mE}{m^2 - 1}(me_y + e_x). \quad \dots\dots\dots \quad (6)$$

From equation (4) we obtain

$$s_{xy} = G\gamma_{xy}. \quad \dots\dots\dots\dots \quad (7)$$

When the axes OX and OY are the principal axes of strain it is evident that equations (1), (2) and (3) give the principal strains in terms of the principal stress intensities and that $\gamma_{xy} = s_{xy} = 0$.

47. Relation between the Modulus of Elasticity and the Modulus of Rigidity. — If we let n_y and n_z equal the principal stress intensities at any point O and e_y and e_z the principal strains at that point, the maximum shearing stress intensity at O (Art. 31) will occur on the planes making angles of 45° with the principal planes and will be equal to

$$s = \frac{n_y - n_x}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the maximum shearing strain (Art. 44) will be equal to

$$\gamma = e_y - e_x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting in (1) the values of n_x and n_y from equations (5) and (6) (Art. 46) and combining (1) and (2) with (7) (Art. 46) we obtain

$$s = \frac{1}{2}\frac{mE}{m^2 - 1}(me_y + e_x - me_x - e_y) = G(e_y - e_x),$$

which reduces to

$$G = \frac{1}{2}\frac{m}{m+1}E, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

or,

$$E = 2G\frac{m+1}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we substitute the value of E in equations (5) and (6) (Art. 46), we obtain

$$n_x = \frac{2G}{m-1}(me_x + e_y), \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$n_y = \frac{2G}{m-1}(me_y + e_x). \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

48. Equations of Equilibrium. — In the discussion of the stress at any point in a body subjected to plane stress we have been concerned with the relations existing between the stress components on different planes passing through the point. If the stress is uniform throughout the body the state of stress at every point will be the same.

In the case of a varying plane stress the law of variation of the stress from point to point in the plane of the stress may be determined by applying the conditions of equilibrium to a small particle which has the shape of a rectangular prism (Fig. 30) with the face *Oabc* parallel to the plane of stress, the dimensions of the particle being taken so small that the stress over each face may be considered to be uniform. Let Δx and Δy be the dimensions of the particle in the directions OX and OY and let the third dimension equal unity. The stress on the face *Oabc* will equal zero and if we let the stress intensities on the X and Y planes at O equal n_x, n_y and s_{xy}, as indicated (Fig. 30), the stress intensities on the faces of the particle parallel to the X and Y planes will be equal to

$$n_x + \Delta n_x, \quad n_y + \Delta n_y \text{ and } s_{xy} + \Delta s_{xy}.$$

It is evident that Δn_x, Δn_y and Δs_{xy} represent the changes in the stress intensities between the X and Y planes through O and the

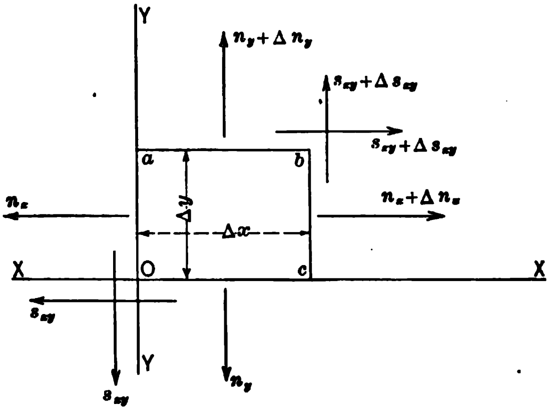


FIG. 30.

X and Y planes through any point b at a small distance from O. Applying the conditions of equilibrium, we have

$$(n_x + \Delta n_x)\,\Delta y - n_x\Delta y + (s_{xy} + \Delta s_{xy})\,\Delta x - s_{xy}\Delta x + X\,\Delta x\,\Delta y = 0$$

and

$$(n_y + \Delta n_y)\,\Delta x - n_y\Delta x + (s_{xy} + \Delta s_{xy})\,\Delta y - s_{xy}\Delta y + Y\,\Delta x\,\Delta y = 0,$$

where X and Y represent the components of any force, other than the stresses on the faces, which may act on the particle. Dividing by $\Delta x\Delta y$ and reducing and passing to the limit we have

$$\frac{\partial n_x}{\partial x} + \frac{\partial s_{xy}}{\partial y} + X = 0, \quad \ldots\ldots \quad (1)$$

$$\frac{\partial n_y}{\partial y} + \frac{\partial s_{xy}}{\partial x} + Y = 0. \quad \ldots\ldots \quad (2)$$

These equations express the law of variation of stress in a body in equilibrium under plane stress. If we neglect the effect of gravity, or other similar forces acting through space, the components X and Y will equal zero.

49. General Relations of Stresses and Strains. — When the stress at a point is not confined to a single plane the relations between stresses on different planes and those between the strains in different directions through the point become more complex and no attempt will be made to deduce them here. It will be of interest, however, to state the following laws governing the state of stress and strain at any point in a body subjected to any system of forces in equilibrium.

(a) Through any point there are three planes at right angles on which there is no shearing stress. On one of the planes the normal intensity of stress is a maximum, on another the normal intensity of stress is a minimum, and on the third it has a value intermediate between the other two. The three planes are called the *principal planes of stress* and the three stresses are called the *principal stresses.*

(b) Through any point there are three lines perpendicular to the principal planes of stress along which there are no shearing strains. Along one of these lines the extension is a maximum, along another a minimum, and along the third it has a value intermediate between the other two. The lines are called the *principal axes of strains* and the extensions are called the *principal strains.*

(c) The following equations express the relations between the principal stresses and strains, the notation being similar to that previously adopted:

$$e_x = \frac{n_x}{E} - \frac{n_y}{mE} - \frac{n_z}{mE}, \quad \dots\dots\dots\dots \quad (1)$$

$$e_y = \frac{n_y}{E} - \frac{n_x}{mE} - \frac{n_z}{mE}, \quad \dots\dots\dots\dots \quad (2)$$

$$e_z = \frac{n_z}{E} - \frac{n_x}{mE} - \frac{n_y}{mE}, \quad \dots\dots\dots\dots \quad (3)$$

$$n_x = \frac{mE}{(m+1)(m-2)}[(m-1)e_x + e_y + e_z], \quad \dots \quad (4)$$

$$n_y = \frac{mE}{(m+1)(m-2)}[(m-1)e_y + e_x + e_z], \quad \dots \quad (5)$$

$$n_z = \frac{mE}{(m+1)(m-2)}[(m-1)e_z + e_x + e_y]. \quad \dots \quad (6)$$

(d) If the stress on each plane of any set of three planes at right angles, passing through any point O, is resolved in a normal component and two shearing components in the directions of the two coördinate axes located in that plane, the intensities or the shearing components on any two of the planes in the direction of the third plane will be equal. If we express this law analytically, using a notation similar to that already adopted, we have

$$s_{xy} = s_{yx}, \qquad (7)$$

$$s_{yz} = s_{zy}, \qquad (8)$$

$$s_{zx} = s_{xz}. \qquad (9)$$

In these equations the first letter of the subscript indicates the plane on which the shear occurs and the second letter the direction of the axis along which the component of the shear is taken.

(e) The shearing strain will be made up of three component shearing strains in the directions of the three pairs of axes, namely:

$$\gamma_{xy}, \qquad (10)$$

$$\gamma_{yz}, \qquad (11)$$

$$\gamma_{zx}, \qquad (12)$$

(f) The relations between the three shearing stress intensities and the three shearing strains will be

$$G = \frac{s_{xy}}{\gamma_{xy}} = \frac{s_{yz}}{\gamma_{yz}} = \frac{s_{zx}}{\gamma_{zx}}. \qquad (13)$$

(g) The relations between the normal stress intensities and the extensions along any three coördinate axes will be the same as the relations of the principal stresses and strains given in section (c).

(h) The equations of equilibrium in terms of the stress intensities on any three planes at right angles through a given point and the components X, Y, Z, per unit of volume, of any force such as gravity acting through space, are the following:

$$\frac{\partial n_x}{\partial x} + \frac{\partial s_{xy}}{\partial y} + \frac{\partial s_{xz}}{\partial z} + X = 0, \qquad (14)$$

$$\frac{\partial n_y}{\partial y} + \frac{\partial s_{yz}}{\partial z} + \frac{\partial s_{yx}}{\partial x} + Y = 0, \qquad (15)$$

$$\frac{\partial n_z}{\partial z} + \frac{\partial s_{zx}}{\partial x} + \frac{\partial s_{zy}}{\partial y} + Z = 0. \qquad (16)$$

CHAPTER III.

UNIFORM STRESS AND UNIFORMLY VARYING STRESS.

50. Uniform Stress. — When the intensities of both the normal and shearing components of a stress on any given plane section are the same at every point in the plane, the stress on the plane is said to be uniform. In such a case the center of the stress coincides with the center of gravity of the section. (Art. 81, Vol. I.) It follows from the laws of equilibrium that in order to produce a uniform stress on any plane section through a body at rest, the resultant of the external forces acting on the part of the body on either side of the plane must be a single force whose line of action passes through the center of gravity of the section.

51. Axial Tension. — When a straight bar of uniform section and material is subjected to a pull in such a manner that the line of action of the resultant force acting on each end coincides with the central axis of the bar, the stress on any cross section, either at right angles to, or inclined to, the axis will be uniform. It is common engineering practice to call the stress on the right cross section in such a case *axial tension.*

This term will also apply in the case of a piece of homogeneous material of non-uniform section, provided the axis passing through the center of gravity of every right cross section is a straight line and the line of action of the resultant force acting at each end of the piece coincides with the axis.

The intensity of the stress at any point O on a cross section at right angles to the central axis will be greater than that on any other cross section through O and will be equal to

$$p = \frac{P}{A}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

where P = the resultant force acting at one end of the piece and A = the area of the cross section. The maximum intensity of stress will evidently occur on the right cross section for which A is a minimum.

The strain at any point O in the direction of the axis will be equal to

$$e = \frac{p}{E}, \quad \ldots \quad \ldots \quad (2)$$

and the strain at O in any direction at right angles to the axis will be equal to

$$-\frac{e}{m} = -\frac{p}{mE}. \quad \ldots \ldots \quad (3)$$

If the piece is of uniform cross section the elongation in any length will be equal to

$$a = el = \frac{pl}{E} = \frac{Pl}{AE}. \quad \ldots \ldots \quad (4)$$

The intensity of the shearing stress on any plane through O inclined at 45° to the central axis of the piece will be greater than that on any other plane through O and will be equal to

$$s = \frac{p}{2} \text{ (Art. 25)}. \quad \ldots \ldots \quad (5)$$

When O is located in the minimum cross section, the value of s will be the maximum intensity of the shearing stress for the entire piece.

52. Axial Compression. — Axial compression may be defined as negative axial tension, it being the uniform compression stress on the right cross section of a piece whose central axis is a straight line. To produce such a stress the line of action of the resultant force acting on each end of the piece must coincide with the axis, the forces being directed so as to produce compression.

The expressions for stress intensities, strains and elongation (Art. 51) will evidently apply in this case, all of the quantities being negative instead of positive.

It is unnecessary to take account of signs, however, in cases where the stress is of the same kind throughout the body.

53. Saint-Venant's Principle. — A careful analysis of either of the cases discussed in Arts. (51) and (52) will show that, in order that stress on every cross section of the piece shall be uniformly distributed, the forces acting on the ends must be distributed in the same manner. Practically, however, the solutions given may be applied to any case in which the lines of action of the resultants of the forces acting at the ends of the piece coincide with the central axis, the distribution of the stresses and strains at different points in the body being independent of the distribution of the terminal forces, except for comparatively small portions near the ends. This statement is in accord with a "principle," first definitely enunciated by Saint-Venant, according to which the strains produced at any point in a body in equilibrium by the action of each of the individual forces, applied to small portions of its surface, are of negligible magnitude compared with the strains produced by the force system as a whole, except for points in comparatively small portions of the body near the places at which the forces are applied.

This principle will be found to have a broad application in the theorems which are discussed later.

54. Thin Hollow Cylinder. — When a hollow circular cylinder, whose thickness is small compared with its radius, is subjected to a *uniform internal pressure* the stress on any cross section through a point in the surface may be considered to be uniform. We will let r = the radius and t = the thickness of such a cylinder (Fig. 31) and assume that the ends are so thick, or of such shape, that the stress in the cylindrical shell will not be affected by the distortion of the ends. Let p = the intensity of the internal pressure.

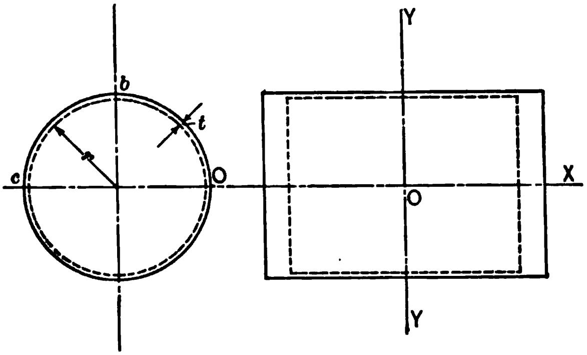


FIG. 31.

To determine the intensity of stress on a longitudinal section OX, through any point O in the surface of the cylinder, we apply the conditions of equilibrium to Obc, a semi-circular section of the shell one unit in width. The forces parallel to the X plane through O acting on this section will be the uniformly distributed internal pressure over the semi-circumference Obc and the uniform stress on the cross sections at O and c. The resultant of the internal pressure may easily be shown to be a force, whose magnitude is equal to 2 pr, acting through the center of the cylinder. Hence the total stresses at the sections O and c will be equal and, if we denote the stress intensity on these sections by the symbol f, we shall have

$$2ft = 2pr$$

and therefore

$$f = \frac{pr}{t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The intensity f is commonly called the *hoop tension* in the cylinder.

The intensity of the stress on a cross section OY, perpendicular to the axis of the cylinder, may be determined by applying the conditions of equilibrium to the portion of the cylinder on either side of the section. The resultant of the internal pressure on the part on one side of the section will evidently be a force of magnitude

$$p\pi r^2,$$

whose line of action coincides with the axis of the cylinder. This force will be balanced by a stress uniformly distributed around the circumference at the section OY. If we denote the intensity of this stress by the symbol f_1, we shall have

$$f_1 2\pi\left(r + \frac{t}{2}\right)t = p\pi r^2,$$

from which we obtain

$$f_1 = \frac{pr}{2t} \text{ (very nearly)}. \quad . \quad . \quad . \quad . \quad (2)$$

The stress intensity f_1 is frequently called the *end tension* in the cylinder. It is evident that its value is not affected by a change in the shape of the ends of the cylinder provided they are strong enough not to distort appreciably under the pressure. A comparison of equations (1) and (2) shows that in any thin cylinder

$$f = 2f_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since there are no shearing stresses on the sections OX and OY, f and f_1 are principal stress intensities and the resultant stress intensity on any other plane through O, containing the axis OZ, will have a value intermediate between them, which is given by the expression

$$p = (f_1^2 \cos^2 \alpha + f^2 \sin^2 \alpha)^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

obtained by substituting the values of f and f_1 in equation (3) (Art. 29).

The hoop tension f is therefore the greatest stress in the shell of the cylinder.

The above formulæ will evidently apply in the case of a thin hollow cylinder subjected to *uniform external pressure,* the outside radius of which is equal to r and thickness equal to t, the normal stresses on the different sections being compression instead of tension.

This important difference between the two cases should be noted, however. When the cross section is not an exact circle, a uniform internal pressure tends to make the section circular, while a uniform external pressure tends to increase the distortion from the circular form, collapsing the cylinder completely at a pressure intensity much smaller than the internal pressure to which it may be subjected without failing.

When the cylinder is subjected to *uniform internal and external pressures simultaneously* the stress intensity on any section can be found by combining the stresses due to the pressures taken separately. If we let r = the mean radius of the shell, p_1 = the intensity of the internal pressure and p_2 = the intensity of the external pressure, the resultant intensity of the stress on the longitudinal section will be very nearly equal to

$$f = \frac{(p_1 - p_2)\,r}{t}, \quad \dots\dots\dots \quad (5)$$

and the resultant intensity of the stress on the cross section perpendicular to the axis will be very nearly equal to

$$f_1 = \frac{(p_1 - p_2)\,r}{2\,t}. \quad \dots\dots\dots \quad (6)$$

55. Thin Hollow Sphere.—When a spherical shell, whose thickness is small compared with its radius is subjected to a *uniform internal pressure,* the stress on any section through the shell may be considered to be uniform. The intensity of the stress on any meridian section of the shell may be determined by applying the conditions of equilibrium to the forces acting on the hemisphere on either side of the section.

Let r = the inside radius of the sphere, t = its thickness, p = the intensity of the internal pressure and f = the intensity of the tension on a meridian section through the shell. The resultant of the internal pressure on the hemisphere may easily be shown to be equal to

$$p\pi r^2,$$

its line of action passing through the center of the sphere.

The total stress on the meridian section of the shell will be equal to

$$f 2\pi\left(r + \frac{t}{2}\right)t = p\pi r^2.$$

Hence

$$f = \frac{pr}{2t} \text{ (very nearly)}. \quad . \quad . \quad . \quad (1)$$

Formula (1) will evidently apply when the sphere is subjected to *uniform external pressure*, r representing the external radius.

The difference between the effects of internal and external pressures when the shell is not exactly spherical in form will evidently be similar to that in the case of the thin hollow cylinder of non-circular section. (Art. 54.)

When the sphere is subjected to *uniform internal and external pressure, simultaneously*, the resultant intensity of stress on any meridian section will be very nearly equal to

$$f = \frac{(p_1 - p_2)\,r}{2t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where p_1 = the intensity of the internal pressure, p_2 = the intensity of the external pressure and r = the mean radius of the shell.

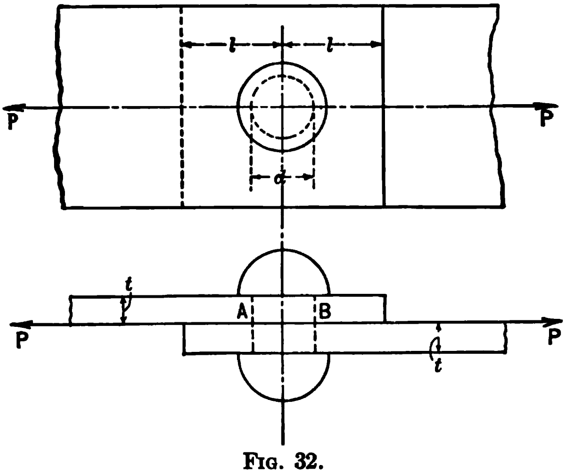


FIG. 32.

56. Uniform Shearing Stress. — When a rivet, or bolt, or pin is used to join two pieces together as in Fig. (32) the intensity of the shearing stress on the section AB due to a pull P is ordinarily assumed to be uniform. This assumption is hardly in accord with the principle stated in Art. (53) but there is no way in which the exact distribution of the stress on the section can be fully determined. Hence, if f_s = the average intensity of the shearing stress, due to the pull P, d = the diameter of the rivet, A = the area of its cross section, we shall have

$$P = f_sA = f_s\frac{\pi d^2}{4} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and

$$f_s = \frac{P}{A}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

The rivet shown (Fig. 32) is said to be subjected to *single shear.*

When two plates are joined to a single plate by a rivet as shown in Fig. 33 the rivet is said to be subjected to *double shear,* and if the rivet is of the same dimensions as before

$$P = 2f_sA = f_s\frac{\pi d^2}{2} \quad . \; . \; . \; . \; . \; . \; . \quad (3)$$

and

$$f_s = \frac{P}{2A}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

the two sections AB and CD, in this case, being subjected to a uniform shearing stress.

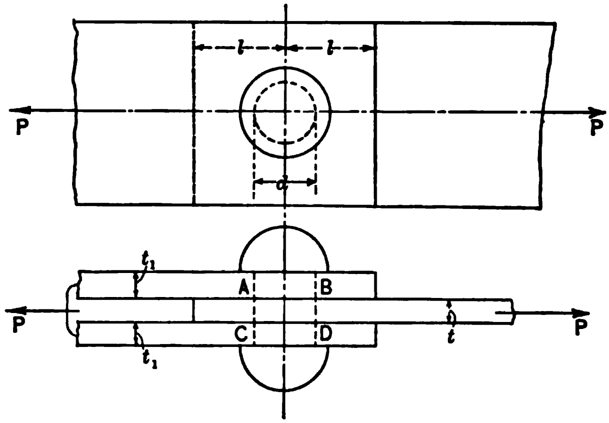


FIG. 33.

57. Uniform Bearing Pressure.—In both of the cases discussed in Art. (56) it is customary to assume that the pressure between either plate and the rivet is uniformly distributed around the semi-circumference of the rivet, there being no shearing stress at the surface of contact.

Hence the relation between the intensity of the pressure and the resultant pressure on the surface of the rivet will be the same as in the case of the circular cylinder subjected to uniform external pressure (Art. 54). Therefore, if we let f_c = the intensity of the pressure between the rivet and either plate (Fig. 32) and t = the thickness of each plate, we shall have

$$P = f_c td \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$f_c = \frac{P}{td}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The quantity of f_c is usually called the *intensity of the bearing pressure* on the rivet. When the rivet is under double shear as shown in Fig. (33), if we let t = the thickness of the middle plate, the intensity of the bearing pressure will be represented by equation (2) as before. If the two outside plates are of equal thickness t_1, the intensity of the bearing pressure between the rivet and the outside plates will be represented by the expression

$$f_c' = \frac{P}{2\,t_1 d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

58. Riveted Joints. — In joining together the plates used in the construction of a boiler, or tank, and in fastening the joints of steel structures, rivets, or in some cases bolts, are employed. In the majority of cases the connection is made in such a manner that the forces acting between the parts joined together produce a shearing stress on the cross sections of the rivets. When the line of action of the resultant of the forces acting on the joint passes through the center of gravity of the combined area of the cross sections of all the rivets in the joint, it is customary to assume that the intensity of the shearing stress on all the rivets is the same. This assumption is practically correct for simple joints with one or two rows of rivets but an analysis of the stress and strains will show that it is not exactly true for the more complicated joints.

In the design of any riveted joint, it is necessary to determine (a) the proper diameter of the rivets; (b) the spacing, or *pitch*, of the rivets; (c) the distance between the rivets and edge of the plate, or the *lap* of the joint as it is called; (d) the distances between the rows of rivets, when more than one row is used.

A method of determining these quantities in a few simple cases will now be given. In each case we will let

f_s = the working strength in shear of the rivet material,
f_c = the greatest allowable intensity of bearing pressure between the plate and a rivet,
f_t = the working tensile strength of the material in the plate,
d = the diameter of a rivet,
A = the area of the cross section of a rivet,
t = the thickness of the plates joined together,
p = the pitch of the rivets; that is, the distance from center to center of the rivets in any one row,
l = the lap; that is, the distance from the center of a rivet to the edge of the plate,
a = the distance between the rows of rivets, when more than one row is used.

Case I. The simplest type of a joint, used for connecting two plates, is known as the *single riveted lap joint* (Fig. 34), in which two plates are lapped together and connected with a single row of rivets spaced at equal distances. We will assume that the joint is subjected to a uniform stress; that is, the stresses on all the rivets are equal.

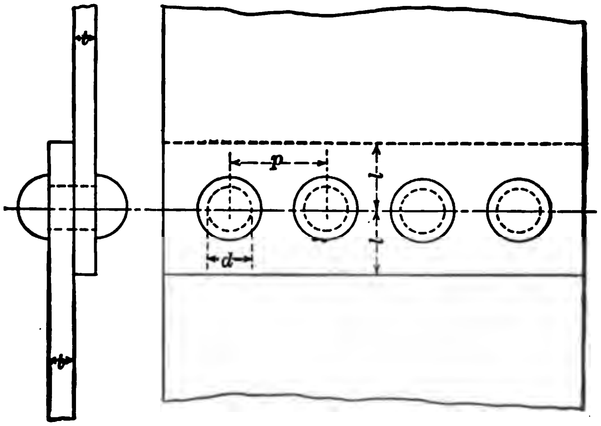


FIG. 34.

If such a joint were subjected to a pull sufficient to break it apart, the manner in which it would fail would depend on the size and spacing of the rivets. If the rivets were too small, or were spaced too far apart, the joint would fail by shearing the rivets. If the rivets were too close together the plate would tear off between the rivet holes. If the lap were too small, the rivets would break through the edge of the plate. In order, therefore, that the joint shall be as strong as it is possible to make it, the strength of one rivet must be equal to the strength of the section of plate between two adjacent rivets, and the strength of the plate in front of a rivet must be equal to, or greater than, the strength of a rivet. If such a joint were pulled apart the plate between the rivets would be at the point of failure when the rivets broke off, or vice versa, provided the lap were made great enough to prevent the rivets from breaking through.

When subjected to its working load the relations between the working stresses in the rivets and the plate would be the same as those between the stresses at breaking. To determine the dimensions we therefore proceed as follows:

(a) *The diameter of the rivets* will depend on the thickness of the plate, the pitch which is desirable and other considerations. The choice of the diameter will usually be based on the results of experience rather than on any particular formula.

The holes in the plate are either punched or drilled larger than the size of the rivets. The rivets on being driven expand to fill the holes. When the holes are drilled in place and accurately in line the diameter of the rivets when driven will equal the diameter of the holes and in making computations it is customary to use the *driven diameter* of the rivet, as it is called.

When the holes in the plate are punched out roughly, the alignment may be such that the smallest section of the driven rivet will not be much greater than the section of the rivet before driving. Hence in such work it is customary to use the original or *nominal diameter* of the rivet in making computations.

Having fixed upon the diameter to use, the working load for a rivet in shear will be equal to

$$W_1 = f_s A = \frac{f_s \pi d^2}{4} \text{ (Art. 56)} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the allowable bearing pressure will be equal to

$$W_2 = f_c t d \text{ (Art. 57)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To find the diameter of the rivet required to make the shearing and crushing resistance equal, equate (1) and (2) and we obtain

$$d = \frac{4 f_c t}{\pi f_s} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

It is evident that if the diameter of the rivets used is greater than that given by (3), $W_2 < W_1$, and if the diameter used is less than the value given by (3) $W_1 < W_2$.

Since the diameters of commercial rivets vary by sixteenths of an inch and practical considerations limit the sizes which can be used, the diameter of the rivets selected for any particular joint will seldom agree with that given by equation (3). Hence, the working load per rivet must be the smaller of the quantities W_1 and W_2.

(b) *The pitch of the rivets* may now be determined by equating the working load on a rivet to the working load on the net section of the plate between a single pair of rivet holes. This will evidently be equal to

$$f_t\,(p - d)\,t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Hence, if $W_1 < W_2$, the pitch of the rivets will be determined from the equation

$$f_sA = f_t\,(p - d)\,t; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

and, if $W_2 < W_1$, the pitch will be determined from the equation

$$f_ctd = f_t\,(p - d)\,t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

(c) When steel plate is used the *lap* of the joint may be determined from the expression

$$l = 0.007\sqrt{\frac{Wd}{t}} + 0.5\,d. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

This equation must for the present be regarded as empirical, the value of W being the smaller of the values W_1 and W_2.

The *efficiency of a riveted joint* is the ratio of the strength of the joint to the strength of the plate of which the joint is made. The *strength of the joint* is understood to be the minimum resistance of the joint to failure. Expressed analytically

$$\text{efficiency} = \frac{\text{minimum resistance of section of width } p}{f_tpt}. \quad . \quad . \qquad (8)$$

If the joint is properly designed equation (8) reduces to

$$\text{efficiency} = \frac{f_t\,(p - d)\,t}{f_tpt} = \frac{p - d}{p}. \quad . \quad . \quad . \quad . \quad . \qquad (8a)$$

Case II. The joints shown in Figs. (35) and (36) are known as *double riveted lap joints.*

In Fig. (35) is shown a case of *zigzag riveting*, or the rivets are said to be staggered, and in Fig. (36) the riveting is known as *chain riveting*. In both cases the net section of plate between two rivets must be equal in strength to the strength of two rivet sections if the joint is to be as strong as possible.

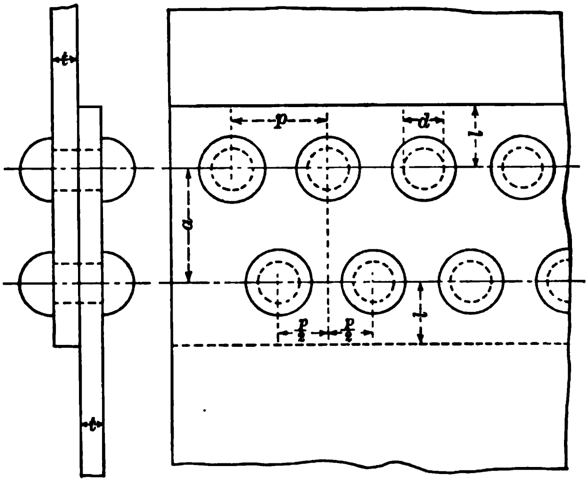


FIG. 35.

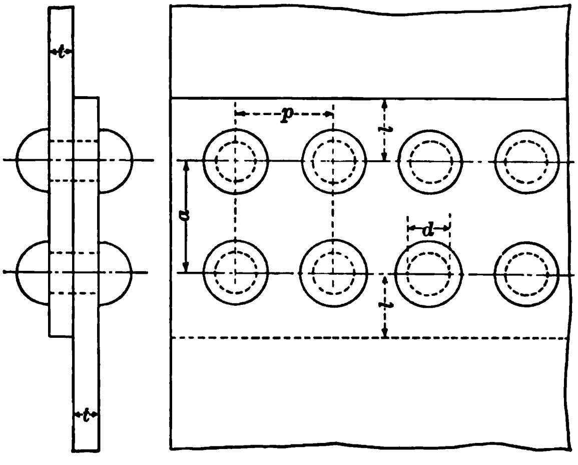


FIG. 36.

(a) Having chosen a diameter, the strength of a single rivet will evidently be the less of the two values W_1 and W_2 (Case I).

(b) If $W_1 < W_2$ the pitch will be obtained from the equation

$$2\,W_1 = 2f_sA = f_t\,(p-d)\,t \quad . \; . \; . \; . \; . \quad (9)$$

and if $W_2 < W_1$ the pitch will be obtained from the equation

$$2\,W_2 = 2f_c\,td = f_t\,(p-d)\,t. \quad . \; . \; . \; . \; . \quad (10)$$

(c) The lap will be determined from equation (7), using the smaller of the two values of W_1 and W_2.

(d) Either of the joints (Figs. 35–36) will be sufficiently strong if the distance a between the rows of rivets is equal to or greater than $l + \frac{d}{2}$.

If the joint is properly designed, the efficiency will evidently be represented by equation (8).

Case III. In Fig. (37) is given an illustration of a ***single riveted butt joint***, in which two plates are butted together and joined by riveting on two cover plates, whose thickness t_1 may be equal to or less than t, the thickness of the main plate, but never as small as $\frac{t}{2}$. The rivets in such a joint are subjected to double shear as shown in Fig. (33).

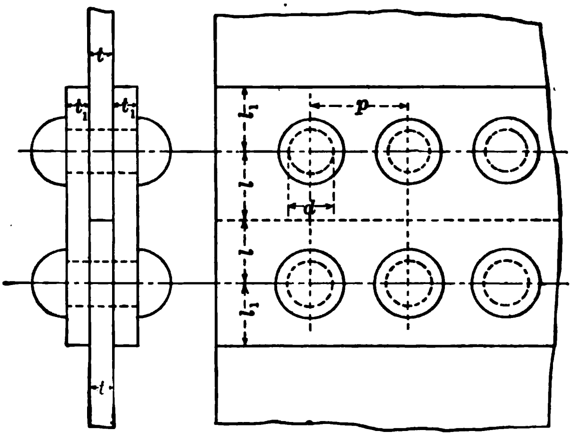


FIG. 37.

(a) The *diameter of the rivet* will be chosen in accord with the considerations stated in Case I. The strength of a single rivet will then be equal to the less of the values

$$W_3 = 2f_sA = \frac{f_s\pi d^2}{2} \text{ (Art. 56)} \quad . \quad . \quad . \quad . \quad (11)$$

and

$$W_4 = f_ctd \text{ (Art. 57)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

(b) The *pitch* will be determined by placing the smaller of the values W_3 and W_4 equal to $f_t\,(p - d)\,t$. Hence if $W_3 < W_4$

$$W_3 = 2f_sA = f_t\,(p - d)\,t \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and if $W_4 < W_3$

$$W_4 = f_ctd = f_t\,(p - d)\,t\,. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

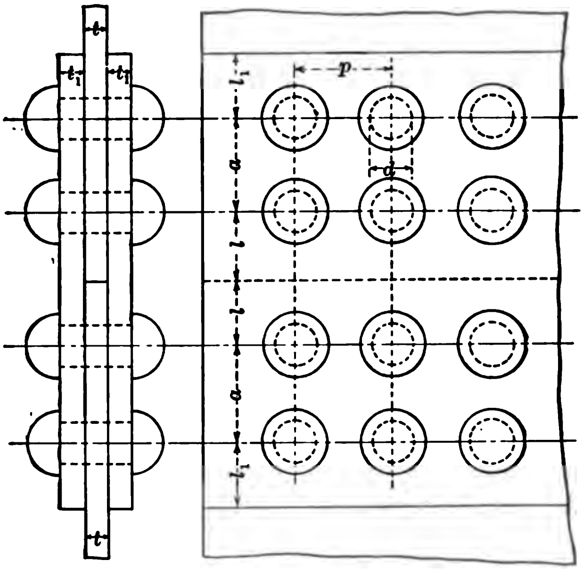


FIG. 38.

(c) The *lap* in the main plate will be determined by substituting the smaller of the values W_3 and W_4 in equation (7); and the lap in the cover plate by substituting t_1 for t and the smaller of the values $\frac{W_3}{2}$ and $\frac{W_4}{2}$ for W in the same equation.

If the joint is properly designed the theoretical efficiency would evidently be determined from equation (8), as before.

Case IV. In Fig. (38) is represented a *double riveted butt joint* with chain riveting. The rivets in such a joint might be staggered as in Fig. (35), but the computation of the dimensions would be the same as for the joint shown.

(a) Having chosen the size of rivet the strength of a single rivet will be the less of the values W_3 and W_4.

(b) In this case the net section of the plate between two rivets must be equal to the strength of two rivets, hence if $W_3 < W_4$ the pitch will be obtained from the equation

$$2W_3 = 4f_sA = f_t(p-d)t. \quad . \quad . \quad . \quad . \quad (15)$$

and if $W_4 < W_3$

$$2W_4 = 2f_ctd = f_t(p-d)t. \quad . \quad . \quad . \quad . \quad (16)$$

(c) The *laps* in the main plate and in the cover plate will be determined by the same equations as in Case III.

(d) The distances between the rows will be determined by the same rule as in Case II.

If properly designed the efficiency will be given by the same expression (equation 8) as that for the preceding cases.

In each of the preceding four cases the object has been the determination of the dimensions of the strongest possible joint of a particular type. It will be found that for any given material the strength of a joint of any one type, in a given thickness of plate, will be varied by changing the diameter of the rivets.

When the dimensions of a joint are determined in accordance with the preceding method it will be found that an increase in the diameter of the rivets will result in an increase in efficiency of the joint until the diameter at which the shearing strength of a rivet is equal to the allowable bearing pressure upon it is reached. Any increase beyond this will not give an increase in efficiency. It will also be found that for any given diameter of rivet, the efficiency of a riveted connection will increase with an increase in the number of rows of rivets used.

The joints given in the preceding four cases represent the simpler types used in the construction of boilers, tanks, pipes, etc. By increasing the number of rows of rivets and varying the numbers of rivets in the different rows, joints of greater strength than any of these can be made. The analysis of the stresses in such cases becomes more complex, however, and will not be undertaken here.

Case V. One other type of joint will be mentioned, namely, the joint used in fastening together two parts of a frame structure when the rivets are placed in such a manner that the stresses on all the rivets are equal. In such a case the line of action of the resultant of the forces acting on the joint must pass through the center of gravity of all the rivet sections combined.

When the rivets are in single shear, as in the illustration of the plate and channel connection given (Fig. 39), the strength of a rivet in single shear will be equal to

$$f_sA = \frac{f_s\pi d^2}{4},$$

and the allowable bearing pressure will be equal to

$$f_ctd$$

where t = the thinner of the two parts.

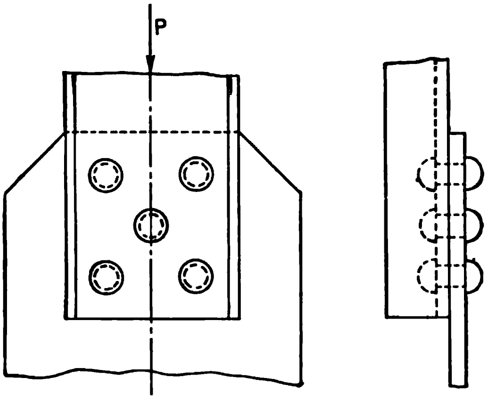


Fig. 39.

The strength of a rivet will evidently be equal to the smaller of these two values and the number of rivets required in a joint will be found by dividing the resultant force acting on the joint by this quantity.

On account of the dissimilar cross sections of the parts connected, the spacing of the rivets in such a joint is usually made according to a certain standard, the standard distances between the rivets and between the rivets and the edge of the plate being based on the results of theoretical computations and data from experiments.

The method of determining the number of rivets when a joint of this type is constructed with the rivets in double shear is self evident.

In practically all joints of this kind the holes in the members are punched and, as previously indicated, the nominal diameter of the rivets should be used in making computations.

59. Uniformly Varying Stress. — When the stress on a plane surface is distributed in such a manner that the intensity at any point is proportional to the distance of the point from a straight line in the plane of the surface the stress is said to be *uniformly varying* (Art. 82, Vol. I). Theoretically, this definition will apply to a normal stress, a shearing stress, or an oblique stress with both a normal and a shearing component; but for the sake of brevity the term *uniformly varying stress*, unless otherwise designated, will be taken to mean a uniformly varying normal stress.

Neutral Axis. — The straight line along which the stress intensity is zero is called the *neutral axis* and this may be located either within or outside of the limits of the surface.

Resultant Stress and Center of Stress. — If the area is referred to the rectangular coördinate axes OX and OY, with OY coinciding with the neutral axis, and if a = the stress intensity at a unit of distance from the neutral axis, the stress intensity at any point in the surface, whose coördinates are (x, y), will be equal to

$$p = ax. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The total stress on the surface will be equal to

$$R = \int p\, dA = a \int x\, dA = ax_0A, \quad . \quad . \quad . \quad . \quad (2)$$

where x_0 = the distance from the neutral axis to the center of gravity of the surface.

The moment of the resultant stress about the neutral axis will be equal to

$$M = \int px\, dA = a \int x^2\, dA = aI, \quad . \quad . \quad . \quad . \quad (3)$$

where I = the moment of inertia of the surface about the neutral axis. The coördinates of the center of stress will be

$$x_1 = \frac{M}{R} = \frac{\int px\, dA}{\int p\, dA} = \frac{a \int x^2\, dA}{a \int x\, dA} = \frac{I}{x_0A}, \quad . \quad . \quad (4)$$

$$y_1 = \frac{\int py\,dA}{\int p\,dA} = \frac{a\int xy\,dA}{a\int x\,dA} = \frac{K}{x_0A}, \quad . \quad . \quad . \quad . \quad (5)$$

where K = the product of inertia of the surface with respect to the X and Y axes.

It is evident that when the axes OX and OY are principal axes of inertia of the surface (Art. 125, Vol. I), $y_1 = 0$ and the center of stress is on the axis OX.

A familiar example of a uniformly varying stress is that of a fluid pressure exerted on any plane submerged surface. In such a case the neutral axis of the stress will be the line of intersection of the plane of the surface with the surface of the fluid and this line may form a boundary of the surface under stress or it may lie entirely outside its limits. The center of stress for such a surface is ordinarily called the *center of pressure.*

If we let p_0 = the intensity of the pressure at the center of gravity of a submerged surface and I_0 = the moment of inertia of the surface about an axis through the center of gravity parallel to the neutral axis, equation (2) may be written

$$R = p_0A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and equation (4) will reduce to

$$x_1 - x_0 = \frac{I_0}{x_0A}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

That is, the resultant pressure on any submerged surface is equal to the product of the pressure intensity at its center of gravity and the area of the surface; and the distance between the center of pressure and the center of gravity is equal to the moment of inertia of the surface about an axis through its center of gravity, parallel to the neutral axis, divided by the moment of the submerged area about the neutral axis.

60. Uniformly Varying Stress Whose Resultant is a Couple. — A uniformly varying stress may be distributed over a plane surface in such a manner that the neutral axis lies within the boundary of the surface, the stress on one side of the neutral axis being tension and on the other compression. A special case arises when the neutral axis passes through the center of gravity of the surface.

In this case the total stress on the surface (equation 2, Art. 59) becomes

$$R = ax_0A = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

and the moment of the stress (equation 3, Art. 59) becomes

$$M = aI_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Hence the resultant of a uniformly varying stress, the neutral axis of which passes through the center of gravity of the surface, is a couple. The magnitude of this couple is represented by equation (2) when the neutral axis is a principal axis of inertia of the surface. In other cases the value of M given by equation (2) is the component couple in a plane perpendicular to the neutral axis, obtained by resolving the resultant couple, formed by the stress, into two components in planes respectively perpendicular and parallel to the neutral axis.

This may be proven in the following manner. Let the plane surface (Fig. 40) be subjected to a uniformly varying stress, YY being the neutral axis passing through the center of gravity O. Let the stress above the neutral axis be tension and below it, compression.

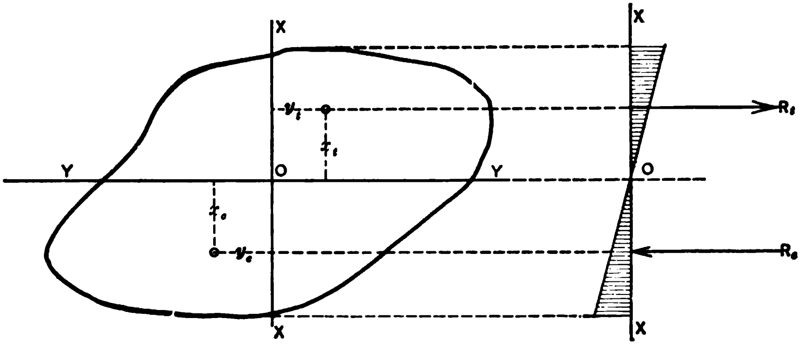


FIG. 40.

Let R_t represent the magnitude of the resultant and x_t, y_t the coördinates of the center of the tension stress; and let R_c represent the magnitude of the resultant and x_c, y_c, the coördinates of the center of the compression stress. Since the resultant of the stress on the entire surface is a couple, $R_t = R_c$.

The sum of the moments of the tension and compression stresses about YY will be equal to

$$M = R_tx_t + R_cx_c = aI_0$$

and this quantity will evidently be equal to the moment of the resultant couple formed by R_t and R_c when $y_t = y_c$ (algebraically). When $y_t >$ or $< y_c$ the couple $M = aI_0$ will evidently be the component of the resultant couple acting in the Y plane.

To determine when $y_t = y_c$ we have from equation (5) (Art. 59)

$$y_t = \frac{K'}{x_0'A'},$$

where K' = the product of inertia, with respect to the X and Y axes, and $x_0'A'$ = the moment, about the axis YY, of the part of the surface above the neutral axis.

Similarly,

$$y_c = \frac{K''}{x_0''A''},$$

where K'' = the product of inertia, with respect to the X and Y axes, and $x_0''A''$ = moment, about the axis YY, of the part of the surface below the neutral axis.

When the axis XX is an axis of symmetry of the surface $K' = K'' = 0$ (Art. 122, Vol. I) and hence $y_t = y_c = 0$, which holds true whether the axis YY is an axis of symmetry or not.

Since the axis YY passes through the center of gravity

$$x_0'A' = -x_0''A''$$

and when YY is an axis of symmetry

$$K' = -K'' \text{ (Art. 121, Vol. I)},$$

and hence $y_c = y_t$, which evidently holds true whether the axis XX is an axis of symmetry or not.

But the product of inertia of the whole surface, with respect to the axes XX and YY, will be equal to

$$K_{xy} = K' + K'' \text{ (Art. 123, Vol. I)}.$$

Hence, in both of the above mentioned cases

$$K_{xy} = 0,$$

the axes XX and YY being principal axes.

Moreover, for a surface of any shape, when the neutral axis passes through the center of gravity and is one of the principal axes, $K_{xy} = 0$ and $K' = -K''$; and hence $y_t = y_c$.

Therefore, *when a plane surface is subjected to a uniformly varying stress, if the neutral axis passes through the center of gravity and is a principal axis of the surface, the resultant couple formed by the stress is in a plane perpendicular to the neutral axis.*

61. Problems. — Uniform and Uniformly Varying Stress.

Problem 1.

A solid cylinder of concrete, 10" diameter and 30" long, is subjected to an axial load in compression of 30,000 lbs. Find the maximum intensity of the compressive stress at any point in the middle portion of the cylinder; also the maximum intensity of the shearing stress at that point.

Problem 2.

Find the required thickness of a steel tube, 6" inside diameter, which is to be subjected to an internal pressure of 400 lbs. per sq. in. Assume the working strength of the material in tension to be 8000 lbs. per sq. in.

Problem 3.

Find the required thickness of the shell of a cylindrical boiler, 5 ft. inside diameter, if the boiler is to be subjected to an internal pressure of 125 lbs. per sq. in. and the efficiency of the longitudinal joint in the shell is 75 per cent. Assume the working strength of the material in tension to be 8000 lbs. per sq. in., and allow $\frac{1}{16}$" for corrosion.

Problem 4.

Find the safe intensity of internal pressure for a hollow sphere, $\frac{1}{4}$" thick and 12" inside diameter; assuming the working strength in tension to be 7500 lbs. per sq. in.

Problem 5.

Find the proper values for the pitch and laps in a single riveted lap joint in $\frac{1}{2}$" steel plate, made up with $\frac{13}{16}$" rivets in $\frac{7}{8}$" drilled holes; also, find the theoretical efficiency of the joint. Use the driven diameter of the rivets in making calculations and assume the following values for working strengths: $f_s = 11{,}000$ lbs. per sq. in.; $f_c = 22{,}000$ lbs. per sq. in.; $f_t = 15{,}000$ lbs. per sq. in.

Problem 6.

Find the proper values for the pitch, laps and distance between the rows of rivets in a double riveted lap joint in $\frac{1}{2}$" steel plate, using $\frac{13}{16}$" rivets in $\frac{7}{8}$" drilled holes; also, find the efficiency of the joint. Use the values for working strengths given in Problem (5).

Problem 7.

Find the proper values for the pitch and laps in a single riveted butt joint in $\frac{1}{2}$" steel plate, made up with $\frac{7}{16}$" cover plates and $\frac{13}{16}$" rivets in $\frac{7}{8}$" drilled holes; also, find the efficiency of the joint. Use the values for working strengths given in Problem (5).

Problem 8.

Find the proper values for the pitch, laps and distance between the rows of rivets in a double riveted butt joint in $\frac{1}{2}$" steel plate, made up with $\frac{7}{16}$" cover plates and $\frac{13}{16}$" rivets in $\frac{7}{8}$" drilled holes; also, find the efficiency of the joint. Use the values for working strengths given in Problem (5).

Problem 9.

Solve Problem (6), using $\frac{15}{16}''$ rivets in $1''$ drilled holes instead of the size given.

Problem 10.

Solve Problem (8), using $\frac{15}{16}''$ rivets in $1''$ drilled holes instead of the size given.

Problem 11.

A standard $4'' \times 4'' \times \frac{1}{2}''$ angle is fastened to $\frac{1}{2}''$ plates at each end with $\frac{3}{4}''$ rivets and acts as a tension member in a frame. How many rivets should be provided at each end if the total tension in the member equals 30,000 lbs. Assume $f_s = 7500$ lbs. per sq. in., $f_c = 15{,}000$ lbs. per sq. in.

Problem 12.

Two standard $6'' \times 6'' \times \frac{3}{4}''$ angles are separated by $\frac{3}{4}''$ plates and riveted together at each end with $\frac{7}{8}''$ rivets. Angles are placed symmetrically relative to the plates. How many rivets should be provided at each end if the total compression in the member equals 105,000 lbs. Assume $f_s = 7500$ lbs. per sq. in., $f_c = 15{,}000$ lbs. per sq. in.

Problem 13.

Find the difference between the original cross section of a rod $4''$ diameter and the area of the cross section when the stress intensity is 12,000 lbs. per sq. in. tension, assuming $m = 3.6$ and $E = 30{,}000{,}000$ lbs. per sq. in.

Problem 14.

Find the increase in the internal diameter of the sphere given in Problem (4) when subjected to the allowable internal pressure, assuming that $E = 30{,}000{,}000$ lbs. per sq. in. and $m = 3.6$.

Problem 15.

A steel tube, 2 ft. inside diameter and $\frac{1}{4}''$ thick, is made up of plates riveted together with spiral joints running at 45° with the right cross sections of the tube. The joints are single riveted lap joints with $\frac{1}{2}''$ rivets in $\frac{9}{16}''$ punched holes. Assuming the following values for working strengths, $f_s = 10{,}000$ lbs. per sq. in., $f_c = 20{,}000$ lbs. per sq. in., $f_t = 16{,}000$ lbs. per sq. in., find the proper pitch of the rivets, using the nominal diameter of the rivets in making calculations; also, find the safe intensity of internal pressure on the tube. Assume *end tension* in all cases, that is, the tube is closed at the ends.

Problem 16.

Deduce the expressions for the resultant pressure and the distance of the center of pressure from the neutral axis for each of the sections A, B, C, D, E (Fig. 41) in terms of the dimensions given. The neutral axis in each case is represented by the line XX.

Problem 17.

Assuming that the sections (Fig. 41) are portions of a vertical surface subjected to water pressure and that the line XX is 3 ft. below the surface of the water, find the total pressure and locate the center of pressure on each surface. Let $h = 6$ ft., $b = 4$ ft. (except for *E* when $b\sqrt{2} = 6$ ft.). Weight of one cubic foot of water = 62.5 lbs.

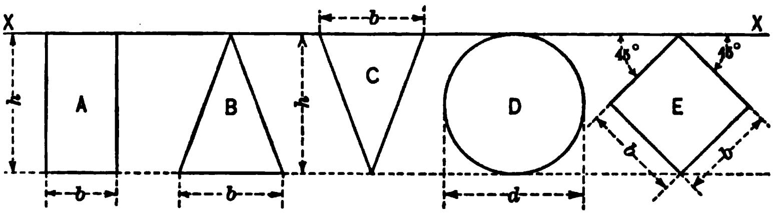


FIG. 41.

Problem 18.

Two circular gates, *A* and *B*, each 5 ft. in diameter, are subjected to water pressure. The surface of the gate *A* is vertical and that of the gate *B* is inclined at an angle of 45° with the vertical. The center of each gate is at a vertical distance of 6 ft. below the surface of the water. Find the total pressure and the distance between the center of gravity and the center of pressure on each gate (Art. 59). Weight of one cubic foot of water = 62.5 lbs.

Problem 19.

A reservoir *ABCD* (Fig. 42) of rectangular cross section 3′ × 8′ is connected with a pipe *E*. If the reservoir and pipe are filled with water up to a point in the pipe 4 ft. above the level of *AD*, find the resultant pressure and the center of pressure on each of the ends *AB* and *CD*. (Art. 59.) Weight of one cubic foot of water = 62.5 lbs.

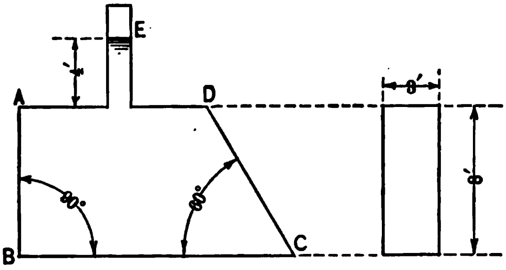


FIG. 42.

Problem 20.

Solve Problem (19), assuming that the cross section of the reservoir is a circle, 8 ft. diameter.

Problem 21.

In Problem (19), assume the ends *AB* and *CD* to be hinged at *A* and *D*, and find the horizontal forces acting at *B* and *C*, respectively, that will hold the ends in place.

Problem 22.

In Problem (19) assume the ends AB and CD to be hinged at B and C and find the forces acting at A and D, perpendicular to AB and CD, respectively, that will hold the ends in place.

Problem 23.

Find the H and V components and the point of application of the resultant thrust on the joint CD due to the water pressure on the face AC and the weight of masonry of cross section $ABDC$ (Fig. 43). Weight of masonry = 150 lbs. per cu. ft. Weight of water = 62.5 lbs. per cu. ft.

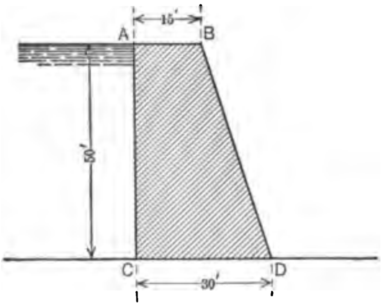


FIG. 43.

Problem 24.

A uniformly varying stress, the resultant of which is equal to 10,000 lbs. is distributed over a square section 10″ × 10″, in such a manner that the center of stress is located 2″ from the center of gravity, on an axis of symmetry parallel to the side of the square. Find the maximum intensity of the stress.

Problem 25.

A 6″ × 12″ wooden beam is supported at the ends and is subjected to a total uniformly distributed load of 8000 lbs. Find the length of the bearing surface necessary at the ends, provided the greatest intensity of compression across the grain does not exceed 150 lbs. per sq. in. by each of the following assumptions: (a) Supporting force uniformly distributed. (b) Supporting force uniformly varying from zero intensity at the end of the beam to the maximum intensity at the inside of the support.

Problem 26.

A uniformly varying stress is distributed over a rectangular section (Fig. 44) in such a manner that the stress below the neutral axis XX is tension and above it, compression. The value of a (Art. 59) is 100 lbs. per sq. in. Find the resultant stress and the center of stress, assuming $c = 4''$, $d = 12''$.

Problem 27.

Solve Problem (26), assuming $c = 7''$, $d = 9''$.

Problem 28.

Solve Problem (26), assuming $c = 8''$, $d = 8''$.

Problem 29.

Solve Problem (26), assuming $c = \frac{16}{3}''$ and $d = \frac{32}{3}''$.

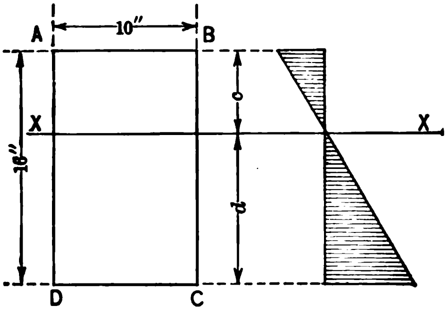


FIG. 44.

Problem 30.

A uniformly varying normal force is distributed in such a manner over the plane surface (Fig. 45), that the resultant P acts on the axis of symmetry XX. Find the extreme limits between which the position of the center of stress may vary under the condition that the stress over the entire surface shall be compression.

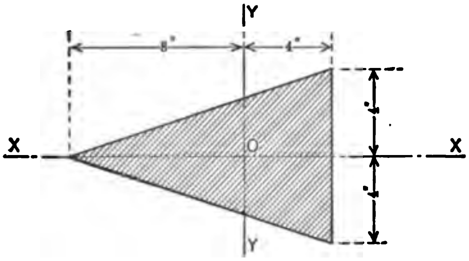


FIG. 45.

Problem 31.

Find the resultant stress and the coördinates x_1 and y_1, of the center of stress, on each of the surfaces shown (Fig. 46), assuming each surface is subjected to a uniformly varying stress with XX as the neutral axis and the stress intensity $a = 100$ lbs. per sq. in., at a distance unity from the neutral axis.

Problem 32.

Find the resultant of the stress on each of the surfaces shown (Fig. 46), assuming each surface is subjected to a uniformly varying stress, part tension

and part compression, with the neutral axis parallel to *XX*, through the center of gravity of the surface, and the stress intensity a = 100 lbs. per sq. in., at a distance unity from the neutral axis.

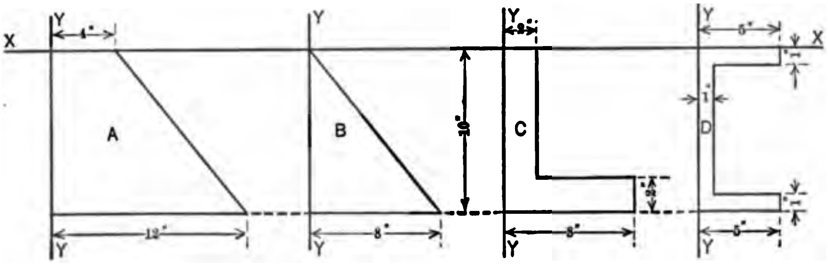


FIG. 46.

CHAPTER IV.

STRESSES IN BEAMS.

62. Definitions. — When a bar of material, straight or curved, is acted upon by a balanced system of external forces which are perpendicular to its central axis, it is called a *beam*.

The forces acting on such a bar cause the straining known as *flexure*, or *bending*.

When the external forces act on the bar obliquely, the components perpendicular to the axis will cause bending and in many, but not all, cases of this kind the bar will be designated as a beam, subjected to oblique loading.

In some cases external forces acting parallel to the axis of the bar will produce bending. In such cases the bar may sometimes be designated as a beam, but more frequently not.

In both of the last mentioned cases the components of the force system acting along the axis of the bar will cause tensile or compressive stresses in combination with the stresses due to the bending; with the exception that, when the system of external forces is so arranged that it can be resolved into couples acting on the ends of the bar, bending only will result.

It is our object in this chapter to discuss the stresses produced in straight, or very nearly straight, bars which are subjected to bending only.

Since a beam is generally placed in a horizontal position, we shall use horizontal beams as illustrations although it will be readily seen that the theory will apply to beams placed in any position whatever, when acted upon by similar force systems. The forces acting on the horizontal beam will ordinarily be designated as *loads* and *supporting forces*, or *reactions*.

When a beam is balanced over one support (Fig. 47) it is known as a *cantilever beam;* or, when it is held by supporting forces at one end (Fig. 48), the free end is called a *cantilever beam*.

When a beam rests on supports at the ends (Fig. 49) it may be called a *simple beam*. When it is securely held by supports at both ends (Fig. 50) it is called a *built-in* or a *fixed beam*.

When a beam is acted upon by terminal couples only (Fig. 51), it is said to be subjected to *uniform* or *simple bending*.

When a beam is supported at more than two points (Fig. 52), it is called a *continuous beam*.

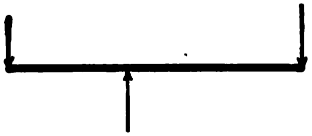


FIG. 47.

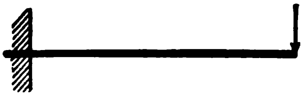


FIG. 48.

In the simplest form, the beam in any one of the above mentioned cases would be a homogeneous bar of material of a uniform cross section of any one of a variety of shapes.

When the distance between supports or the magnitude of the loads is large, however, it becomes necessary to use a large rolled section, or to *build up* a beam by riveting together rolled sections of different shapes and plates.

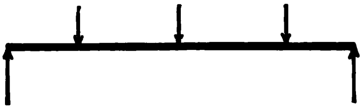


FIG. 49.

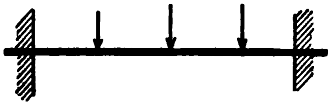


FIG. 50.

Such beams are called *girders*, *girder beams*, *plate girders*, *beam girders*, etc., according to the size of the beam and the type of cross section.

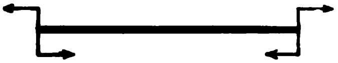


FIG. 51.

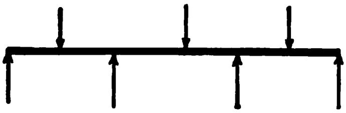


FIG. 52.

63. Limitations. — Throughout the discussion of the theory for determining the stresses due to bending the following limitations will be imposed, unless otherwise stated:

(a) The material of the beam is homogeneous.

(b) The beam is straight and of uniform cross section throughout.

(c) The external forces are in equilibrium and are applied in such a manner that they can be considered equivalent to a system of forces acting in a single plane which will be known as the *plane of loading*.

(d) The plane of loading intersects every cross section at an axis of symmetry and, wherever the term *cross section* is used, the section at right angles to the central axis is to be understood.

(e) The length of the beam is large in proportion to the greatest dimension of its cross section and the difference between the depth and greatest width of the cross section is not excessive.

In order to fix ideas it will be convenient to consider a beam as being made up of a bundle of fibers parallel to its central axis, some of which are stretched while others are compressed by the bending, and the terms fiber and fiber stress will be frequently used in the discussion. This will not mean that the material is fibrous in texture, however, for it will be seen that the conception of a beam as being composed of longitudinal fibers is entirely unnecessary in the development of the theory of stress. Unless specifically stated, the weight of the material in the beam itself will be neglected throughout the discussion.

64. Uniform or Simple Bending. — Whenever a beam is subjected to the action of equal and opposite couples, at its ends only, it is said to be subjected to *uniform* or *simple bending*. Such a case is represented in Fig. (53), the terminal couples M being shown by the usual convention of two equal and opposite parallel forces, perpendicular to the sections on which the couples are assumed to act. It should be borne in mind, however, that the couples M may be taken to represent the resultants of a large variety of force systems, acting at the sections CD and EF.

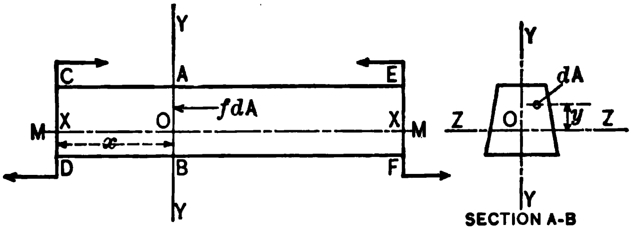


FIG. 53.

The axes of reference OX, OY and OZ are so chosen that OX coincides with the central axis passing through the centers of gravity of the cross sections of the beam; OY is the axis of symmetry of any cross section AB and OZ is perpendicular to the plane of loading.

Since the beam is in equilibrium, it is evident that the resultant of the stress on the cross section AB must be in equilibrium with the couple M acting on the end CD. Hence the resultant of the stress on AB must be a couple, which, according to the limitations imposed (Art. 63), will be equivalent to a couple acting in the plane of symmetry XOY (Fig. 53.)

65. Bending Moment and Moment of Resistance — Simple Bending. — The couple, comprising the external forces acting on the part of the beam to the left of the section AB (Fig. 53), is called the *bending moment* at the section AB and the equal and opposite couple, comprising the stress on the section AB, is called the *moment of resistance* at the section AB.

It is evident that if we apply the conditions of equilibrium to the forces acting on the part of the beam between the section AB and the right hand end EF, the bending moment and the moment of resistance at the section AB will be couples of the same magnitude but of opposite sign to those obtained above.

It is customary, therefore, in order to avoid confusion of signs, to denote the bending moment at any section of a horizontal beam as positive where the external forces act in such a manner that the bending is as shown in Fig. (54), and as negative when the bending is as shown in Fig. (55), irrespective of the sign of the couple obtained by the application of the condition of equilibrium to the portion of the beam between the cross section AB and either end.

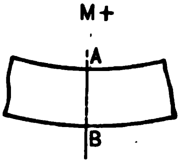


Fig. 54.

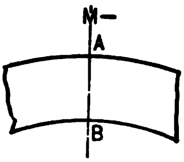


Fig. 55.

Since in the case shown (Fig. 53) the bending moment will be the same, whatever the position of the cross section AB, the beam is said to be subjected to *uniform bending*, and the state of stress on every cross section must, evidently, be the same.

It is hardly necessary to state that the application of the conditions of equilibrium shows that the stress on the section AB has no shearing component and hence the moment of resistance is the resultant of a normal stress, the distribution of which is determined by the theory of uniform bending.

66. Theory of Uniform or Simple Bending. — In the ordinary theory for determining the distribution of the stress on a cross section of a beam bent by terminal couples (Fig. 53) it is necessary to make the following assumptions, based upon the fact that the material is not absolutely rigid, but bends slightly under the action of the external forces.

(1) That plane cross sections remain plane and normal to the longitudinal fibers after bending. This is known as Bernouilli's assumption.

(2) That the material obeys Hooke's law; that is, the stress intensity is proportional to the strain throughout the beam.

(3) That every longitudinal fiber is free to extend, or contract, under stress as if separate from the other fibers and the ratio of stress intensity to strain is the same in every fiber; that is, $E_t = E_c = E$.

The beam shown (Fig. 53) will bend slightly under the action of the terminal couples M into a curve, as illustrated in Fig. (56). It will follow from the first assumption that any two parallel cross sections AB and GH, originally at a small distance l apart, will be inclined to each other and, if produced, will intersect along a line perpendicular to the plane of loading whose trace, in the plane of loading, is the point K. Since the bending moments at all cross sections are equal, the distortion of the portion between the cross sections AB and GH will be the same, in whatever part of the beam the sections are taken, and hence all such pairs of sections will intersect along the line perpendicular to the plane of loading and passing through the point K. Therefore, all the longitudinal fibers of the beam will become circular in form, with the center of curvature at K.

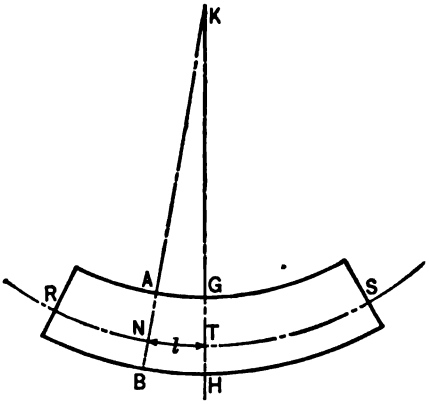


FIG. 56.

The fibers near the top of the beam will be compressed and those near the bottom elongated, and somewhere between there will be a layer of fibers which are neither compressed nor elongated. This layer is called the *neutral layer* and its trace on the plane of loading is represented by the line RS (Fig. 56) called the *neutral line*. Owing to the symmetry of the cross section and the loading, the neutral layer evidently has the form of a circular cylinder with the axis passing through K.

An inspection of Fig. (56) will show that the elongations, or compressions, of the fibers between the sections AB and GH are proportional to their distances from the *neutral fibers* NT and hence *the longitudinal strains* (tensile, or compressive) *in the fibers at the intersection of any cross section are proportional to their distances from the neutral layer.*

Therefore, it will follow from the second and third assumptions that *the normal stress on any cross section is uniformly varying, with the neutral axis at the line of intersection of the cross section and the neutral layer of the beam.*

The moment of resistance being a couple (Art. 65) the *neutral axis of the normal stress on any cross section must pass through the center of gravity of the section* (Art. 60) and hence the neutral layer will contain the central axis XX (Fig. 53).

The moment of resistance will be equal to

$$M = aI \text{ (Art. 60)}, \quad \ldots\ldots \quad (1)$$

where M is equal in magnitude to the bending moment, a = the stress intensity at a distance unity from the neutral axis and I = the moment of inertia of the cross section about the neutral axis.

If we let f = the intensity of the normal stress at a point in the cross section at a distance y from the neutral axis,

$$a = \frac{f}{y}, \quad \ldots\ldots\ldots \quad (2)$$

and therefore the expression

$$f = \frac{My}{I} \quad \ldots\ldots\ldots \quad (3)$$

will give the *normal intensity of stress* at any point in a cross section of the beam. Since the value of f varies as the value of y, the greatest intensity of the normal stress will occur at the point, or points, in the cross section farthest from the neutral axis. If we let c = the greatest value of y, the ***greatest intensity of the normal stress*** will be represented by the expression

$$f = \frac{Mc}{I}. \qquad (4)$$

The stress intensity f is frequently called the ***fiber stress*** and its value, as given by equation (4), the ***outside fiber stress*** in the beam.

Emphasis should be laid on the fact that M in the foregoing formulas denotes the moment of resistance, which is equal in magnitude and of opposite sign to the bending moment at any given section and hence, if we follow the system of signs previously adopted, calling y positive when measured upwards, the sign of f will be positive when the normal intensity of stress is tension and negative when the normal intensity is compression.

This fact will be important in the development of the theory of flexure (Art. 95) but, in ordinary computations for fiber stresses it will be unnecessary to take the signs of the different quantities into account.

If the resultant of the tensile stress on the section is designated by the symbol R_t, and the resultant of the compressive stress by R_c (Art. 60),

$$R_t = R_c \qquad (5)$$

and, since the cross section is symmetrical with respect to OY the couple formed by R_t and R_c is in the plane of loading.

Attention may again be called to Saint-Venant's principle (Art. 53) according to which the above formulas will not give correct values for the stress intensities at sections very near to the ends of the beam, unless the terminal couples are the resultants of stresses distributed over the ends in the same manner as the stress on the middle sections of the beam.

67. Ordinary Bending. — The bar bent by terminal couples has been discussed in the preceding articles as representative of the simplest type of bending. Most commonly, bending is produced by forces acting at right angles to the central axis of the beam. Hence we shall speak of flexure produced by force systems acting in this manner as *ordinary bending*, in distinction from the less common case of simple, or uniform, bending.

A case of this kind is represented in Fig. (57), in which we have a simple beam subjected to the loads W_1, W_2, W_3 and supported at the ends by the reactions R_1 and R_2. The vectors W_1, R_1, etc., are to be taken to represent the resultants of forces distributed over small portions of the surface of the beam, which can be treated as concentrated forces, acting in the plane of loading, without introducing any appreciable error in the results of the theory.

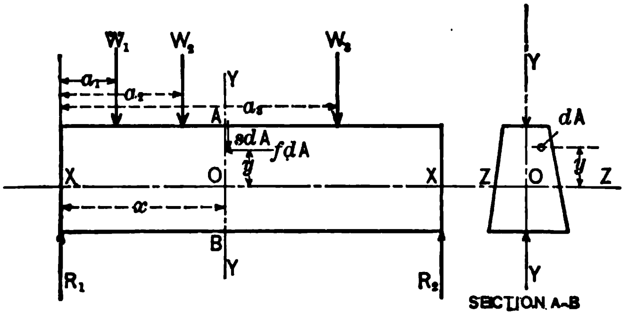


FIG. 57.

As in the case of simple bending, the axes of reference are so chosen that OX coincides with the central axis through the centers of gravity of the cross sections, OY is the axis of symmetry of any cross section AB and OZ is perpendicular to the plane of loading.

The stress on the cross section AB must be in equilibrium with the forces acting on the portion of the beam to the left of the section, but in this case it will be found that the resultant of the stress on AB is not ordinarily a couple, simply, but is made up of both normal and shearing components.

Irrespectively of the manner of distribution of the stress on the section, the stress on each element dA, of the cross section AB, can be resolved into a normal component $f\,dA$ and a shearing component $s_r\,dA$, where f and s_r are the intensities of the normal and shearing components.

If the shearing component $s_r\,dA$ is not vertical, it can be resolved into a vertical component $s\,dA$ and a horizontal component $s_1\,dA$, where s and s_1 represent the vertical and horizontal components, respectively, of the resultant shearing stress intensity s_r. Under the limitations imposed (Art. 63), either $s_1 = 0$ for all points in any cross section or the resultant horizontal shearing stress $\int s_1\,dA = 0$, depending on the shape of the cross section. As ordinarily the intensity s_1 is small compared with the other components it is neglected and the vertical component s only is taken into consideration.

If we apply the conditions of equilibrium, $\Sigma H = 0$, $\Sigma V = 0$ and $\Sigma M = 0$, to the forces acting on the part of the beam to the left of AB, the first condition will give

$$\sum H = \int f\,dA = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Therefore, the resultant of the normal stresses on the elements of the cross section AB is a couple.

The second condition of equilibrium will give

$$\sum V = R_1 - W_1 - W_2 - \int s\,dA = 0. \quad . \quad . \quad . \quad (2)$$

Solving equation (2) we obtain the value of the resultant of the vertical shearing stress on the cross section AB,

$$\int s\,dA = R_1 - W_1 - W_2. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The quantity $\int s\,dA$ is called the *shearing stress*, or the *resistance to shear* at the section AB.

The third condition of equilibrium, taking the axis of moments at O and the location of the loads as indicated (Fig. 57), will give

$$\sum M = R_1x - W_1(x - a_1) - W_2(x - a_2) - \int fy\,dA = 0. \quad (4)$$

Solving equation (4) we obtain the value of the resultant couple, formed by normal components of the stress on the elements of AB,

$$\int fy\,dA = R_1x - W_1(x - a_1) - W_2(x - a_2). \quad . \quad . \quad (5)$$

The quantity $\int fy\,dA$ is called the *moment of resistance* at the section AB, as in the case of simple bending (Art. 65).

If we resolve the forces acting on the part of the beam to the left of AB (Fig. 57) into equal and parallel forces acting along the axis OY and couples (Art. 48, Vol. I), the resultant of the force system will ordinarily be found to comprise a single force, acting along OY, equal to

$$R_1 - W_1 - W_2 = \Sigma F, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and a couple equal to

$$R_1x - W_1(x - a_1) - W_2(x - a_2) = \Sigma F(x - a), \quad . \quad . \quad (7)$$

where ΣF may be taken to represent the algebraic sum of a system of any number of vertical forces acting to the left of AB and $\Sigma F\,(x - a)$ may be taken to represent the algebraic sum of their moments about the neutral axis of AB.

68. Shearing Force, Bending Moment and Moment of Resistance.—*The component force* ΣF (Art. 67), *acting along the axis* OY, *is called the shearing force at the section* AB *and the component couple* $\Sigma F\,(x - a)$ *is called the bending moment at* AB. Denoting the former quantity by S, we shall have

$$S = \Sigma F \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and denoting the latter quantity by M, we shall have

$$M = \Sigma F\,(x - a). \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Hence, in the case of a beam subjected to ordinary bending, *the shearing force at any cross section will be equal to the algebraic sum of the vertical forces acting on the portion of the beam to the left of the section; and the bending moment at any cross section will be equal to the algebraic sum of the moments, about the neutral axis of the section, of the forces acting on the portion of the beam to the left of the section.*

It is evident that the *shearing force* at any cross section will be balanced by the *shearing stress* on the section and that the *bending moment* will be balanced by the *moment of resistance*, or the couple formed by the normal stress on the section.

The magnitude of the bending moment and also of the shearing force at any section AB might obviously be obtained by applying the conditions of equilibrium to the portion of the beam between the section and the right hand end; but the sign of each quantity would be opposite to that obtained by the application of the conditions of equilibrium to the forces acting on the part to the left of the section if the usual system of signs for couples and forces were followed.

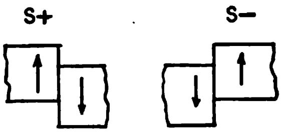


Fig. 58.

Hence, in the horizontal beam, the sign of the bending moment is determined by the manner of bending, as in the case of simple bending (Art. 65), and the sign of the shearing force is determined by the manner in which the tendency to shear is exerted, the shearing force at any section being called positive when the tendency is as shown in Fig. (58a) and negative when the tendency is as shown in Fig. (58b).

It is evident from the preceding discussion that in the case of *ordinary bending* the stress on any cross section will be comprised of a shearing component in addition to a couple formed by the normal stress, whereas in the case of *simple bending* the couple formed by the normal stress is the only stress component.

69. Theory of Ordinary Bending. — In the theory for determining the distribution of the normal stress on a cross section of a beam subjected to ordinary bending it is customary to make the same assumptions as are made in the case of simple bending (Art. 66). A more exact and complex analysis of the relations between stresses and strains in beams, bent by transverse loads, will show that all these assumptions are not exactly correct but that the results obtained by the ordinary theory in all cases coming within the limitations imposed (Art. 63) are in very close agreement with those obtained by the more exact theory and also with those obtained from bending experiments.

It is evident that these assumptions will affect only the longitudinal strains in the fibers and the normal components of the stress at any section, being entirely independent of the strains and stresses due to shear. Hence the normal stress on any cross section will be distributed in the same manner as in the case of simple bending.

Therefore, if we follow the notation adopted in that case (Art. 66), the intensity of the normal or fiber stress at any point in a cross section at a distance y from the neutral axis, through its center of gravity, will be equal to

$$f = \frac{My}{I} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and the outside fiber stress at any section will be equal to

$$f = \frac{Mc}{I} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

The observation in regard to the sign of f made in the case of simple bending will apply in this case and also that in regard to the relative position of the couple, formed by the normal stress, and the plane of loading.

It will be convenient to delay the discussion of the theory for determining the intensity of the shearing stress at any section of a beam until later (Art. 86), with the comment here that in beams of the simpler types the greatest intensity of the shearing component of the stress on any section is so small that it is unnecessary to take it into account in making computations for strength or stability.

Attention may again be called to the application of Saint-Venant's principle in the case of ordinary bending. Moreover, in some cases, it will be found necessary in making computations for strength to take into consideration the so-called "local stresses" due to the concentration of loads, or supporting forces, over small portions of the surface of the beam in addition to the stresses due to bending.

70. Ordinary Bending under Distributed Loads. — In the discussion of ordinary bending thus far we have, for the sake of simplicity, considered the forces acting on the beam as concentrated at single points, although it has been evident that the forces were actually distributed over small portions of the surface.

In many cases, however, the loads are distributed over a considerable portion of, or over the entire length of, the beam. When the weight of the beam itself is to be taken into account, it evidently must be treated as a distributed load.

Ordinarily in such cases the supporting forces can be considered as concentrated, but cases sometimes arise in which the supporting forces, as well as the loads, must be considered as distributed.

The stress on any cross section of such a beam might be determined to a very close degree of accuracy by dividing the loads and reaction on the part of the beam on one side of the section into a number of equivalent concentrated forces, as indicated in Fig. (59), where the load is represented as equivalent to a force distributed along the central axis of the beam varying in intensity, as indicated by the ordinates to the line *abc*, and the supporting forces as equivalent to forces distributed in a similar manner, with intensities varying as the ordinates to the lines *ef* and *gh*.

The load on the portion of the beam to the left of any section *AB* may be divided into any number of parts, the resultants of which are represented by the vectors W_1, W_2, W_3, etc., and, when necessary, the supporting force may be divided into similar parts, represented by the vectors R_1, etc. The stress on the cross section *AB*, due to the equivalent system of concentrated forces, can evidently be obtained by the method already given and the accuracy of the result will depend on the number of divisions made in the distributed load.

In many cases it will be convenient to obtain the exact values of the shearing force and bending moment, at a cross section of a beam subjected to distributed loads, by subdividing the loads and reactions into elementary forces, acting on very small portions of the beam, and summing up by the methods of integration.

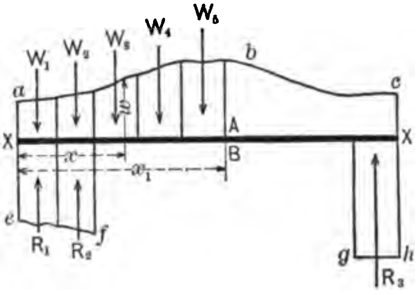


FIG. 59.

If we let w equal the resultant intensity of the external force acting at any point at a distance x from the end of the beam (Fig. 59); that is, w equals the resultant intensity of the load and supporting force, over the support, and of the load only between the supports, the value of the shearing force at a cross section, at any distance x_1 from the end of the beam, will be represented by the expression

$$S = \int_{x=0}^{x=x_1} w\,dx, \qquad \ldots \ldots \qquad (1)$$

and the value of the bending moment at the section by the expression

$$M = \int_{x=0}^{x=x_1} \int_{x=0}^{x=x} w\,dx\,dx. \qquad \ldots \ldots \qquad (2)$$

In order to follow the system of signs already adopted in designating shearing forces and bending moments (Art. 68), it is evident that w must be taken positive when representing the intensity of an upward force and negative for the intensity of a downward force acting on the part of the beam to the left of any cross section.

In cases in which the value of w is constant for different portions of the length of the beam, and also in cases in which the supporting forces can be considered as concentrated and the value of w as constant for different portions, or for the entire length, of the beam, the solution of equations (1) and (2) can be easily made.

Since a so-called concentrated load, or supporting force is always a force distributed over a small portion of the length of the beam, the foregoing analysis will evidently apply to a beam which is subjected to any system of distributed and concentrated loads and equations (1) and (2) may be considered as the general equations for shearing force and bending moment, of which equations (1) and (2) (Art. 68) are special, slightly approximate, forms.

71. Relation of Bending Moments, Shearing Forces and Loads. — If we differentiate equation (2) (Art. 70) we obtain

$$\frac{dM}{dx} = \int w\,dx = S, \quad \ldots\ldots \quad (1)$$

and a second differentiation will give

$$\frac{d^2M}{dx^2} = \frac{dS}{dx} = w; \quad \ldots\ldots\ldots \quad (2)$$

and conversely, integration gives, as already shown (Art. 70),

$$S = \int w\,dx \quad \ldots\ldots\ldots\ldots \quad (3)$$

and

$$M = \int S\,dx = \int\int w\,dx\,dx. \quad \ldots\ldots \quad (4)$$

The foregoing relations between loads and shearing forces and between shearing forces and bending moments are of great importance in computations for the strength and stability of beams.

Since w is a function of x only, it is evident that where $w = 0$ the value of S will be either a maximum or a minimum, and that where $S = 0$ the value of M will be either a maximum or a minimum.

72. Graphical Representation of Loads, Shearing Forces and Bending Moments. — A graphical representation of the values of w, S and M for a typical case may add to the clearness of the statement in the preceding article. Let the diagram (Fig. 60a) represent, in the conventional manner, the loads and supporting forces acting on a simple beam, each load and each supporting force being uniformly distributed.

The load intensity diagram (Fig. 60b) will be constructed by plotting ordinates w, equal to the difference in the intensities of the load and supporting forces at the ends of the beam and equal to the resultant intensity of the loads in the portion between the supports.

If we divide the load diagram into small sections, of length Δx, and plot the ordinates obtained by adding together the successive increments of load $w\Delta x$ and draw a line through the ends of the ordinates we shall obtain the shearing force diagram (Fig. 60c). In plotting this diagram it should be noted that an upward force acting on the part of the beam to the left of the section will tend to produce a positive shear, and a downward force a negative shear at the section, while the reverse is true for the forces acting on the part to the right of the section.

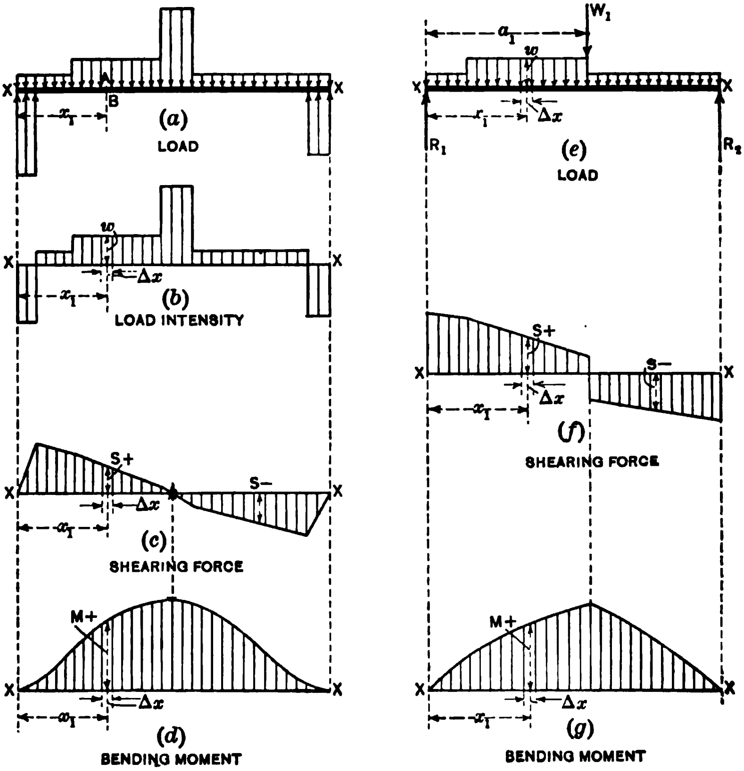


FIG. 60.

It is evident that the ordinates of the diagram might also be obtained by plotting values of the integral

$$S = \int_{x=0}^{x=x_1} w\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for successive values of x_1. The maximum and minimum values of S, that is, the greatest positive and negative values, will occur at the cross sections of the beam at which w changes sign. If the change in the intensity w, from negative to positive values and vice versa, were gradual instead of abrupt, the maximum and minimum values of S would evidently occur at the sections where $w = 0$.

The bending moment diagram (Fig. 60d) may be obtained by dividing the shearing force diagram into small sections, of length Δx, and plotting the ordinates obtained by adding together the successive increments $S\Delta x$ and drawing a line through the ends of the ordinates.

The ordinates of the diagram might also be obtained by plotting the values of the integral

$$M = \int_{x=0}^{x=x_1} S\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

for successive values of x_1.

Still another method, based on the definition of the bending moment and indicated by the form of the integral

$$M = \int_{x=0}^{x=x_1} \int_{x=0}^{x=x} w\,dx\,dx, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

will be to determine the ordinates of the bending moment diagram directly from the load diagram by adding the moments of the forces, acting on the part of the beam either to the right or the left of each section, about an axis in the section. In this case an upward force acting on either side of a cross section will evidently tend to cause a positive bending moment and a downward force a negative bending moment at the section.

The maximum and minimum values of the bending moment will evidently occur at the sections where $S = 0$, or where the value of S changes sign.

Ordinarily, in cases like the one just described, a sufficiently accurate solution can be made by considering the reactions and the load distributed over the small portion of beam as concentrated forces R_1, R_2 and W_1, acting at the same points as the resultants of the supporting forces and the load.

The load diagram would then be modified to the form shown (Fig. 60e). Following the same methods of procedure as before, the shearing force diagram would take the form (Fig. 60f) and the bending moment diagram the form (Fig. 60g).

The equation representing the value of S at any cross section at a distance x_1, from the left end of the beam (Fig. 60f) would be written in the form

$$S = R_1 - \int_{x=0}^{x=x_1} w\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

for all sections to the left of the load W_1, and in the form

$$S = R_1 - W_1 - \int_{x=0}^{x=x_1} w\,dx, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

for all sections to the right of W_1.

The equations representing the values of the bending moments at the same sections would be written in the form

$$M = R_1x - \int_{x=0}^{x=x_1}\int_{x=0}^{x=x} w\,dx\,dx, \quad . \quad . \quad . \quad . \quad (6)$$

for sections to the left of the load W_1, and in the form

$$M = R_1x - W_1(x - a_1) - \int_{x=0}^{x=x_1}\int_{x=0}^{x=x} w\,dx\,dx, \quad . \quad (7)$$

for sections to the right of W_1.

It is important to note that in writing equations (4) to (7) inclusive we have followed the method, adopted in the case of concentrated loads (Art. 67), of indicating the manner of the shear and the bending, which the reaction and the loads tend to produce, by the signs of the terms in the algebraic equations and hence positive values would be substituted for both upward and downward forces in making the solutions of these equations.

73. Relations of Shearing Forces and Bending Moments at Two Different Cross Sections.—Let AB and CD represent any two cross sections of a beam subjected to ordinary bending by the action of a system of both distributed and concentrated loads, as indicated (Fig. 61a).

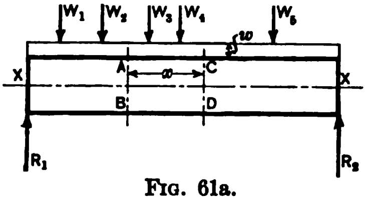


FIG. 61a.

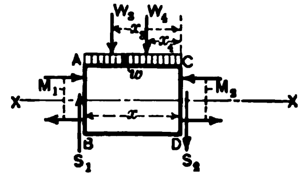


FIG. 61b.

The forces acting on the portion of the beam between AB and CD are in equilibrium and for the sake of clearness may be represented in the conventional manner, as in Fig. 61b, where the couple M_1 represents the resultant of the normal stress and S_1 the resultant of the shearing stress acting on the left side of the section AB, and the couple M_2 represents the resultant of the normal stress and S_2 the resultant of the shearing stress acting on the right side of CD. We will let w equal the intensity of any distributed load between the sections.

The couples M_1 and M_2 will evidently be equal in magnitude to the bending moments at the two sections, which will both be positive if the resultants of the normal stresses on the two sections tend to rotate in the directions indicated. The resultant shearing stresses S_1 and S_2 will be equal in magnitude to the shearing forces at the two sections, which will both be positive if the shearing stresses act in the directions indicated.

Applying the condition of equilibrium $\Sigma V = 0$ to the forces acting (Fig. 61b), we have

$$\sum V = S_1 - W_3 - W_4 - \int w\,dx - S_2 = 0. \quad . \quad . \quad (1)$$

Solving for S_2 we obtain

$$S_2 = S_1 - W_3 - W_4 - \int w\,dx = S_1 - \sum W - \int w\,dx, \quad . \quad (2)$$

where $\Sigma W = W_3 + W_4 + \text{etc.};$

that is, the shearing force at CD is equal to the shearing force at AB minus the vector sum of the loads between the sections.

Applying the condition of equilibrium $\Sigma M = 0$, taking the axis in the section CD, we have

$$\sum M = M_1 + S_1 x - W_3 x_3 - W_4 x_4 - \int\int w\,dx\,dx - M_2 = 0. \quad (3)$$

Solving for M_2 we obtain

$$\begin{aligned} M_2 &= M_1 + S_1 x - W_3 x_3 - W_4 x_4 - \int\int w\,dx\,dx \\ &= M_1 + S_1 x - \sum Wx - \int\int w\,dx\,dx, \quad . \quad . \quad . \quad . \quad (4) \end{aligned}$$

where $\Sigma Wx = W_3 x_3 + W_4 x_4 + \text{etc.};$

that is, the bending moment at CD is equal to the bending moment at AB, plus the moment of the shearing force at AB about an axis in CD, minus the algebraic sum of the moments of the loads between the sections about an axis in CD.

Solving for S_1 we obtain an expression for the shear at any section AB, in terms of the bending moments at two sections and the loads between the sections,

$$S_1 = \frac{M_2 - M_1}{x} + \frac{\sum Wx + \int\int w\,dx\,dx}{x}. \quad . \quad . \quad (5)$$

If there were no external forces acting on the portion of the beam between the sections, equations (2), (4) and (5) would reduce to

$$S_2 = S_1, \quad \ldots\ldots\ldots \quad (6)$$

$$M_2 = M_1 + S_1x, \quad \ldots\ldots\ldots \quad (7)$$

$$S_1 = \frac{M_2 - M_1}{x}. \quad \ldots\ldots\ldots \quad (8)$$

The preceding equations afford in many cases a convenient method of determining the variation in the shearing force and bending moment from section to section of a beam.

74. Problems. — Shearing Force and Bending Moment. — The general formulas for shearing force and bending moment are most easily obtained, in any case, by applying the rules given in Art. (68). The magnitude of the shearing force or the bending moment at any section of a beam subjected to a given system of loads can usually be most readily obtained in the same manner. The convention of signs for positive and negative values (Art. 68) should be observed in any case.

There are certain types of loading, however, of which the following problems, (1–9) inclusive, are given as examples, for which the algebraic expressions for the values of the greatest shearing force and bending moment will be convenient for reference. The diagrams showing the location of the loads and supporting forces and the variation in the magnitudes of the shearing forces and the bending moments are given. The loads W, acting over small portions of the surface, and the reactions are considered as concentrated forces and all distributed loads are of uniform intensity w. The shearing force diagrams are represented by dotted lines and the bending moment diagrams by full lines in each case.

Maximum values of shearing forces and bending moments are denoted by the symbols S_0 and M_0, respectively.

The formulas for shearing forces and bending moments in the cantilever beams (Fig. 62) and (Fig. 63) are determined by dealing with the forces acting on the portion of the beams between the section and the free end. The formulas for each of the remaining cases (Figs. 64–70) are obtained by dealing with the forces acting on the portion to the left of the section.

Problem 1.

Cantilever beam with a concentrated load W at the free end (Fig. 62).

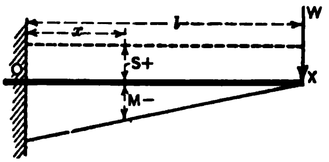


FIG. 62.

For values of x from 0 to l,

$$S = W, \quad M = -W(l - x).$$

When $x = 0$ to l,

$$S_0 = W.$$

When $x = 0$,

$$M_0 = -Wl.$$

Problem 2.

Cantilever beam with a total load W uniformly distributed over the entire length (Fig. 63).

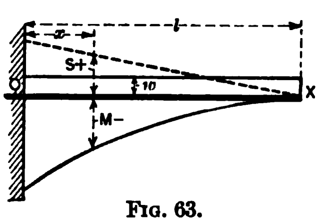


FIG. 63.

Let w = the intensity of the distributed load.

For values of x from 0 to l,

$$S = w(l - x), \quad M = -\frac{w}{2}(l - x)^2.$$

When $x = 0$,

$$S_0 = wl = W, \quad M_0 = -\frac{wl^2}{2} = -\frac{Wl}{2}.$$

Problem 3.

Simple beam with a concentrated load W at the middle of the span (Fig. 64).

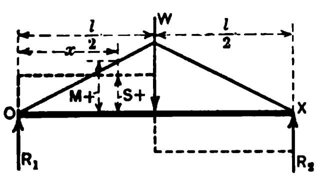


FIG. 64.

$$R_1 = R_2 = \frac{W}{2}.$$

For values of x from 0 to $\frac{l}{2}$,

$$S = \frac{W}{2}, \quad M = \frac{W}{2}x.$$

When $x = 0$ to $\frac{l}{2}$,

$$S_0 = \frac{W}{2}.$$

When $x = \frac{l}{2}$,

$$M_0 = \frac{Wl}{4}.$$

Owing to the symmetry of the loading the magnitudes of S and M, for sections between the load and the right hand end, can evidently be obtained from the above equations by letting x equal the distance from the right hand support to the section.

Problem 4.

Simple beam with a total load W uniformly distributed over the entire length (Fig. 65).

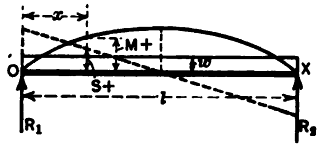


FIG. 65.

Let w = the intensity of the distributed load.

$$R_1 = R_2 = \frac{wl}{2} = \frac{W}{2}.$$

For values of x from 0 to l,

$$S = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right),$$

$$M = \frac{wl}{2}x - \frac{wx^2}{2} = \frac{w}{2}(lx - x^2).$$

When $x = 0$,

$$S_0 = \frac{wl}{2} = \frac{W}{2}.$$

When $x = \frac{l}{2}$,

$$M_0 = \frac{wl^2}{8} = \frac{Wl}{8}.$$

Problem 5.

Simple beam with a single concentrated load W not at the middle of the span (Fig. 66).

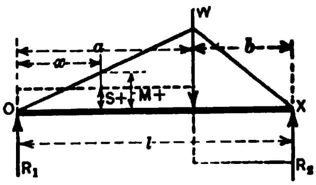


FIG. 66.

$$R_1 = \frac{Wb}{l}, \qquad R_2 = \frac{Wa}{l}.$$

For values of x from 0 to a,

$$S = \frac{Wb}{l}, \qquad M = \frac{Wb}{l}x.$$

When x = 0 to a,

$$S_0 = \frac{Wb}{l}.$$

When $x = a$,

$$M_0 = \frac{Wab}{l}.$$

The above value of S_0 will be the greatest shearing force in the beam when $b > a$. When $a > b$ the greatest shearing force will occur at sections between the load and the right hand support and will be equal to

$$S_0 = -\frac{Wa}{l}.$$

The value of the bending moment at any section between the load and the right hand support may evidently be obtained by taking x as the distance of the section from the support R_2 and substituting a for b in the above equation for M.

Problem 6.

Simple beam subjected to a total load W concentrated at two points at equal distances from the supports (Fig. 67).

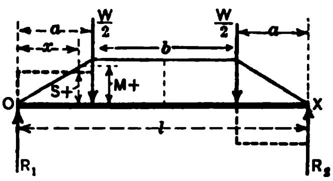


FIG. 67.

$$R_1 = \frac{W}{2}, \qquad R_2 = \frac{W}{2}.$$

For values of x from 0 to a,

$$S = \frac{W}{2}, \qquad M = \frac{Wx}{2}.$$

For values of x from a to $(a + b)$,

$$S = 0, \qquad M = \frac{Wa}{2}.$$

For values of x from $(a + b)$ to l,

$$S = -\frac{W}{2}, \qquad M = \frac{W}{2}(l - x).$$

When $x = 0$ to a,

$$S_0 = \frac{W}{2}.$$

Also, when $x = (a + b)$ to l,

$$S_0 = -\frac{W}{2}.$$

When $x = a$ to $(a + b)$,

$$M_0 = \frac{Wa}{2}.$$

The portion of the beam between the loads is evidently subjected to uniform bending.

Problem 7.

Simple beam subjected to a total load W, uniformly distributed over a part of its length, the load being symmetrically placed with respect to the middle point in the span (Fig. 68).

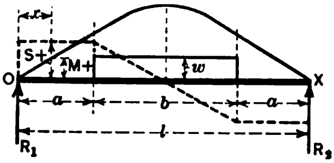


FIG. 68.

Let w = the intensity of the distributed load.

$$R_1 = \frac{wb}{2} = \frac{W}{2}, \qquad R_2 = \frac{wb}{2} = \frac{W}{2}.$$

For values of x from 0 to a,

$$S = \frac{wb}{2}, \qquad M = \frac{wb}{2}x.$$

For values of x from a to $(a + b)$,

$$S = \frac{wb}{2} - w(x - a) = w\left(a + \frac{b}{2} - x\right),$$

$$M = \frac{wb}{2}x - \frac{w}{2}(x - a)^2 = \frac{w}{2}[bx - (x - a)^2].$$

For values of x from $(a + b)$ to l,

$$S = -\frac{wb}{2}, \qquad M = \frac{wb}{2}(l - x).$$

When $x = 0$ to a,

$$S_0 = \frac{wb}{2} = \frac{W}{2}.$$

Also, when $x = (a + b)$ to l,

$$S_0 = -\frac{wb}{2} = -\frac{W}{2}.$$

When $x = \frac{l}{2}$,

$$M_0 = \frac{wbl}{4} - \frac{w}{2}\left(\frac{l}{2} - a\right)^2 = \frac{wb}{4}\left(l - \frac{b}{2}\right) = \frac{W}{4}\left(l - \frac{b}{2}\right).$$

Problem 8.

Beam overhanging the two supports equal amounts with equal concentrated loads $\frac{W}{2}$ at the ends (Fig. 69).

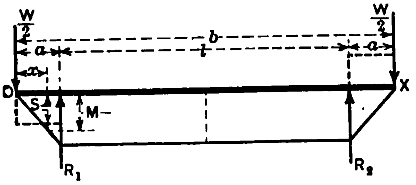


FIG. 69.

$$R_1 = \frac{W}{2}, \qquad R_2 = \frac{W}{2}.$$

For values of x from 0 to a,

$$S = -\frac{W}{2}, \qquad M = -\frac{Wx}{2}.$$

For values of x from a to $(a + l)$,

$$S = 0, \qquad M = -\frac{Wa}{2}.$$

For values of x from $(a + l)$ to $(l + 2a)$,

$$S = \frac{W}{2}, \qquad M = -\frac{W}{2}(l + 2a - x).$$

When $x = 0$ to a,

$$S_0 = -\frac{W}{2}.$$

Also, when $x = (l + a)$ to $(l + 2a)$,

$$S_0 = \frac{W}{2}.$$

When $x = a$ to $(a + l)$,

$$M_0 = -\frac{Wa}{2}.$$

The portion of the beam between the supports is evidently subjected to uniform bending.

Problem 9.

Beam overhanging the two supports equal amounts and subjected to a total load W, uniformly distributed over its entire length (Fig. 70).

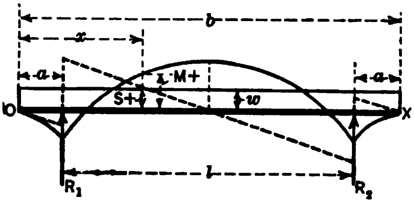


FIG. 70.

$$R_1 = \frac{wb}{2} = \frac{W}{2}, \qquad R_2 = \frac{wb}{2} = \frac{W}{2}.$$

For values of x from 0 to a,

$$S = -wx, \qquad M = -\frac{wx^2}{2}.$$

For values of x from a to $(l + a)$,

$$S = \frac{wb}{2} - wx = w\left(\frac{b}{2} - x\right), \qquad M = \frac{wb}{2}(x - a) - \frac{wx^2}{2} = \frac{w}{2}(bx - ba - x^2).$$

For values of x from $(l + a)$ to b,

$$S = w(b - x), \qquad M = -\frac{w}{2}(b - x)^2.$$

When $x = a$,

$$S_0 = -wa, \qquad S_0' = w\left(\frac{b}{2} - a\right).$$

Also, when $x = (l + a)$,

$$S_0 = wa, \qquad S_0' = -w\left(\frac{b}{2} - a\right).$$

The magnitude of the greatest shearing force will be given by the expression for S_0, when $a > \frac{b}{4}$, and by the expression for S_0', when $a < \frac{b}{4}$.

When $x = a$,

$$M_0 = -\frac{wa^2}{2}.$$

Also, when $x = (l + a)$,

$$M_0 = -\frac{wa^2}{2}.$$

When $x = \frac{b}{2}$,

$$M_0' = \frac{wb}{2}\left(\frac{b}{4} - a\right) = \frac{W}{2}\left(\frac{b}{4} - a\right)$$

The magnitude of the greatest bending moment will be given by the expression for M_0', when $a < \frac{b}{2}(\sqrt{2}-1)$, and by the expression for M_0, when $a > \frac{b}{2}(\sqrt{2} - 1)$.

Note. — When constructing the diagrams it is of importance that the following relations of shearing forces and bending moments, which have been brought out during the discussion of the theory, be kept in mind.

When a beam is subjected to concentrated forces only, the shearing force has a constant value and the bending moment a uniformly varying value at sections between any two adjacent forces. In such a case the shearing force diagram is a series of straight lines, parallel to the central axis of the beam, and the bending moment diagram is a series of straight lines, intersecting on the lines of action of the external forces. It is therefore necessary to determine the values of the shearing forces and bending moments at the sections at the loads and supports only in order to construct the diagrams.

When a beam is subjected to uniformly distributed loads only, the shearing force has a uniformly varying value for any portion of the beam over which the load is distributed and the bending moment diagram for that portion is a part of a parabola, the axis of which coincides with the ordinate through the section at which the shearing force line (extended if necessary) crosses the axis. The shearing force and bending moment diagrams for portions of the beam over which there is no load are made up of straight lines, similar to the diagrams for concentrated loads.

In such a case, after having determined the straight lines representing the shearing force and bending moment diagrams for the portions of the beam over which there is no load and locating the vertex of the parabola, the remainder of the bending moment diagram can be constructed by the use of a graphical method for plotting a parabola.

In any case, since

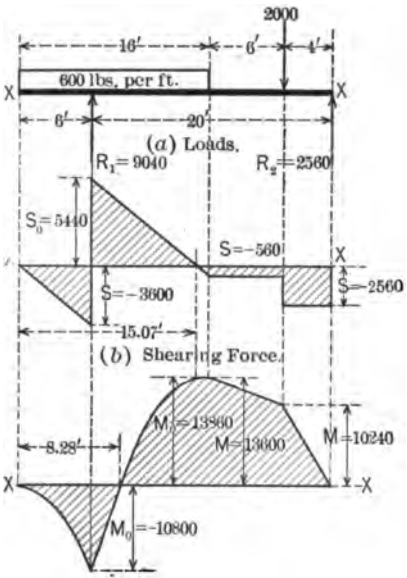
$$S = \frac{dM}{dx},$$

the ordinates of the shearing force diagram may be taken to represent the tangents of the angles of slope at the corresponding points in the bending moment diagram and hence the form of the shearing force diagram will always show whether the lines in the bending moment diagram are tangent, or intersect at an angle, at the points at which they meet.

For example, the form of the shearing force diagram (Problem 7) indicates that the parabola, forming the middle portion of the bending moment diagram for that case, is tangent to the straight lines forming the remainder of the diagram. In Problems (3), (5), (6), (8) and (9), the abrupt change in the value of the shearing force, between the sections on either side of the concentrated forces, indicates that the lines of the bending moment diagrams intersect at an angle at these points.

In Problems (2) and (9), since the shearing force is zero at the ends of the beams, the axes of the parabolas representing the bending moment diagrams for the overhanging portions will evidently coincide with the ordinates through the ends.

It should be borne in mind, however, that in the five last mentioned cases the diagrams are slightly approximate, since all the forces are really distributed over small portions of the beams (Art. 68) and that, if accurate bending moment diagrams could be constructed for each case, they would have no sharp corners.



(c) Bending Moment.

FIG. 71.

In the construction of diagrams for more complicated systems of loading the foregoing relations will be found to have additional value. In any such case the bending moment diagrams for each portion of the beam subjected to a uniformly distributed load will be a part of a parabola, and, when the portion does not contain the section of zero shear, the vertex of the parabola, of which the bending moment diagram is a part, can easily be located by extending the straight line, representing the shearing force diagram for that portion of the beam, until it intersects the central axis.

Problem 10.

Given a beam subjected to a uniformly distributed load of 600 lbs. per ft. and a concentrated load of 2000 lbs., supported as shown (Fig. 71a):

(a) Find the supporting forces.

(b) Write the general formulas for shearing forces and bending moments throughout the length of the beam.

(c) Plot the shearing force diagram and determine the location and magnitude of the greatest shearing force.

(d) Plot the bending moment diagram and determine the location and magnitude of the greatest bending moment.

Treat the supporting forces and all loads, represented by single vectors, as concentrated forces and neglect the weight of the beam.

Solution. — (a) By the application of the conditions of equilibrium for a system of parallel forces in a single plane we obtain $R_1 = 9040$, $R_2 = 2560$ (very nearly).

(b) Fig. (71a) will evidently represent the load diagram for the beam, and, if we let $x =$ the distance of any cross section from the left hand end of the beam and write the equations for shearing forces and bending moments, using the numerical values of the external forces we shall have four sets of equations, as follows:

For values of x from 0 to 6,

$$S = -600x, \qquad (1)$$

$$M = -300x^2. \qquad (2)$$

For values of x from 6 to 16,

$$S = 9040 - 600x, \qquad (3)$$

$$M = 9040(x-6) - 300x^2. \qquad (4)$$

For values of x from 16 to 22,

$$S = 9040 - 9600 = -560, \qquad (5)$$

$$M = 9040(x-6) - 9600(x-8) = 22{,}560 - 560x, \qquad (6)$$

or, if we determine the values of S and M from the forces acting on the portion of the beam between a cross section and the right hand end,

$$S = -2560 + 2000 = -560, \qquad (7)$$

$$M = 2560(26-x) - 2000(22-x) = 22{,}560 - 560x. \qquad (8)$$

For values of x from 22 to 26,

$$S = -2560, \qquad (9)$$

$$M = 2560(26-x). \qquad (10)$$

(c) The shearing force diagram will be made up of straight lines and the necessary points may be obtained by making the following substitutions.

When $x = 0$,

$$S = 0. \qquad (11)$$

When $x = 6$, equation (1) gives

$$S = -3600, \qquad (12)$$

which is the shearing force at a section to the left of the support R_1.

When $x = 6$, equation (3) gives

$$S = 5440, \qquad (13)$$

which is the shearing force at a section to the right of the support R_1.

When $x = 16$ to 22, the shearing force has the constant value

$$S = -560 \text{ (equation 5)}. \qquad (14)$$

When $x = 22$ to 26, the shearing force has the constant value

$$S = -2560 \text{ (equation 9)}. \qquad (15)$$

Plotting these values the shearing force diagram (Fig. 71b) can be constructed.

The greatest shearing force will evidently be located at the cross section just to the right of the support R_1 and will be equal in magnitude to

$$S_0 = 5440 \text{ lbs}. \qquad (16)$$

Attention may again be called to the abrupt change in the value of the shearing force, between the sections to the left and right of support R_1 and also between the sections to the left and right of the load of 2000 lbs. and that, if the actual distribution of these two forces were taken account of, the change in the shearing force from the one value to the other would be a gradual one in each case.

The shearing force is evidently zero at some point over support R_1 and also at a point at a distance

$$x = 15.07 \text{ ft.}$$

from the left end of the beam, the value of x being obtained by placing the expression for S (equation 3) equal to zero.

(d) It is evident, from the loads on the beam as well as the forms of the equations, that the diagrams of the bending moments represented by equations (2) and (4) will be parabolas, with the axes vertical, and that the diagrams of equations (6) and (10) will be straight lines.

The vertex of the parabola (equation 2) will evidently be located at the point

$$x = 0, \qquad M = 0. \quad \dots\dots\dots \quad (17)$$

When $x = 6$, equation (2) gives

$$M = -10{,}800, \quad \dots\dots\dots \quad (18)$$

and by computing values of M for one or two values of x between 0 and 6 the necessary ordinates for plotting the curve may be obtained.

When $x = 6$, equation (4) will also give

$$M = -10{,}800.$$

The vertex of the parabola will be on the ordinate through the section of zero shearing force at the distance $x = 15.07$ ft. from the left end of the beam.

When $x = 15.07$, equation (4) gives

$$M = 13{,}860. \quad \dots\dots\dots \quad (19)$$

When $x = 16$, equation (4) gives

$$M = 13{,}600. \quad \dots\dots\dots \quad (20)$$

Since the bending moment changes from negative to positive between the values $x = 6$ and $x = 16$, the point at which the diagram will cross the axis XX may be found by placing the expression for M (equation 4) equal to zero and solving for x, whence we obtain

$$x = 8.28 \text{ ft.}$$

By computing values of M for one or two additional values of x the necessary ordinates for plotting the curve are obtained.

When $x = 16$, equation (6) will also give the value

$$M = 13{,}600,$$

and when $x = 22$, equation (6) will give

$$M = 10{,}240. \quad \dots\dots\dots \quad (21)$$

When $x = 22$, equation (10) will also give

$$M = 10{,}240,$$

and when $x = 26$,

$$M = 0. \quad \dots\dots\dots \quad (22)$$

In plotting the diagram (Fig. 71c) it should be noted that the form of the shearing force diagram shows that at the point $x = 16$ the parabola and the straight line are tangent.

The greatest bending moment is evidently located at the cross section at a distance

$$x = 15.07 \text{ ft.}$$

from the left hand end of the beam and is equal in magnitude to

$$M_0 = 13{,}860 \text{ ft. lbs.} \qquad (23)$$

The point at the distance $x = 8.28$ ft. from the left end at which $M = 0$ is called a *point of inflexion*, the bending moments being negative at sections on one side of the point and positive at sections on the other side. It is evident that the fiber stress is zero over the entire cross section through this point.

Problems 11–24.

Note. — In Problems (11–24), inclusive, the equations for shearing forces, bending moments, the shearing force and bending moment diagrams and the values of the greatest shearing force and bending moment are to be determined in a manner similar to that indicated in the solution of Problem (10). In each case the weight of the beam is to be neglected unless otherwise noted.

Problem 11.

The beam (Fig. 72) is subjected to a system of concentrated loads.

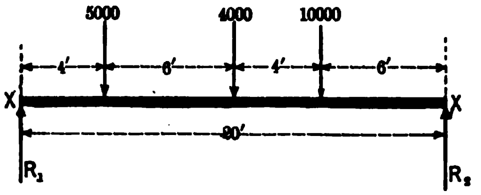


Fig. 72.

Problem 12.

The beam (Fig. 73) is subjected to a system of concentrated loads

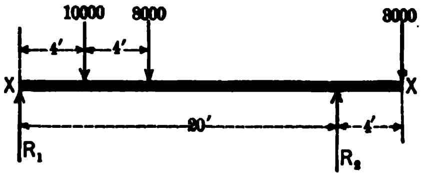


Fig. 73.

Problem 13.

The beam (Fig. 74) is subjected to a concentrated load of 10,000 lbs. and a uniformly distributed load of 800 lbs. per ft.

Problem 14.

Solve Problem (13), substituting a concentrated load of 1000 lbs. for the load of 10,000 lbs.

Problem 15.

The beam (Fig. 75) is subjected to two uniformly distributed loads.

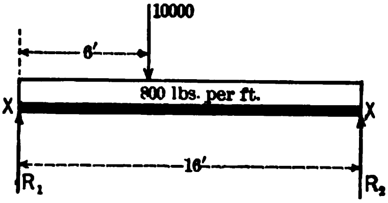


Fig. 74.

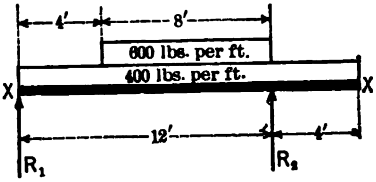


Fig. 75.

Problem 16.

The beam (Fig. 76) is subjected to a system of uniformly distributed and concentrated loads as indicated.

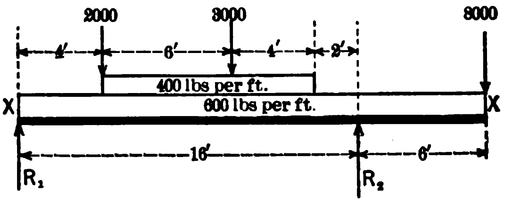


Fig. 76.

Problem 17.

The beam (Fig. 77) is subjected to a system of uniformly distributed and concentrated loads as indicated.

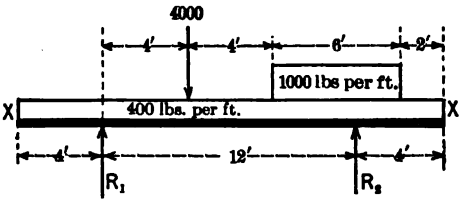


Fig. 77.

Problem 18.

The cantilever beam (Fig. 78) is subjected to a system of uniformly distributed and concentrated loads as shown.

Problem 19.

The beam (Fig. 79) is subjected to a uniformly distributed load. Assume that the reaction is also uniformly distributed over the entire length of the beam.

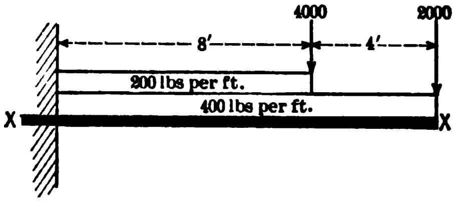


FIG. 78.

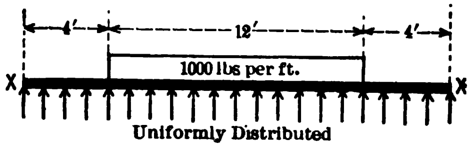


FIG. 79.

Problem 20.

The beam (Fig. 80) is subjected to a load over a part of its length, varying uniformly in intensity from 0 to 1000 lbs. per ft.

Solution. — The diagram (Fig. 80a) will evidently represent the load diagram and the resultant of the load will be equal to 5000 lbs. acting at a distance of $\frac{35}{3}$ ft. from the left end of the beam.

(a) The supporting forces will be

$$R_1 = 1111 \text{ lbs.},$$
$$R_2 = 3889 \text{ lbs.}$$

(b) Let x = the distance of any section from the left end. Then for values of x from 0 to 5,

$$S = 1111, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$M = 1111\,x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For values of x from 5 to 15,

$$S = 1111 - 100\frac{(x-5)^2}{2} = 1111 - 50\,(x-5)^2, \quad . \quad . \quad . \quad . \quad (3)$$
$$M = 1111\,x - 100\frac{(x-5)^3}{6} = 1111\,x - \frac{50}{3}\,(x-5)^3. \quad . \quad . \quad . \quad (4)$$

(c) For values of x from 0 to 5, the shearing force diagram will evidently be a straight line, the shearing force having the constant value given by equation (1),

$$S = 1111. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

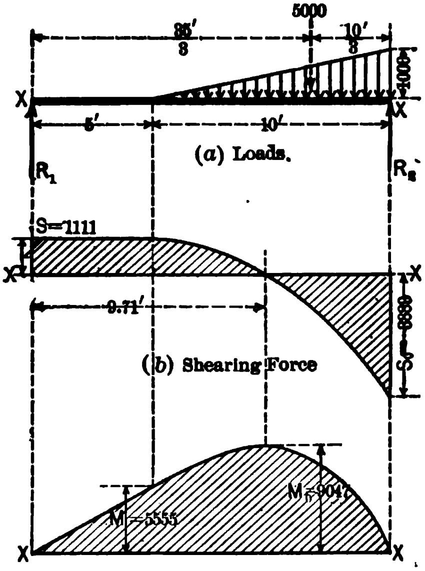
When $x = 5$, equation (3) will also give

$$S = 1111.$$

When $x = 15$, equation (3) gives

$$S = -3889. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Since the shearing force changes from positive to negative between the values $x = 5$ and $x = 15$, the distance of the cross section of zero shear from the left end of the beam can be determined by placing the value of S given by equation (3) equal to zero and solving for x, whence we obtain $x = 9.71$ ft.



(c) Bending Moment.

FIG. 80.

The shearing force diagram for this portion of the beam is evidently a parabola and the ordinates for two or three additional points should be obtained in order to determine the curve (Fig. 80b).

The greatest shearing force will evidently be located at the section to the left of the right hand support and will be equal to

$$S_0 = 3889 \text{ lbs.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

(d) For values of x from 0 to 5, the bending moment diagram is evidently a straight line and when

$$x = 0, \quad M = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and when

$$x = 5, \quad M = 5555. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For values of x from 5 to 15, the bending moment diagram is a curve and the maximum value will occur at the section of zero shear, where $x = 9.71$.

When $x = 5$, equation (4) will give the same value as equation (2),

$$M = 5555.$$

When $x = 9.71$,

$$M = 9047. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When

$$x = 15, \quad M = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

By computing the values of M at two or three additional points the necessary ordinates for plotting the diagram (Fig. 80c) may be obtained.

The greatest bending moment will evidently be located at the section 9.71 ft. distant from the left end and will be equal in magnitude to

$$M_0 = 9047 \text{ ft. lbs.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

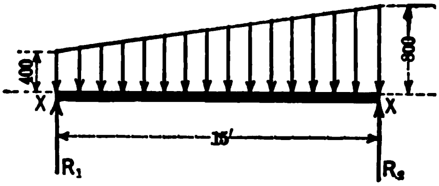


FIG. 81.

Problem 21.

The beam (Fig. 81) is subjected to a distributed load, the intensity of which varies uniformly from 400 lbs. per ft. at the left end to 800 lbs. per ft. at the right end.

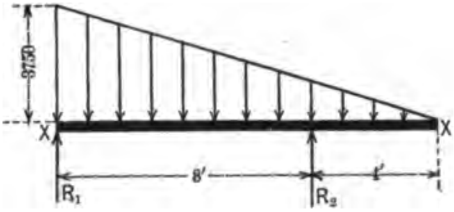


FIG. 82.

Problem 22.

The beam (Fig. 82) is subjected to a distributed load, the intensity of which varies uniformly from 3750 lbs. per ft. at the left end to 0 at the right end.

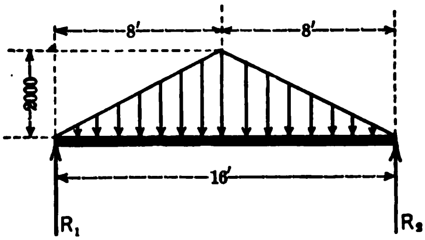


FIG. 83.

Problem 23.

The beam (Fig. 83) is subjected to a distributed load, the intensity of which varies uniformly from 0 at the supports to 2000 lbs. per ft. at the middle of the span.

Problem 24.

The beam (Fig. 83) is subjected to a distributed load, the intensity of which varies uniformly from 3000 lbs. per ft. at the supports to 0 at the middle of the span.

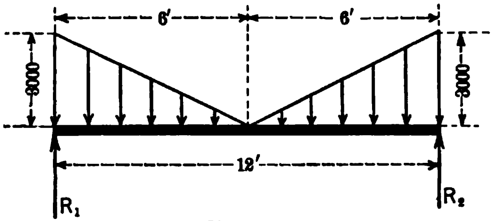


FIG. 84.

75. Combining Shearing Force and Bending Moment Diagrams.

— When a beam is subjected to a system of two or more loads the shearing force and bending moment diagrams may be determined by constructing the diagrams representing the shearing forces and bending moments due to each load acting alone and then combining by adding together the ordinates. In certain cases this method may be preferable to obtaining the diagrams for the entire load system by the method indicated in Art. (72).

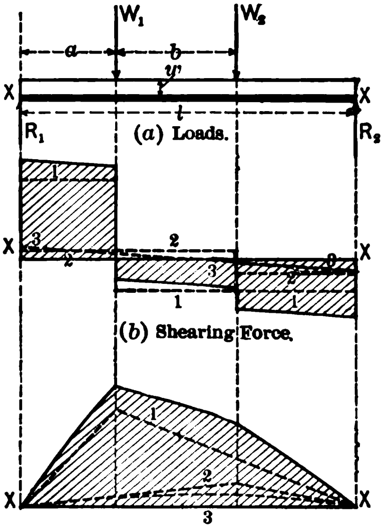
Two illustrations of its application in comparatively simple cases follow.

(a) *A simple beam subjected to a uniformly distributed load of intensity w and two concentrated loads W_1 and W_2* (Fig. 85a). To avoid confusion the diagrams are constructed separately, the shearing force diagram being shown in Fig. 85b and the bending moment diagram in Fig. 85c.

The shearing force diagram for the load W_1 will be of the same general form as that for the beam in Problem (5) (Art. 74) and the ordinate at any point may be computed by the formulas given for that case. The diagram is shown by the dotted line marked 1 (Fig. 85b). The diagram for the load W_2 is of the same type as that for the load W_1 and is shown by the dotted line marked 2 (Fig. 85b). The diagram for the uniformly distributed load is of a type similar to that for the beam in Problem (4) (Art. 74) and the ordinates may be computed by the formulas given for that case. The diagram is shown by the dotted line marked 3 (Fig. 85b).

By adding the ordinates of the diagrams 1, 2 and 3 algebraically, the shearing force diagram for the entire system of loads, shown by the full line (Fig. 85b), is obtained.

Proceeding in the same manner, the bending moment diagrams for the loads W_1 and W_2, of the same general type as the diagram for the beam in Problem (5) (Art. 74) and shown by the dotted lines 1 and 2 (Fig. 85c), are obtained; and the bending moment diagram for the uniformly distributed load, shown by the dotted line marked 3 (Fig. 85c), will be of the same form as that for the beam in Problem (4) (Art. 74). The values of the ordinates to these diagrams may be computed by the formulas given for the cases mentioned.

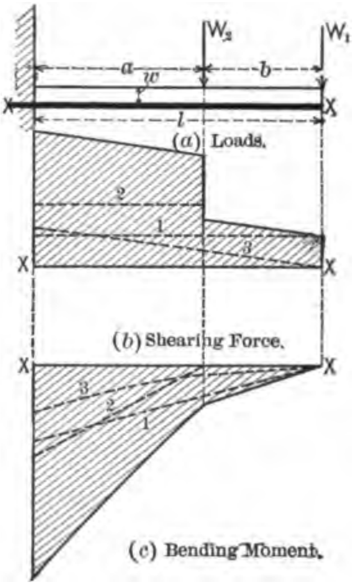


(a) Loads.

(b) Shearing Force.

(c) Bending Moment.

FIG. 85.



(a) Loads.

(b) Shearing Force.

(c) Bending Moment.

FIG. 86.

By adding the ordinates of the diagrams 1, 2 and 3, algebraically, the bending moment diagram for the entire system of loads, shown by the full line (Fig. 85c), is obtained.

(b) *A cantilever beam subjected to a uniformly distributed load of intensity w and two concentrated loads* W_1 *and* W_2 (Fig. 86a).

The diagrams for this case are constructed in the same manner as those for Case (a).

In the shearing force diagram (Fig. 86b) the dotted lines 1 and 2, representing the shearing force diagrams for loads W_1 and W_2 acting separately, are of a form similar to that for the beam in Problem (1) (Art. 74). The diagram for the uniformly distributed load is represented by the dotted line 3, of a form similar to that for the beam in Problem (2) (Art. 74). By adding the ordinates, algebraically, the shearing force diagram for the entire load system, represented by the full line, is obtained.

In the bending moment diagram (Fig. 86c) the dotted line 1 represents the bending moment diagram for load W_1, the dotted line 2 that for load W_2 and the dotted line 3 that for the uniformly distributed load. These are similar in form to the bending moment diagrams for the beams in Problems (1) and (2) (Art. 74) and by adding the ordinates, algebraically, the diagram for the entire load system, represented by the full line, is obtained.

76. Problems. — Combining Shearing Force and Bending Moment Diagrams. — The following problems will serve to illustrate the methods explained in Art. (75).

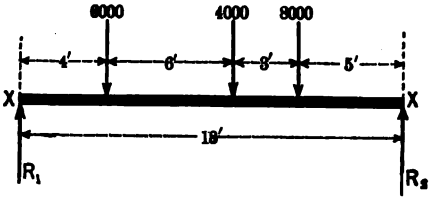


FIG. 87.

Problem 1.

Construct the shearing force and bending moment diagrams for the beam subjected to concentrated loads as shown (Fig. 87) by constructing the diagrams for each load acting separately and adding the ordinates. Determine the values of the greatest shearing force and the greatest bending moment.

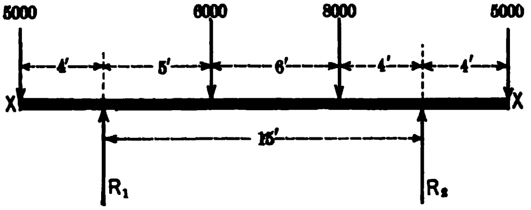


FIG. 88.

Problem 2.

Construct the shearing force and bending moment diagrams for the beam subjected to the concentrated loads shown (Fig. 88) by constructing separate diagrams for each load between the supports and for the loads at the ends acting together and adding the ordinates. Determine the values of the greatest shearing force and the greatest bending moment.

Problem 3.

Construct the shearing force and bending moment diagrams for the cantilever beam (Fig. 89) by constructing the diagram for each load acting separately and adding the ordinates. Determine the greatest value of the shearing force and the bending moment.

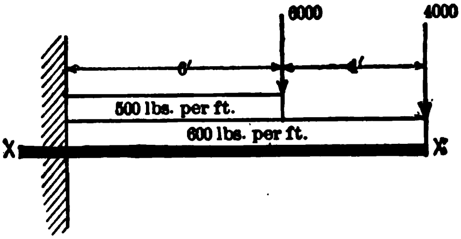


FIG. 89.

Problem 4.

Solve Problem (1), assuming that the beam is subjected to a uniformly distributed load of 500 lbs. per ft. in addition to the concentrated loads shown (Fig. 87).

Problem 5.

Solve Problem (2), assuming that the beam is subjected to a uniformly distributed load of 600 lbs. per ft. in addition to the concentrated loads shown (Fig. 88).

77. Greatest Outside Fiber Stress. — Section Modulus. — The relation between shearing forces and bending moments, discussed in Arts. (71–72), is of great importance in the determination of the maximum stress intensity in a beam subjected to ordinary bending. We have seen that the greatest intensity of the normal stress at a given cross section is at the point, or points, farthest from the neutral axis (Art. 69) and is equal in magnitude to

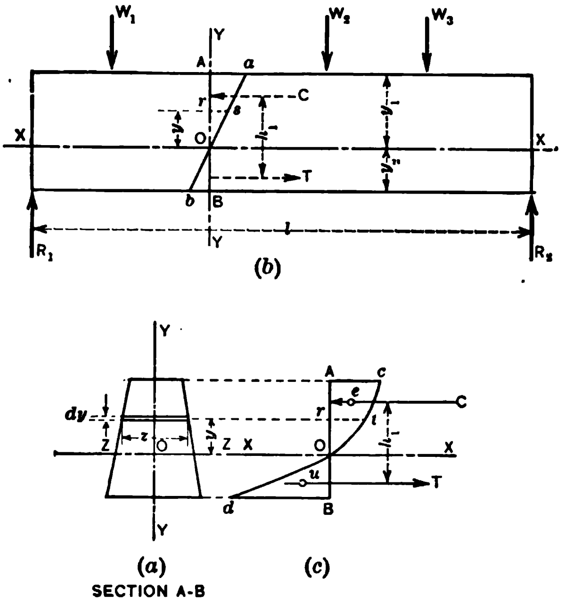
$$f = \frac{Mc}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For a beam of uniform cross section the value of $\frac{c}{I}$ is evidently the same for all cross sections. Therefore, the value of f will vary as the value of M and the outside fiber stress at the section at which the bending moment is a maximum will be greater than that at any other cross section. Its value is known as the *greatest intensity of normal stress, or the greatest outside fiber stress*, in the beam.

Since $f \propto M$, the bending moment diagrams may be taken to represent the value of the outside fiber stress at the different cross sections of a beam of uniform section, by simply measuring the ordinates to the suitable scale.

Section Modulus. — The reciprocal of the quantity $\frac{c}{I}$ is known as the *section modulus* of the beam. When the value of I is expressed in (ins.)^4 and the value of c in (ins.) the *units* in which the value of the section modulus $\frac{I}{c}$ is expressed will evidently be (ins.)^3.

78. Graphical Representation of the Normal Stress. — In the discussion of the theories both of simple and of ordinary bending it has been shown that, if certain limitations are imposed (Art. 63) and certain assumptions hold true (Art. 66), the normal stress on a cross section of a beam is a uniformly varying stress, with the neutral axis passing through the center of gravity of the section, at right angles to its vertical axis of symmetry, and that the resultant of the normal stress is a couple in the plane of loading perpendicular to the neutral axis of the section (Art. 60).



(a) SECTION A-B (c)

FIG. 90.

Hence, if z = the length and dy = the width of an elementary strip at a distance y from the neutral axis of a cross section AB of a beam subjected to ordinary bending (Fig. 90), the total normal stress on the strip will be equal to

$$fz\,dy = \frac{My}{I}z\,dy.$$

The stress on every strip above the neutral axis will be compression and the magnitude of the resultant of the compressive stress on the cross section will evidently be equal to

$$C = \int_{y=0}^{y=y_1} fz\,dy = \frac{M}{I}\int_{y=0}^{y=y_1} yz\,dy, \quad . \quad . \quad . \quad . \quad (1)$$

where y_1 = the distance from the neutral layer to the outside fibers under compression.

Similarly the magnitude of the resultant of the tensile stress on the cross section will be equal to

$$T = \int_{y=0}^{y=y_2} fz\,dy = \frac{M}{I}\int_{y=0}^{y=y_2} zy\,dy, \quad . \quad . \quad . \quad . \quad (2)$$

where y_2 = the distance from the neutral layer to the outside fibers under tension.

Since the resultant of the entire normal stress is a couple,

$$T = C \text{ (Art. 66)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

in magnitude and, if h_1 = the distance between the center of the compressive stress and the center of the tensile stress on the section, the magnitude of the moment of resistance will be equal to

$$M = Th_1 = Ch_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The distribution of the normal stress over the section may be represented graphically in the following manner. The intensity of the normal stress at any point r in the cross section, at a distance y from the neutral axis through O, may evidently be represented by the ordinate $f = rs$ (Fig. 90b), from the trace AB of the section on the plane of loading to a straight line ab, intersecting AB at O and sloping in such a manner that

$$Aa = \frac{Mc}{I} \text{ (Art. 69)}.$$

Compressive stress intensities will then be represented by ordinates to the right of AB and tensile stress intensities by ordinates to the left of AB. The product fz of the stress intensity and the width of the cross section at any point *r* will be a quantity which we may designate as the *stress per unit of depth of the section* at *r*. A diagram, constructed (Fig. 90c) with each ordinate *rt* equal to the value of *fz* at the point *r*, will represent the distribution of the normal stress above and below the neutral axis.

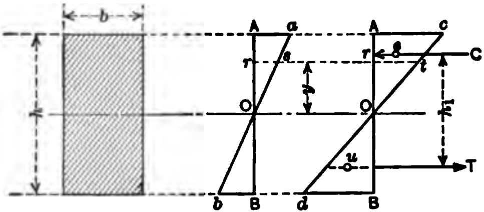


FIG. 91.

The area *AOc* will evidently represent the magnitude of the resultant compressive stress

$$C = \int_{y=0}^{y=y_1} fz\,dy$$

and the distance of the center of the compressive stress from the neutral axis will be equal to the perpendicular distance between the axis *XX* and *e*, the center of gravity of *AOc*.

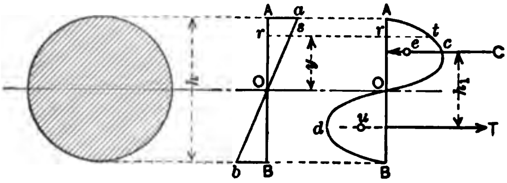


FIG. 92.

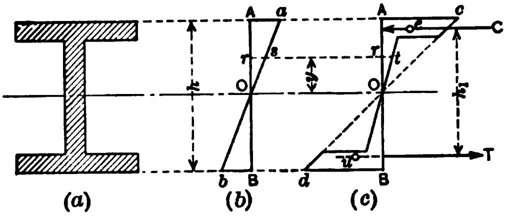
Similarly the magnitude of the resultant tensile stress

$$T = \int_{y=0}^{y=y_2} fz\,dy$$

will be represented by the area *BOd* and the distance of *u*, the center of gravity of *BOd*, from *XX* will be equal to the distance of the center of the tensile stress from the neutral axis.

Hence h_1, the vertical distance between the centers of gravity *e* and *u*, will be equal to the arm of the couple formed by the resultants of the tensile and compressive stresses.

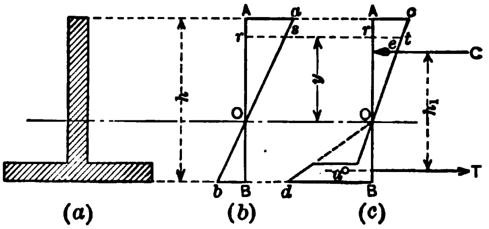
It is evident that when the intensity f at any point in a given cross section is known the moment of resistance M of the section can be determined by an analysis similar to the above without using the moment of inertia of the section. Moreover, since the moment of resistance is represented by the sum of the moments of the areas AOc and BOd (Fig. 90c), about the axis XX, or any parallel axis, in determining its magnitude it is unnecessary to determine the values of T or C.



(a) (b) (c)

FIG. 93.

Similar diagrams representing the distribution of stress when the section AB is a rectangle, a circle, an I section and a ⊥ section are shown in Figs. (91–94). In each case the depth of the section is equal to h and the diagram marked (b) represents the variation in stress intensity over the section and the diagram marked (c) the distribution of the total stress.



(a) (b) (c)

FIG. 94.

In the case of the rectangle (Fig. 91) z is a constant and hence the areas AOc and BOd will become equal triangles. Therefore, the arm of the resultant couple will be equal to

$$h_1 = \tfrac{2}{3} h.$$

In each of the other cases the area AOc will equal the area BOd

and the value of h_1 will vary according to the shape of the section, being smallest for the circle and greatest for the I section.

The preceding discussion will evidently apply equally as well in the case of simple bending as in that of ordinary bending.

79. Problems. — Moment of Resistance. — In Problems (1–5), which follow, find the moment of resistance of the sections by plotting the normal stress after the method given in Art. (78) and check the results obtained by using the formula for the moment of resistance

$$M = \frac{fI}{c}.$$

Solve the remaining problems by the latter method only.

Problem 1.

Given a wooden beam with a cross section 4″ × 12″, find the moment of resistance, provided the greatest allowable fiber stress = 1000 lbs. per sq. in.

Problem 2.

Given a round bar 4″ diameter, find the moment of resistance, provided the greatest allowable fiber stress = 16,000 lbs. per sq. in.

Problem 3.

Given an I-beam with a cross section similar to Fig. (93), depth = 8″, thickness of web = ½″, thickness of flanges = 1″, width of flanges = 4″, find the moment of resistance if the greatest fiber stress = 16,000 lbs. per sq. in.

Problem 4.

Given a T-beam with a cross section similar to Fig. (94), width of flange = 4″, depth 5″, thickness of web = ½″, thickness of flange = ½″, find the moment of resistance, provided the greatest allowable fiber stress = 16,000 lbs. per sq. in.

Problem 5.

Given a beam with a square cross section 4″ × 4″, find the moment of resistance, when the plane of loading coincides with a diagonal of the square, provided the greatest allowable fiber stress = 16,000 lbs. per sq. in.

Problem 6.

In a built-up plate girder the web plate is 37″ × ½″. Find the moment of resistance of the cross section of the web plate provided the greatest fiber stress is 12,000 lbs. per sq. in.

Problem 7.

In Problem (6), assume that the section is through a vertical row of ⅞″ rivets spaced 3″ on centers, with the outside rivets 2″ from the edges of the plate. Find the moment of resistance of the net section, provided the greatest fiber stress is 12,000 lbs. per sq. in. Assume the rivet holes to be ⅛″ larger in diameter than the rivets.

Problem 8.

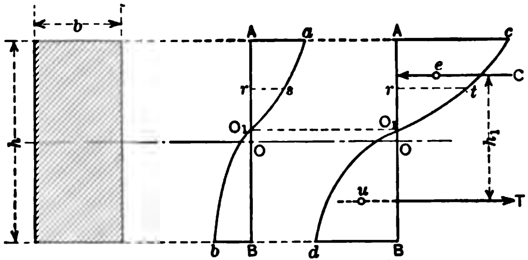
If each of the flanges of the plate girder given in Problem (6) are made up of two 6″ × 4″ × $\frac{3}{4}$″ angles find the moment of resistance of the entire section assuming the greatest fiber stress is 12,000 lbs. per sq. in. Total width of flanges = 12½″.

Problem 9.

Solve Problem (8) making the allowance for rivet holes called for in Problem (7).

80. Bending which Produces Stress Intensities above the Elastic Limit. — If the forces acting on a beam are large enough, fiber stresses in excess of the elastic limit of the material will be produced at some of the cross sections. Where the stress intensity on a cross section exceeds the elastic limit the second assumption of the theory evidently will not hold true and the stress will not be uniformly varying over the entire section. Hence, in such a case the formulas (Art. 69) will not give correct values for the normal stress intensity and the distribution of the stress over a cross section will be different from that illustrated in Art. (78).

If we hold the first and third assumptions (Art. 66) to be correct when the fiber stresses exceed the elastic limit, the general form of the stress intensity diagram for cross sections which are symmetrical with respect to the horizontal axis through the center of gravity may be determined from the stress-strain diagrams for the material in tension and compression.



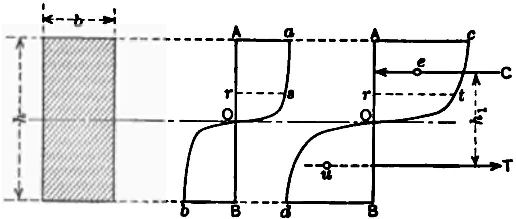
(a) (b) (c)

FIG. 95.

For example, in Fig. (7) (Art. 8) are given the stress-strain diagrams for cast iron in tension and in compression. These diagrams may be combined to represent the variation in stress intensity over a rectangular cross section AB (Fig. 95) by plotting compressive stress intensities to the right of AB and tensile stress intensities to the left, as in the diagrams (Art. 78).

Since cast iron does not follow Hooke's law and the ratio of stress intensity to strain for a tensile stress is less than the ratio for a compressive stress of equal intensity (Art. 11) the line *ab* will be a curve and the ordinate *Aa* will not be equal to *Bb*, as would be the case if the variation in stress intensity were the same in tension and compression. Owing to this fact, the neutral axis of the cross section will not pass through its center of gravity *O*, but will be raised above it to some point O_1. Since the section is rectangular the location of O_1 must be such that the area AO_1a will be equal to the area BO_1b. If the diagram representing the total stress on the cross section (Fig. 95c) is plotted in the same manner as in Art. (78), the area AO_1c will represent the total compressive stress and the area BO_1d the total tensile stress.

Diagrams representing the distribution of the normal stress on a cross section of a bar of a medium grade of steel of rectangular cross section may be constructed in a similar manner. Referring again to Fig. (7) we note that for this grade of steel the elastic limit and the yield point in compression are practically the same as in tension and that the ratios of stresses to strains below the elastic limits are also the same for tension and compression. Hence, if a stress intensity diagram for a cross section, symmetrical with respect to a horizontal axis through its center of gravity, is constructed with compressive stress intensities represented by ordinates to the right of *AB* and tensile stress intensities by ordinates to the left (Fig. 96), the two parts of the diagram *AOa* and *BOb* will be equal and the neutral axis will pass through the center of gravity *O* of the section. The diagram (Fig. 96c) representing the total stress on the section may be constructed in the same manner as before.



(a) (b) (c)

FIG. 96.

It should be noted that in both of the cases represented (Figs.

95 and 96) the resultant of the compressive stress C will act through the center of gravity of the area AOc and the resultant of the tensile stress T through the center of gravity of the area BOd, that $C = T$ and, if h_1 = the vertical distance between the centers of gravity of the areas AOc and BOd, the moment of resistance at the section will be equal to

$$M = Th_1 = Ch_1.$$

It is evident that, for a given value of M, the distance h_1 will be less than if the stress on the cross section were uniformly varying and that the value of the outside fiber stress, if computed by the formula

$$f = \frac{Mc}{I},$$

would be greater than the actual outside fiber stress in the beam.

Similar diagrams might be constructed for cross sections of different shapes, as in Art. (78). Similar results would be obtained in each case, but it would be found that for any given outside fiber stress, where the material does not follow Hooke's law, the difference between the actual intensity of stress and the value given by the formula

$$f = \frac{Mc}{I}$$

would vary with the shape of the cross section.

Since under a working load a material like steel is never stressed beyond the elastic limit and for practical purposes a material like cast iron may be assumed to follow Hooke's law for stresses below its working strength, the foregoing discussion is of value merely in showing the limitations of the theory of bending.

81. Modulus of Rupture. — When a beam is loaded to the breaking point, the value obtained by the application of the formula

$$f = \frac{Mc}{I} \quad \text{. . . . .} \quad (1)$$

at the section of greatest bending moment is called the transverse *modulus of rupture.*

It is evident from the preceding discussion (Art. 80) that the modulus of rupture is not the actual maximum outside fiber stress, which always is less than the value given by the formula.

Since no general formula can be obtained, however, which will give the greatest intensity of fiber stress at breaking for beams of different materials and cross sections, it is customary to use the modulus of rupture as if it were a breaking strength and to obtain the value of the *working strength*, or *working outside fiber stress*, in a given beam by dividing its modulus of rupture by a proper factor of safety.

When used in this way formula (1) must evidently be regarded as an empirical formula which affords a means of comparison of the strengths of beams of different materials and shapes.

Emphasis should be laid on the fact, brought out in the discussion (Art. 80), that when a material does not follow Hooke's law the difference between the value given by formula (1) and the actual fiber stress, for any given stress intensity, varies with the shape of the cross section. Hence the *modulus of rupture will not be constant for all beams of the same material but different values will be obtained for different shapes of cross section.*

The analysis given (Art. 80) shows that when Hooke's law fails the fibers near the neutral layer carry stress intensities which are greater in proportion to the intensities at the outside fibers than if the stress were uniformly varying. Hence beams in which a considerable portion of the material is located near the neutral layer, like round or square bars, show a higher modulus of rupture than beams, like the I beam, in which the larger portion of the material is located near the outside layers.

A large amount of experimental data is obtainable giving values of the modulus of rupture of beams of all the common materials and cross sections. An investigation of such data will show that definite values for the breaking load and hence of the modulus of rupture can be obtained for beams which are composed of more or less brittle materials. Such beams ultimately fail by the fracture of the fibers which are subjected to tension.

When a bar is composed of a more ductile material, such as a medium grade of steel, it merely buckles after the fibers are stressed beyond the yield point and, if the ductility is great enough, the buckling will continue as the load increases until the bar is no longer a beam in the ordinary sense. In such a case no definite value can be obtained for the modulus of rupture and it is more satisfactory to determine the working strength, by making it a fractional part of the value of the greatest outside fiber stress at the load at the yield point, than to attempt to assign a definite value for the modulus of rupture.

82. The Design of Beams for Fiber Stress. — Thus far we have discussed the methods of determining the fiber stresses in beams of various cross sections, when the loads and the supporting forces are known.

The more common problem in practice is the determination of the size and type of a beam to use when the loads which it must carry and the supporting forces are known. When ordinary beams, that is, beams with simple (not built-up) sections, are used, the solution of the problem is comparatively simple. In such cases, after the loads and supporting forces have been determined, the section of zero shear and the greatest bending moment can be found.

The formula for the greatest outside fiber stress (Art. 77) may be written

$$\frac{I}{c} = \frac{M}{f}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

whence it follows that the section modulus of the beam required to carry the load is equal to the quotient obtained by dividing the greatest bending moment by the value of the working strength (the maximum allowable fiber stress) for the material in the beam.

If the cross section of the beam is to be a rectangle, of breadth b and depth h, the value of the section modulus will be equal to

$$\frac{I}{c} = \frac{bh^2}{6}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Hence the value of the product

$$bh^2 = \frac{6\,M}{f} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

can be determined and suitable values for b and h, required to satisfy the equation, can be found.

The values of the section modulus $\frac{I}{c}$ for all of the standard shapes of steel beams may be found in handbooks published by the manufacturers. Hence, if a steel beam is to be used, it is only necessary to choose from the list of suitable sections one whose section modulus is at least as great as the calculated value.

The greatest bending moment which a beam of any given cross section will safely carry will evidently be equal to the product of the working strength of the material and the section modulus

$$M_0 = \frac{fI}{c}. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Having obtained the value of M_0, the working load for any distribution of the load and manner of supporting the beam can be found.

83. Problems. — Fiber Stresses in Beams of Uniform Section. — Problems involving the determination of fiber stresses in beams may be divided into three general classes.

First: Where all the loads, the supporting forces, the dimensions of the beam, including the span and cross section, are given and the fiber stresses in the beam are to be determined. To obtain the magnitude of the greatest fiber stress, the section of greatest bending moment must be found.

Second: Where all the loads, the supporting forces and the span are given and the type and dimensions of the cross section are to be determined. To solve such a problem the material of the beam and the value of the working fiber stress for the type of cross section to be used must first be decided upon. Then, by finding the greatest bending moment and dividing by the value of the greatest allowable fiber stress, the required value of the section modulus can be obtained (Art. 82).

Third: Where the material and the dimensions of the beam, including the span and cross section, and the manner of loading are given and the greatest allowable load is to be determined.

To solve such a problem the value of the working fiber stress, for the material and the type of cross section, must first be decided upon. The magnitude of the greatest bending moment will then be equal to the product of the working fiber stress and the section modulus, and, having obtained its value, the greatest allowable load can be found.

In the solution of any problem involving the determination of the stresses in a beam, a rough sketch, at least, of the diagrams showing the variation in shearing forces and bending moments will usually aid in arriving at a correct solution.

It is generally most convenient, when computing the values of the bending moments from the loads on the beam, to use ft. lbs. for units; but in using the formulas involving fiber stress it must be remembered that the stress intensities are practically always expressed in lbs. per sq. in. and values of the section modulus are expressed in $(\text{ins.})^3$ and hence values of the bending moment to use in these formulas must be reduced to in. lbs.

Problem 1.

Find the greatest fiber stress in a wooden beam, of cross section 8″ × 10″, supported at the ends and subjected to a single concentrated load of 6000 lbs., at a distance of 6 ft. from the left hand support. Span = 16 ft.

Problem 2.

A floor, supported by wooden beams, 15 ft. span, spaced 16 ins. on centers, is designed to carry a total uniformly distributed load of 100 lbs. per sq. ft. of floor area. The working stress of material is 1000 lbs. per sq. in. Which of the following sections would be suitable to use: 2″ × 10″; 3″ × 10″; 3″ × 12″; 4″ × 12″?

Solution. — The total uniformly distributed load, supported by one beam, will equal

$$W = \frac{15 \times 16 \times 100}{12} = 2000 \text{ lbs.}$$

The greatest bending moment will be equal to

$$M_0 = \frac{Wl}{8} = \frac{2000 \times 15}{8} = 3750 \text{ ft. lbs.} = 45{,}000 \text{ in. lbs.}$$

For a rectangular section we have

$$bh^2 = \frac{6M_0}{f} = \frac{6 \times 45{,}000}{1000} = 270 \ (\text{ins.})^3,$$

and for the given sections we have the following values for bh^2:

$$\begin{aligned} 2'' \times 10'' \quad & bh^2 = 200\ (\text{ins.})^3, \\ 3'' \times 10'' \quad & bh^2 = 300\ (\text{ins.})^3, \\ 3'' \times 12'' \quad & bh^2 = 432\ (\text{ins.})^3, \\ 4'' \times 12'' \quad & bh^2 = 576\ (\text{ins.})^3. \end{aligned}$$

Hence the section 3″ × 10″ is the smallest one which can be used.

Problem 3.

A floor is supported on wooden beams, 4″ × 12″ cross section and 16 ft. span, spaced 3 ft. on centers. Find the safe uniformly distributed live load per sq. ft. of floor area, provided the greatest allowable fiber stress is 1200 lbs. per sq. in. Assume the dead load, including the weight of the beams, to be equal to 25 lbs. per sq. ft. of floor area.

Solution. — The section modulus of the beams

$$\frac{I}{c} = \frac{bh^2}{6} = \frac{4 \times 12 \times 12}{6} = 96\ (\text{ins.})^3.$$

The greatest allowable bending moment,

$$M_0 = f\frac{I}{c} = 1200 \times 96 = 115{,}200 \text{ in. lbs.} = 9600 \text{ ft. lbs.}$$

Transposing the formula for the greatest bending moment in terms of a uniformly distributed load (Problem 4, Art. 74), we obtain for the value of the total load on a single beam,

$$W = \frac{8\,M_0}{l} = \frac{8 \times 9600}{16} = 4800 \text{ lbs.}$$

Since each beam supports a floor area of

$$3 \times 16 = 48 \text{ sq. ft.,}$$

the total load per sq. ft. of floor area will be equal to

$$\frac{4800}{48} = 100 \text{ lbs.}$$

and the net, or live load, will be equal to

$$100 - 25 = 75 \text{ lbs. per sq. ft.}$$

Problem 4.

A standard 12" I beam, weighing 40 lbs. per ft., is subjected to a uniformly distributed load of 2000 lbs. per ft. in addition to its own weight. Find the greatest fiber stress. Span = 20 ft. $I = 269 \text{ (ins.)}^4$

Problem 5.

Find the section modulus of the timber beam which would be required to support the loads given in Problem (14) (Art. 74). Assuming the working strength of the material $f = 1200$ lbs. per sq. in., select a suitable cross section from the following list of sizes: 4" × 12"; 6" × 12"; 8" × 12"; 6" × 16"; 8" × 16"; 10" × 16".

Problem 6.

Find the section modulus of the steel I beam which would be required to support the loads given in Problem (16) (Art. 74). Assume the working strength of the material $f = 14{,}000$ lbs. per sq. in.

From a manufacturers' handbook select a suitable section.

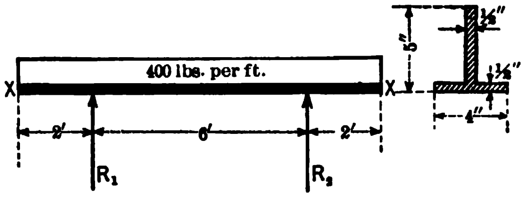


FIG. 97.

Problem 7.

A steel T beam is subjected to a uniformly distributed load of 400 lbs. per ft. as shown (Fig. 97). The total depth of the section is 5", the width of the flange is 4" and the thickness of the metal is ½" in both the stem and the flange. Distance of center of gravity from back of flange = 1.57". Moment of inertia about neutral axis = 10.5 (ins.)^4. Find the greatest fiber stresses, in tension and in compression.

Problem 8.

A floor is supported by wooden beams, of cross section 3″ × 10″, placed 2 ft. on centers, with a span of 12 ft. The weight of the floor is 10 lbs. per sq. ft. Find the uniformly distributed live load which can be placed on the floor, assuming a working strength $f = 1000$ lbs. per sq. in.

Problem 9.

A wooden beam, of cross section 4″ × 10″, with a span of 12 ft., fails under a breaking load of 12,000 lbs. concentrated at the middle of the span. Find the modulus of rupture of the beam.

Problem 10.

Find the section modulus of the steel I beam required to support the loading given in Problem 10 (Art. 74). Assume a working strength $f = 16{,}000$ lbs. per sq. in.

From a manufacturers' handbook select a suitable section.

Problem 11.

A wooden beam, of cross section 6″ × 8″, with a 12 ft. span, is subjected to a concentrated load W, at a distance of 4 ft. from one support. Assuming a working strength $f = 1000$ lbs. per sq. in., find the greatest allowable value of W.

Problem 12.

Find the greatest allowable value of W if the beam given in Problem (11) is subjected to an additional uniformly distributed load of 200 lbs. per ft.

Problem 13.

A 6″ steel I beam (Fig. 98) is subjected to a uniformly distributed load. Find the allowable value of the intensity w, if the working strength of the material is assumed to be 12,000 lbs. per sq. in. The value of $\frac{I}{c} = 7.3\,(\text{ins.})^3$.

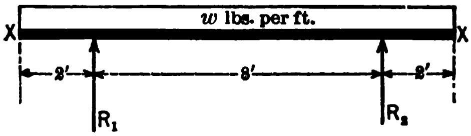


FIG. 98.

Problem 14.

Solve Problem (13), assuming that two additional loads of 2000 lbs. each are concentrated at the ends of the beam.

Problem 15.

Solve Problem (13), assuming that the support R_1 is moved to the left end of the beam.

Problem 16.

Solve Problem (14), assuming that the support R_1 is moved to the left end of the beam.

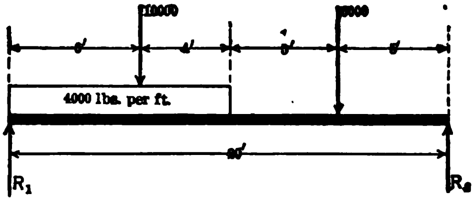
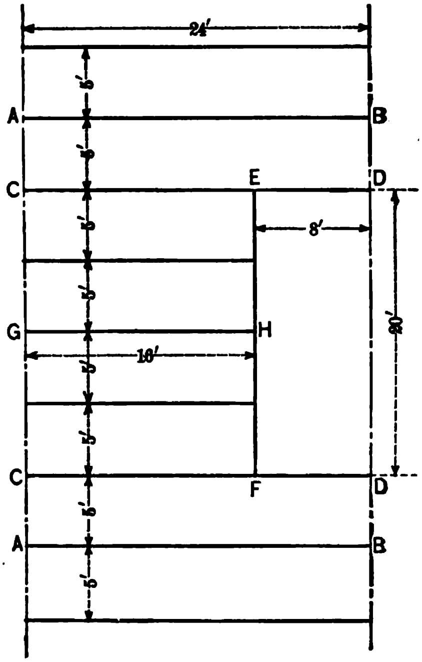


FIG. 99.

Problem 17.

A floor, designed to carry a uniformly distributed load of 300 lbs. per sq. ft., is supported on wooden beams of 6″ × 12″ cross section, with a span of 14 ft. Find the proper spacing of the beams, center to center, assuming a working strength of 1200 lbs. per sq. in.



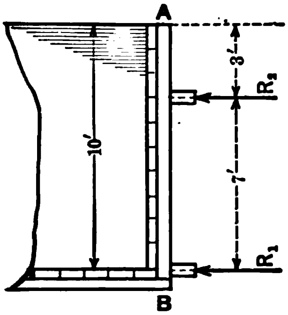


FIG. 100. FIG. 101.

Problem 18.

Find the allowable span for a 10″ steel I beam, $\frac{I}{c} = 24.4$ (ins.)^3, when subjected to a uniformly distributed load of 1000 lbs. per ft., assuming a working strength of 15,000 lbs. per sq. in.

Problem 19.

Two standard I beams are placed side by side and act as a single beam. If the beams are supported at the ends and loaded, as shown (Fig. 99), find which of the following sections will be suitable to use, provided the greatest allowable fiber stress is 12,000 lbs. per sq. in. The section modulus for a 10″ I beam is 31.7 $(\text{ins.})^3$; a 12″ is 53.5 $(\text{ins.})^3$; a 15″ is 81.2 $(\text{ins.})^3$ and an 18″ is 88.4 $(\text{ins.})^3$.

Problem 20.

A floor, designed to carry a total uniformly distributed live load of 200 lbs. per sq. ft., is constructed around an opening as shown (Fig. 100). Assuming that the weight of the floor itself is 80 lbs. per sq. ft., find the section moduli of the steel I beams required for the beams AB, CD, EF and GH. The working strength = 16,000 lbs. per sq. in. From a manufacturer's handbook select the lightest section that will be suitable in each case.

Problem 21.

The sides of a rectangular wooden tank, 10 ft. deep, are constructed of horizontal planks supported by wooden uprights AB, placed 5 ft. from center to center. The uprights are supported by horizontal braces R_1 and R_2, as shown (Fig. 101). Find the size of timber required for the uprights AB to support the sides when the tank is full of water. Assume the working strength of the material f = 800 lbs. per sq. in. and the weight of the water = 62.5 lbs. per cu. ft.

84. Beams of Varying Cross Section. — Throughout the discussion of the theory of bending thus far we have imposed the limitation that the cross section of the beam is uniform throughout its entire length.

It is customary to assume that the theory of bending will apply when the different cross sections of a beam vary in size and shape, *provided each section is symmetrical with respect to the plane of loading* and the other limitations imposed (Art. 63) still hold.

In other words, when the shape of a beam varies under the above conditions, the stress intensity on any cross section is assumed to vary in the same manner as if the beam were of uniform section throughout and hence the outside fiber stress at any cross section will be given by the formula

$$f = \frac{Mc}{I},$$

where $\frac{I}{c}$ is the section modulus for the section.

Since the value of $\frac{I}{c}$ *will vary through the length of the beam it will follow that the value of the greatest outside fiber stress will not necessarily occur at the section of greatest bending moment.* If the section at which the outside fiber stress is a maximum cannot be located by an inspection of the beam and its loading, the values of f at a number of different sections can be obtained, and, if necessary, a *diagram of the outside fiber stresses* at the different cross sections can be plotted. The greatest outside fiber stress can then be determined from the plot.

An exact determination of the maximum value of f can evidently be made, when both the bending moment and the section modulus at any section can be expressed in terms of the distance x of the section from the end of the beam, by differentiating and placing the derivative equal to zero.

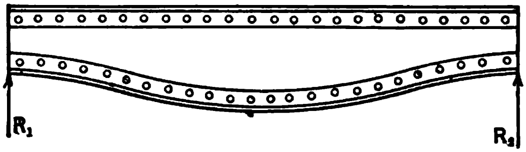


FIG. 102.

The usual object of making a beam of varying cross section is the saving of material and weight and in some cases the improvement of the appearance. When beams are made of cast metal it is a simple matter to make the patterns conform to any desired

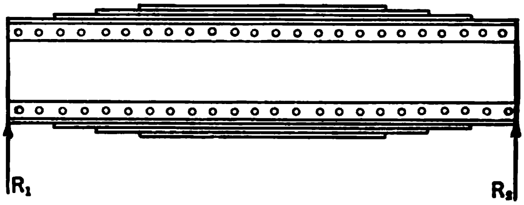


FIG. 103.

shape. Forged beams can also be made in a like manner. It evidently would not be practicable, however, to vary the section of a rolled beam and hence the simple types of steel beams are always made of uniform section.

In the case of large built-up girders a saving in weight can be made by varying the depth of the web as indicated (Fig. 102), or by varying the cross sections of the flanges as indicated (Fig. 103), or by varying the dimensions of the web and flanges together.

All beams of varying cross section should be so designed that the section at which the value of $\frac{I}{c}$ is a maximum is the section at which the bending moment is greatest and all other sections should be so proportioned that the value of the outside fiber stress at any one of them will not be greater than the outside fiber stress at the section of greatest bending moment.

85. Beams of Uniform Strength. — A beam of uniform strength is one in which the size of the cross section is varied in such a manner that the outside fiber stress is constant throughout the beam, that is

$$f = \frac{Mc}{I} = \text{a constant.}$$

Hence the values of the section moduli for the different cross sections of such a beam must vary directly as the values of the bending moments, that is

$$\frac{I}{c} \propto M.$$

A beam which exactly fulfills this condition is an impossibility, since the condition makes no allowance for the shearing stress which occurs at or, more correctly, near the section of zero bending moment where the value of $\frac{I}{c}$ must evidently be zero.

For example, if the beam supporting a load evenly divided between two points, equidistant from the supports (Fig. 104), is to be designed as a beam of uniform strength with a rectangular cross section throughout, the value of $\frac{I}{c}$ may be varied by varying the breadth only, or by varying the depth only, or by varying both dimensions together.

If we choose the first method, letting h = the depth of each cross section, it is evident from the form of the bending moment diagram that, between the cross sections C and D, the value of

$$\frac{I}{c} = \frac{b_1h^2}{6} = \frac{M}{f} = \text{a constant,}$$

and hence the breadth b_1 will be constant. Between the support R_1 and the cross section C the value of the bending moment at any section will be proportional to the distance of the section from the support and hence

$$\frac{I}{c} = \frac{bh^2}{6} = \frac{M}{f} = Kx,$$

where K = a constant $= \frac{W}{2f}$.

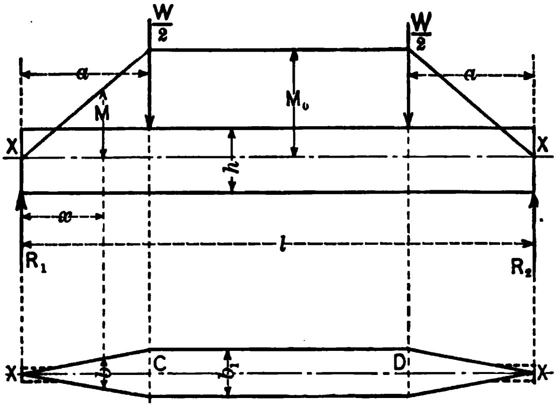


Fig. 104.

Hence

$$b = \frac{6K}{h^2}x = K_1x,$$

where $K_1 = \frac{6K}{h^2}$ = a constant, and therefore the breadth b will vary uniformly from 0, at the support, to b_1, at the cross section C. In order to resist the shearing stresses at sections near the support the cross section would have to be widened as indicated by the dotted lines. If this is done at each end, the remainder of the beam can be designed to satisfy the condition for a beam of uniform strength.

It may be noted that, if the distribution of the supporting force R_1 were taken into account, the value of b would be zero at the left side of the support but not zero at the right side. Unless the supporting force were distributed over a considerable distance, however, the bending moment at the right side of the support R_1 would be so small that the value obtained for b would not be great enough to make the section sufficiently large to stand the shearing stress, and the end of the beam would still have to be widened out as indicated.

In a similar manner the beam might be designed with the cross sections of uniform width and varying depth, or with both dimensions varying.

The forms of beams of uniform strength to carry other load systems can be determined in the same manner as above.

86. Cross Sections of Equal Strength. — If the cross section of a beam is designed in such a manner that

$$\frac{y_c}{y_t} = \frac{f_c}{f_t},$$

where y_c = the distance of the outside fibers in compression from the neutral layer, y_t = the distance of the outside fibers in tension from the neutral layer, f_c = the working strength of the material in compression and f_t = the working strength of the material in tension, the beam is said to have a cross section of equal strength.

Owing to the fact that the modulus of rupture of a beam is not the same as the breaking strength of the material in either tension or compression (Art. 81) and, furthermore, varies for different cross sections, it is impracticable to obtain values for the working strengths f_c and f_t, in both compression and tension, sufficiently definite to make the design of a cross section of equal strength of any value except in a very limited way.

In the design of steel beams and girders it is customary to make the cross sections symmetrical with respect to the neutral axis, although this results in making the tension side stronger than the compression side of the beam. Practically it is found to be better economy to do this than to attempt to vary the values of y_c and y_t.

87. Problems. — Fiber Stresses in Beams of Varying Sections. —

Problem 1.

A circular shaft is tapered as shown in Fig. (105). Plot a bending moment diagram and a diagram showing the variation in the outside fiber stress in the shaft due to a load of 400 lbs. at the center of the span. Find the greatest fiber stress.

Problem 2.

Plot the bending moment diagram and the diagram of outside fiber stress for the shaft given in Problem (1), taking into account the weight of the shaft, assuming the weight of the material equals 0.3 lbs. per cu. in.

Problem 3.

Deduce the expression for the diameter d, at any point at a distance x from the support, for a circular shaft designed as a beam of uniform strength, supported at the ends and subjected to a load W at the center of the span. Let l = the length of the span and neglect the weight of the shaft.

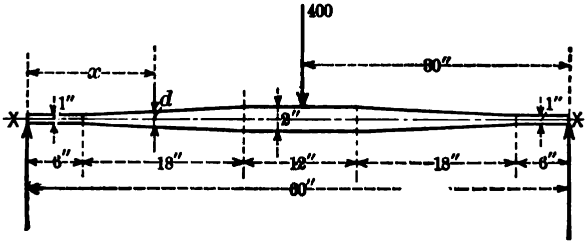


FIG. 105.

Problem 4.

Sketch a plan showing the variation in breadth b of the cross sections of a cantilever beam of uniform strength subjected to a single concentrated load W at the free end, all the cross sections of the beam being rectangular and of uniform depth h. Let l = the length of the beam and neglect its weight. Deduce the expression for the value of b at any cross section in terms of x, the distance of the section from the load W.

Problem 5.

Sketch an elevation showing the variation in depth of the cross sections of cantilever beam of uniform strength subjected to a single concentrated load W at the free end, all the sections of the beam being rectangular and of uniform breadth b, letting l = the length of the beam and neglecting its weight. Deduce the expression for the value of h at any cross section in terms of x, the distance of the section from the load W.

Problem 6.

Solve Problem (4), replacing the concentrated load W with a uniformly distributed load of intensity w extending over the entire length of the beam.

Problem 7.

Solve Problem (5), replacing the concentrated load W with a uniformly distributed load of intensity w extending over the entire length of the beam.

Problem 8.

Show that the sketches, Problems (4) and (5), will represent a half plan and half elevation, respectively, of a beam of uniform strength supported at the ends and subjected to a concentrated load 2 W at the middle of the span, the length of the span being 2 l.

88. Longitudinal Shearing. — Whenever a beam is subjected to ordinary bending, shearing stresses are produced on different longitudinal sections through the beam. The common method of determining the intensity of the shearing stress at any point in a longitudinal section, parallel to the neutral layer, is the following:

Let AB and CD be any two cross sections, at a small distance x apart (Fig. 106), of a beam subjected to ordinary bending. Let the axis OX coincide with the central axis of the beam, OY coincide with the axis of symmetry of the cross section AB and OZ coincide with the neutral axis of AB and let y_2 = the distance of the neutral layer from the top of the beam. Let mn be the trace on the plane of loading and gk the trace on a cross section of any longitudinal plane parallel to the neutral layer of the beam at a distance y_1.

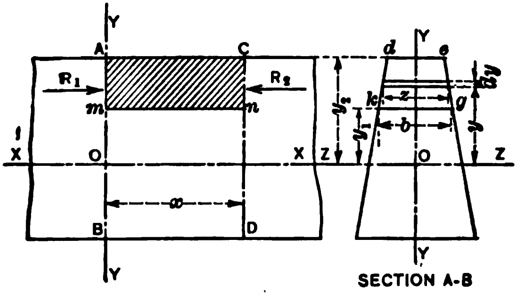


FIG. 106.

The prism $ACnm$, bounded by the outside surface of the beam, the longitudinal plane mn and the two cross sections, will be in equilibrium under forces acting upon it and, if we apply the condition of equilibrium $\Sigma H = 0$, it is evident that the total shearing stress on the longitudinal plane mn will be equal to the difference of the resultants R_2 and R_1 of the normal stresses on the ends of the prism. Hence, if we let s = the average intensity of the shearing stress on mn and $b = gk$, the width of the section mn, we shall have

$$sbx = R_2 - R_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let M_1 = the bending moment at the section AB and M_2 = the bending moment at the section CD. Then, *if the outside fiber stress does not exceed the elastic limit*, the resultant of the normal stress on Am will be equal in magnitude to

$$R_1 = \int_{y=y_1}^{y=y_2} f_1\,dA = \frac{M_1}{I}\int_{y=y_1}^{y=y_2} yz\,dy, \quad . \quad . \quad . \quad . \quad (2)$$

where I = the moment of inertia of the cross section and z = the length of an elementary strip of width dy, parallel to the neutral axis.

The resultant of the normal stress on Cn will be equal in magnitude to

$$R_2 = \frac{M_2}{I}\int_{y=y_1}^{y=y_2} yz\,dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But $\int_{y=y_1}^{y=y_2} yz\,dy$ = the moment of the portion $degk$ of the cross section (between its intersection gk with the longitudinal layer and the top of the beam) about the neutral axis and, if we denote the value of this moment by the letter Q, we shall have

$$R_1 = \frac{M_1}{I}Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$R_2 = \frac{M_2}{I}Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting these values in equation (1) we have

$$sbx = \frac{M_2 - M_1}{I}Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and solving for s we obtain

$$s = \frac{M_2 - M_1}{x}\,\frac{Q}{bI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

When there are no external forces acting on the portion of the beam between the sections AB and CD

$$\frac{M_2 - M_1}{x} = S \text{ (Art. 73)},$$

where S = the shearing force at either section. Therefore in that case equation (7) reduces to

$$s = \frac{SQ}{bI}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and, since the value $S = \frac{M_2 - M_1}{x}$ will be the same whatever the value of x, it will follow that the shearing stress on the plane mn will be uniformly distributed, so long as there are no external forces acting on the portion of the beam between m and n.

When external forces act on the portion of the beam between m and n, the shearing stress on the longitudinal plane will not be uniform, and to determine its intensity at any point we may take the two sections AB and CD intersecting the longitudinal layer near that point at a very small distance dx apart, in which case the difference of the bending moments $M_2 - M_1$ will become equal to dM. Proceeding in the same manner as before, we obtain in place of equation (7)

$$s = \frac{dM}{dx}\frac{Q}{bI} = \frac{SQ}{bI}. \quad \ldots\ldots \quad (9)$$

In either case the product sb of the stress intensity and the width of the longitudinal section will represent a quantity which may be called the *longitudinal shearing stress per unit of length of the beam.* If we represent the value of this quantity by

$$Z = sb,$$

equation (9) will reduce to

$$Z = \frac{SQ}{I}. \quad \ldots\ldots \quad (10)$$

An inspection of equations (9) and (10) will show that for any longitudinal section the values of both s and Z will vary as the value of S and hence the maximum values of s and Z will occur at the intersection of the longitudinal section with the cross section at which the shearing force is a maximum.

Also, if the shearing stresses on different longitudinal sections at the points of intersection with a given cross section are compared, the section on which the value of Z will be the greatest is the one for which Q is a maximum, which is evidently the neutral layer.

For ordinary types of beams the value of s will be a maximum on the neutral layer also, but it is possible that the value of b may vary in such a manner that the maximum intensity of the longitudinal shearing stress will occur on some other layer than the neutral layer.

It follows that the *greatest value of the longitudinal shearing stress per unit of length of a beam* will occur on the neutral layer at its intersection with the cross section at which the shearing force is a maximum and, ordinarily, *the maximum intensity of longitudinal shear* will occur at the same place.

When the cross section of a beam is a rectangle of breadth b and depth h, the greatest values of s and Z will be equal to

$$s = \frac{SQ}{bI} = \frac{S\frac{bh}{2}\cdot\frac{h}{4}}{b\cdot\frac{bh^3}{12}} = \frac{3}{2}\frac{S}{bh} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

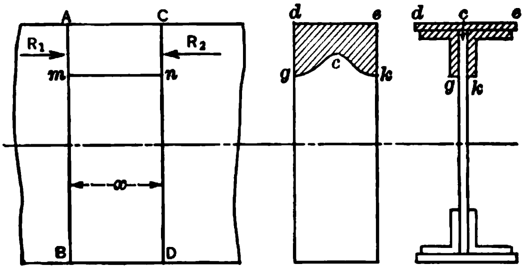
$$Z = sb = \frac{3}{2}\frac{S}{h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It should be observed that equation (1) may be written in the form

$$Zx = R_2 - R_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and, if a longitudinal section which is not a plane is taken through a beam subjected to ordinary bending, an equation in this form will result from the application of the condition of equilibrium $\Sigma H = 0$ to the forces acting on the prism bounded by the longitudinal section, the outside surface and two cross sections of the beam, at a distance x apart.

Two cases of this kind are represented in Fig. 107, the line *gck* (Fig. 107b) being taken to represent the intersection of an irregular longitudinal section of a rectangular beam with a given cross section and the line *gck* (Fig. 107c) being taken to represent the intersection of a longitudinal section between the parts of a built-up girder with a given cross section.



(a) (b) (c)

FIG. 107.

If Z is taken to represent the *average shearing stress per unit of length of the longitudinal section* between the two cross sections, equation (13) will evidently apply in either case and, by following the same method of reasoning as was used in the previous case, it may be shown that the value of Z for any loading will be given by equation (10).

The intensity of the shearing stress will not be uniform across the section, however, as in the case when the longitudinal section is a plane section.

The determination of the value of Z where the longitudinal section is not a plane is of importance in the calculation of the stresses on the connecting rivets of a built-up beam, or girder. For example, if the angles (Fig. 107c) are connected to the web plate by a single row of rivets, spaced at a distance p on centers, the total stress carried by one rivet will be equal to the total shearing stress on the longitudinal section gck between one pair of rivets. Hence, if the value of Z is constant, the total stress per rivet will be equal to

$$W = pZ = p\frac{SQ}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If Z is not constant the average value between the cross sections at two adjacent rivets may be taken.

Similar computations for other types of riveted connections may easily be made.

It is important to observe that the value of $R_2 - R_1$ (equation 1) will have the same sign for all longitudinal sections between any two given cross sections of a beam and hence the directions of the resultant shearing forces on all longitudinal sections between two cross sections will be the same.

Therefore, the shearing stresses on every longitudinal section intersecting a given cross section will have the same direction.

89. Intensity of the Shearing Stress on a Cross Section of a Beam. — In the discussion of the case of ordinary bending (Arts. 67–68) we have shown that the resultant of the shearing stress on any cross section of a beam is equal in magnitude to the shearing force at the section or that

$$\int s\,dA = S. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In order to completely determine the stress on any cross section it is evidently necessary to know the value of the intensity s at every point in the section.

It has been shown that when a body is under stress the intensities of the component shearing stresses at any point on two planes at right angles, in directions parallel to the third coördinate plane, are equal (Arts. 24 and 49).

Therefore the intensity of the vertical shearing stress at any point in a cross section of a beam will be equal to the intensity of the shearing stress on a longitudinal section, parallel to the neutral layer, cutting the cross section at that point.

Hence the formula for the intensity of the shearing stress at any point in a plane longitudinal section,

$$s = \frac{SQ}{bI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

will give the vertical shearing stress intensity at the same point on a cross section and the formula

$$Z = sb = \frac{SQ}{I} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

will represent a quantity which may be called the *intensity of the shearing stress per unit of depth* of the section.

It will follow from the relation of the shearing stresses on any two planes at right angles (Art. 24) that, since the directions of the shearing stresses on all longitudinal sections intersecting a given cross section are the same, the direction of the vertical shearing stress at every point in a cross section will be the same.

Therefore, at any section at which the value of $S = 0$ the value of $s = 0$ at every point, and at any cross section where $S \gtrless 0$ the value of s will vary with the value of Q, being greatest in magnitude at the neutral axis, ordinarily, and zero at the top and bottom of the section.

If the cross section of a beam is a rectangle, of breadth b and depth h, the value of Q for any point at a vertical distance y from the neutral axis of a cross section will be equal to

$$Q = b\left(\frac{h}{2} - y\right)\frac{1}{2}\left(\frac{h}{2} + y\right) = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right), \quad . \quad . \quad . \quad (4)$$

and substituting this value, together with the value of I, in equation (2), we obtain for the intensity of the shearing stress

$$s = \frac{S\frac{b}{2}\left(\frac{h^2}{4} - y^2\right)}{b \cdot \frac{bh^3}{12}} = \frac{3\,S}{2\,bh^3}(h^2 - 4\,y^2)\,. \quad . \quad . \quad . \quad (5)$$

It should be noted, as a result of the foregoing theory and the theory for determining the fiber stress (Art. 69), that, at every cross section at which $S = 0$, the stress intensity is zero at the neutral axis and the stress at every other point in the section has a normal component only; also that, at every cross section at which $S \gtrless 0$ the stress at the top and bottom of the section has a normal component only, the stress at the neutral axis a shearing component only and the stress at every other point in the section has both a normal and a shearing component.

In the foregoing discussion of the manner of distribution of the vertical shearing stress on a cross section of a beam, subjected to ordinary bending, no mention has been made of a *horizontal shearing component* on any cross section (Art. 67). If such a component exists at any point in a cross section there must also be a shearing stress of equal intensity, in a horizontal direction, on a vertical longitudinal plane through the point (Art. 49) and also, in order to have equilibrium, the resultant of the horizontal shearing stress on the entire cross section must be zero.

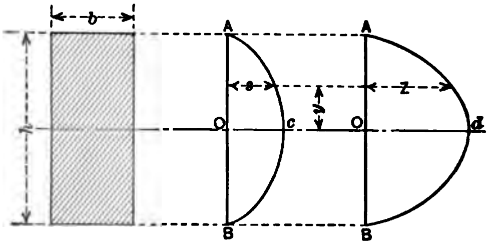
In a homogeneous beam of rectangular section there will be no horizontal shearing stresses on vertical longitudinal planes and hence no horizontal shearing components on the cross sections.

On cross sections of certain types, however, horizontal shearing components will exist. For example, according to the theory of longitudinal shearing, there will be horizontal shearing stresses on vertical longitudinal sections through the flanges of an I beam and hence there will be horizontal shearing components on the cross sections through the flanges. The intensities of these components will be so small, however, that the determination of the manner of their distribution is of no practical value. The web of the I beam may be considered to be subjected to plane stress.

90. Graphical Representation of the Shearing Stress. — If the values of the shearing stress intensity at different points of a rectangular cross section are laid off as ordinates from the trace AB, of the cross section on the plane of loading (Fig. 108b), and a line be drawn through the ends of the ordinates, a diagram showing the intensity s of the shearing stress at all points in the cross section will result.

If each ordinate of the diagram for s is multiplied by the width of the section at that point, the ordinates for a diagram representing the value Z (Fig. 108c) will be obtained.

It is evident from the form of the equations (Art. 89) that the curves AcB and AdB, representing the values of s and Z, respectively, will be parabolas with their axes coinciding with the central axis XX of the beam.



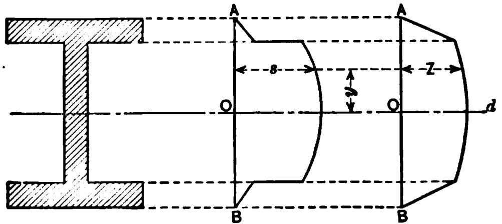
(a) (b) (c)

FIG. 108.

The area AdB (Fig. 108c) will evidently represent the value of the resultant shearing stress on the section

$$S = \int s\,dA = \int_{y=-\frac{h}{2}}^{y=\frac{h}{2}} Z\,dy. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The diagrams for s and Z for an I section (Fig. 109) are constructed in the same way as those for the rectangular section.



(a) (b) (c)

FIG. 109.

It should be noted that a large portion of the resultant shearing stress, represented by the area AdB (Fig. 109c), acts on the cross section of the web of the beam and that an approximately correct value for the greatest intensity of the shearing stress may be obtained by dividing the resultant shearing stress by the area of the cross section of the web, that is

$$s = \frac{S}{A_{\text{web}}} \text{ (approx.)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similar diagrams may be constructed to represent the distribution of the shearing stress on other shaped cross sections as was done to represent the distribution of the normal stress (Art. 78).

91. Problems. — Longitudinal Shearing Stresses in Beams. — The following problems will serve to illustrate a few applications of the theory of longitudinal shearing in beams.

Problem 1.

The vertical shearing force on a given section of a 6″ × 12″ beam is 4800 lbs. Find the longitudinal shearing force per unit of length = vertical shearing force per unit of depth, also the intensity of the longitudinal shearing stress on the horizontal plane = the intensity of the shearing stress on the vertical plane, at the neutral axis and at points 1″, 2″, 3″, 4″, 5″ and 6″ from the neutral axis.

Solution. — By use of equations (3), (4) and (5) (Art. 89) the results given in the following table can be easily obtained. In the case of the rectangular cross section the least amount of work will be involved if the values of s are obtained first by the use of Equation (5). The values of Z can then be readily found and it is unnecessary to compute the values of Q. For other cross sections, however, the simplest solution will be made by finding the values of Q, Z and s in the order given.

Points.	Q (ins.)^3	$Z = \frac{SQ}{I}$ lbs. per in.	$s = \frac{SQ}{bI}$ lbs. per sq. in.
Neutral axis	108	600	100.0
1	105	583	97.2
2	96	533	88.8
3	81	450	75.0
4	60	333	55.6
5	33	183	30.5
6	0	0	0.0

Plot the values of Z and s and show that the area under the $Z = \frac{SQ}{I}$ line represents 4800 lbs.

Problem 2.

If the outside fiber stress on the section given in Problem (1) is 600 lbs. per sq. in. find the magnitude and direction of the resultant of the normal and shearing intensities of the stress on the cross section at the neutral axis and at points 2″, 4″ and 6″, from the neutral axis.

Problem 3.

Find the greatest intensity of the longitudinal shearing stress in the beam given in Problem (1) (Art. 76). Assume a cross section 8″ × 16″.

Problem 4.

The shearing force at a cross section of a wooden beam, 6″ × 12″, is 8000 lbs. Find the maximum intensity of the shearing stress on the section.

Problem 5.

Make diagrams showing the intensities of the shearing stress at different points in the cross section given (Problem 4); also make a diagram showing the distribution of the total shearing stress.

Problem 6.

A beam of 10″ × 12″ rectangular cross section is supported at the ends and is subjected to a uniformly distributed load, including its own weight, of 1000 lbs. per ft. of length. If the span is 20 ft., find the total longitudinal shearing force at the neutral layer between sections 4 ft. and 6 ft. from one support.

Problem 7.

A total load W is divided equally between two points equidistant from the middle of a 6″ × 12″ wooden beam having a 12 ft. span, supported at the ends. Find the magnitude of W and the distance of the points of application from the middle of the span, provided the conditions that maximum intensity of longitudinal shearing stress = 100 lbs. per sq. in. and the maximum fiber stress = 1200 lbs. per sq. in. are both satisfied.

Problem 8.

Find the magnitude of the total load W which the beam (Fig. 110) will support when the greatest intensity of the longitudinal shearing stress is 100 lbs. per sq. in. Assume the cross section of beam to be 4″ × 12″.

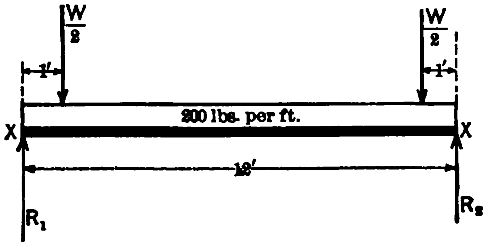


FIG. 110.

Problem 9.

The beam shown (Fig. 111) is built up by bolting four wooden planks, 2″ × 8″, together with ⅝″ bolts placed in pairs (Fig. 111b), the spacing from center to center of the pairs being 8″. Find the magnitude of the greatest shearing stress on the bolts. Assume a pair of bolts at each end section in lines of action of R_1 and R_2.

Problem 10.

The dimensions of the cross section of a standard 15″ I beam are approximately those shown in Fig. (112). Assuming that the resultant shearing stress on the cross section is 25,000 lbs., calculate the intensities of the shearing stress at the neutral axis and at points 2″, 4″, 6½″ and 7½″ distant from the neutral axis. Plot a diagram showing the variation in the shearing stress intensity; also plot a diagram showing the distribution of the total shearing stress on the section. Calculate the percentage of the total shear taken by the web.

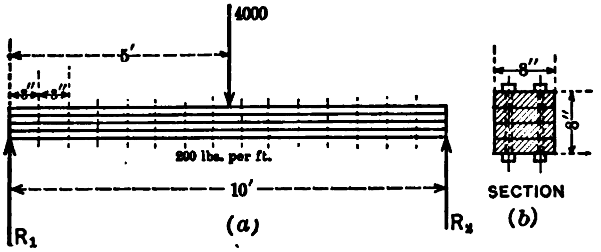


Fig. 111.

Problem 11.

Determine the average value of the shearing stress intensity on the cross section given (Problem 10), assuming that the total stress is uniformly distributed over the cross section of the web only.

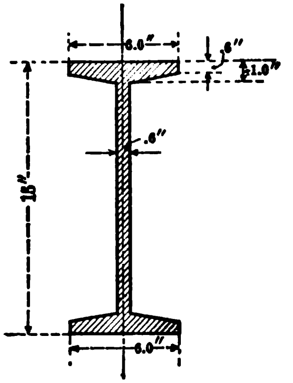


Fig. 112

Problem 12.

A plate girder, of the cross section shown (Fig. 113), is made up of a web plate 36″ × ½″, 4 angles 5″ × 3½″ × ½″, and 2 flange plates 12″ × ½″. The span = 30 ft. The rivets are ¾″ diameter. The girder is subjected to a uniformly distributed load of 4000 lbs. per ft. Assuming that the allowable stress intensities are f_s = 8000 lbs. per sq. in., f_c = 16,000 lbs. per sq. in., computed for the nominal diameter of the rivets, find the allowable spacing of the rivets near the ends of the girder.

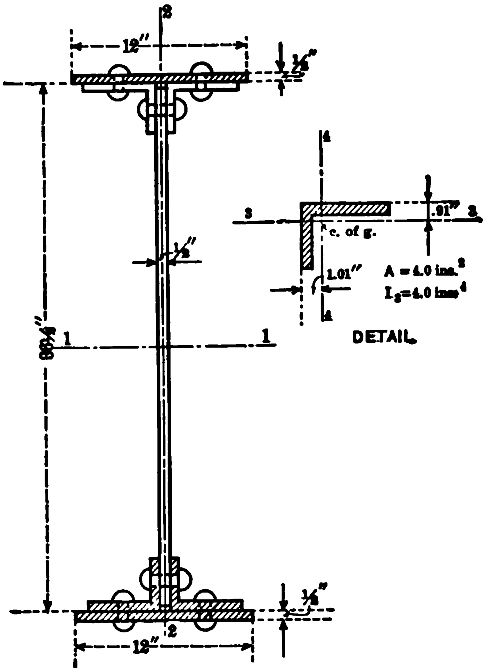


FIG. 113.

Solution. — The moment of inertia of the section with respect to the neutral axis 1-1 may be easily determined and will be found to be equal to

$$I = 10,880 \text{ (ins.)}^4$$

The resistance of a rivet to single shear will be equal to

$$W_1 = f_s \frac{\pi d^2}{4} = 3540 \text{ lbs.}$$

and the resistance to double shear

$$W_2 = 2 f_s \frac{\pi d^2}{4} = 7080 \text{ lbs.}$$

The allowable bearing pressure of a rivet in ½" plate will be equal to

$$W_3 = f_c t d = 6000 \text{ lbs.}$$

In computing the spacing near the supports we will use the maximum value of the shearing force

$$S = 60,000,$$

assuming that the value is constant for a small distance near the support.

This approximation will make the computed stresses on the rivets slightly greater than the actual stresses.

Web Rivets. — If we let p = the pitch of the rivets, the total stress carried by one rivet will be equal to

$$\frac{SQ}{I} \times p \text{ (Art. 88)}$$

and this quantity must not be greater than the allowable load on a rivet.

The value of Q will evidently be equal to

$$Q = 12 \times \tfrac{1}{2} \times 18\tfrac{1}{2} + 2 \times 4 \times 17.3 = 250 \text{ (ins.)}^3$$

Hence
$$p\frac{SQ}{I} = p\,\frac{60000 \times 250}{10880} = p\,1380.$$

Since the web rivets are in double shear and the resistance to double shear is greater than the allowable bearing pressure, the allowable load per rivet will be equal to

$$W_3 = 6000$$
$$\therefore\ p\,1380 = 6000$$

and solving for p we obtain

$$p = 4.35''.$$

Flange Rivets. — Let p_1 = the pitch of the rivets. Since the rivets will be placed in pairs on either side of the web the total stress carried by one pair of rivets will be equal to

$$p_1 \times \frac{SQ_1}{I},$$

where

$$Q_1 = 12 \times \tfrac{1}{2} \times 18\tfrac{1}{2} = 111 \text{ (ins.)}^3$$

Hence
$$p_1 \times \frac{SQ_1}{I} = \frac{60000 \times 111}{10880}\, p_1 = 612\, p_1.$$

Since the rivets are in single shear the allowable load on a pair of rivets will be equal to

$$2\ W_1 = 7080$$
$$\therefore\ 612\, p_1 = 7080$$

and solving for p_1 we obtain

$$p_1 = 11.6''.$$

It is customary in a girder of this type to make the pitch of the flange rivets the same as that of the web rivets and the foregoing solution shows that, when a single row of web rivets is used, these rivets will carry the greatest stress.

The solution might have been made by computing the values of I and Q for the net section of the girder, at a cross section through the centers of the rivets, by deducting the area of the rivet holes. The results obtained by this method would differ little, however, from those given above.

Problem 13.

A built-up beam is made by riveting a plate, 10″ × ¾″, to each flange of a standard 20″ I beam, weighing 80 lbs. per ft. The moment of inertia of the I beam about its neutral axis = 1466 (ins.)^4. The built-up beam is supported at the ends and a total load W is concentrated at two points dividing the span

into thirds. Find the magnitude of W, provided the maximum fiber stress in the built-up beam is 14,000 lbs. per sq. in. If there are two lines of $\frac{7}{8}''$ rivets in each flange and the spacing of the rivets, along the beam, is 6″ on centers, find the total load carried by each rivet due to longitudinal shearing. Assume the span = 30 ft. and neglect the weight of beam. Use the net section through the rivet holes in making all calculations, allowing 1″ for the diameter of the rivet holes.

92. Principal Stresses in Beams. — In the course of the preceding discussion the formulas for the intensities of the normal and shearing components of the stress at any point in a cross section at right angles to the axis of a beam have been obtained. Hence, if required, the resultant intensity of stress

$$p_r = \sqrt{f^2 + s^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

at any point in a cross section can be determined.

Except for the compressive stresses due to the external forces acting on the beam, which may ordinarily be neglected (Art. 53), the stress on any longitudinal section has been shown to consist of a shearing component only, the intensity of which can easily be determined.

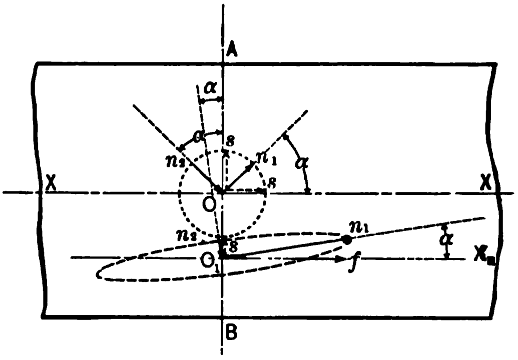


Fig. 114.

The intensities of the stress components on two planes at right angles through any point being known, the principal planes of stress and the intensities of the principal stresses (Art. 28) at the point can always be determined in any case in which a beam is subjected to plane stress.

For example, if AB (Fig. 114) represents any cross section of a rectangular beam, of breadth b and depth h, at which the shearing force and bending moment are equal to S and M, respectively, the normal stress intensity at any point O_1 at a distance y from the neutral axis will be equal to

$$f = \frac{My}{I} = \frac{12\,My}{bh^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the shearing stress intensity at the point will be equal to

$$s = \frac{SQ}{bI} = \frac{3\,S}{2\,bh^3}(h^2 - 4\,y^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

both of which quantities are represented as positive.

The shearing stress intensity on the horizontal plane through O_1 also will be given by equation (3).

Hence the principal stress intensities at O_1 will be equal to

$$n_1 = \frac{f}{2} + \frac{1}{2}\sqrt{f^2 + 4\,s^2}\,(\text{Art. } 32) \quad . \quad . \quad . \quad . \quad (4)$$

and

$$n_2 = \frac{f}{2} - \frac{1}{2}\sqrt{f^2 + 4\,s^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the angles α between the coördinate axes and the normals to the principal planes of stress can be determined from the equation

$$\tan 2\,\alpha = \frac{2\,s}{f}\,(\text{Art. } 32). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

An inspection of equations (4) and (5) will show that n_1 is the intensity of a tensile stress and n_2 that of a compressive stress.

The ellipse of stress at the point O_1 can be constructed with the vectors n_1 and n_2 as semi-major and semi-minor axes.

The planes of maximum shear through O_1 will make angles of 45° with the principal planes and the intensities of the shearing stresses on these planes will be equal to

$$s_1 = \frac{n_1 - n_2}{2}\,(\text{Art. } 31). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

At the intersection of the cross section and the neutral layer the above equations reduce to

$$n_1 = +s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$n_2 = -s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\tan 2\,\alpha = \infty. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Therefore the principal stresses at this point are equal in magnitude and of opposite sign and the principal planes of stress make angles of 45° with the coördinate axes (Art. 32), the cross section and the neutral plane being in this case the planes of maximum shear. The ellipse of stress at O is evidently a circle.

At the bottom and the top of the cross section $s = 0$; and hence the outside fiber stresses are principal stresses at the points B and A. There being no stress on the horizontal planes at B and A, the ellipse of stress becomes a straight line in each case.

Since there is no shearing stress at any point in a cross section at which the bending moment is a maximum, the section is a principal plane of stress at every point through which it passes and hence the fiber stress at any point in the section is greater than the stress intensity on any other plane through the point.

In general, therefore, when a cross section of a beam is subjected to combined shearing and normal stresses the planes of principal stress through different points in the cross section are inclined, one at an angle α and the other at an angle $90° + \alpha$, with the section and the value of α will vary from zero, at points farthest from the neutral axis of the section, to a maximum of 45°, at points on the neutral axis. The magnitude of the maximum principal stress intensity for all the points located in a given cross section will be found to be greatest at points which are farthest from the neutral axis, except that, in cases where the bending moment at the section is sufficiently small, the greatest intensity of the principal stress will occur at points on the neutral axis or at points between the neutral axis and the outside of the section, depending on the shape of the section.

In simple types of beams it will be found that the greatest principal stress intensity at any point near the cross section at which the bending moment is zero is much less than that at the outside fiber stress at the section at which the bending moment is a maximum. In such a beam, therefore, the value of the maximum outside fiber stress

$$f = \frac{M_0c}{I}$$

is greater than the stress intensity on any plane passing through any other point in the beam.

In the design of more complex built-up beams, however, the determination of the principal stresses, after the manner indicated above, at points in other sections than the section of the greatest bending moment, will be found to be necessary.

93. Problems. — Stresses in Beams. — Principal Stresses. — Problem 1.

A standard 24″ I beam, 80 lbs. per ft., is supported at the ends and carries, in addition to its own weight, a single concentrated load of 34,000 lbs. at the center of the beam. Span = 20 ft. I_1 = 2087.2 (ins.)^4. Section modulus = 173.9 (ins.)^3. Cross section = 23.32 sq. in. (Fig. 115).

Find the following quantities, for points on the neutral axis and at 4″, 8″, 10.86″ and 12″ from the neutral axis at each of the following sections, viz., (1) Just to right of left hand support, (2) ⅛ of span from left hand support, (3) Just to the left of the single concentrated load W:

(a) The intensity f of the normal stress on the cross section;

(b) The intensity s of the vertical shearing stress on the cross section;

(c) The angles α_1 and α_2 that the normals to the principal planes of stress make with the axis of the beam;

(d) The principal stress intensities n_1 and n_2.

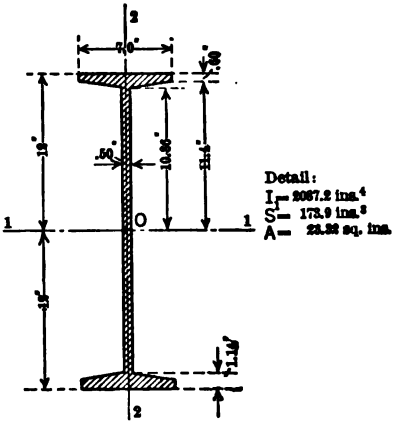


Fig. 115.

Solution. — The values of the shearing forces and bending moments for the sections (1), (2) and (3) will be the following:

(1) S = 17,800 lbs. M = 0.
(2) S = 17,600 lbs. M = 531,000 in. lbs.
(3) S = 17,000 lbs. M = 2,088,000 in. lbs.

(a) and (b). By use of the formulas $f = \frac{My}{I}$ and $s = \frac{SQ}{bI}$ the values of f and s given in the following table may be readily obtained.

Normal and Shearing Stress Intensities
(Pounds per square inch)

Points.	Q (ins.)^3	Section 1.		Section 2.		Section 3.	
		f	s	f	s	f	s
0	101.4	0	− 1730	0	− 1710	0	− 1651
4	97.4	0	− 1661	− 1017	− 1642	− 4,000	− 1586
8	85.4	0	− 1456	− 2035	− 1440	− 8,000	− 1390
10.86	71.9	0	− 1226	− 2762	− 1212	− 10,860	− 1171
12.0	0.0	0	0	− 3052	0	− 12,000	0

(c) The angles made by the principal axes of stress with the axis of the beam may be calculated from the formula,

$$\tan 2\alpha = \frac{2s}{f},$$

and the following results obtained.

Angles Made by the Principal Axes

Points.	Section 1.		Section 2.		Section 3.	
	α_1	α_2	α_1	α_2	α_1	α_2
0	45°	135°	45°	135°	45°	135°
4	45°	135°	36° 24′	126° 24′	19° 13′	109° 13′
8	45°	135°	27° 23′	117° 23′	9° 35′	99° 35′
10.86	45°	135°	20° 38′	110° 38′	6° 7′	96° 7′
12.0	....	....	0°	90°	0°	90°

(d) The principal stress intensities may be computed from the formulas,

$$n_1 = \frac{f}{2} + \frac{1}{2}\sqrt{f^2 + 4s^2}, \qquad n_2 = \frac{f}{2} - \frac{1}{2}\sqrt{f^2 + 4s^2}.$$

Principal Stresses
(Pounds per square inch)

Points.	Section 1.		Section 2.		Section 3.	
	n_1	n_2	n_1	n_2	n_1	n_2
0	– 1730	1730	– 1710	1710	– 1,651	1651
4	– 1661	1661	– 2227	1211	– 4,553	553
8	– 1456	1456	– 2781	746	– 8,235	235
10.86	– 1226	1226	– 3218	456	– 11,000	137
12.0	0	0	– 3042	0	– 12,000	0

Problem 2.

From the results given for Problem (1), make a sketch showing the position of the principal axes of stress and the general form of the ellipse of stress for each of the points given; and also for the points symmetrically located on the opposite side of the neutral layer of the beam.

Problem 3.

From the results given for Problem (1), compute the following:

(a) The percentage of the total shearing stress, on the cross section, taken by the web and flanges, respectively;

(b) The percentage of the total moment of resistance of the section taken by the web and flanges, respectively;

(c) The average intensity of the vertical shearing stress in the web, assuming that all the shear is taken by the web and that the stress is uniformly distributed;

(d) The average intensity of the shearing stress actually carried by the web.

Problem 4.

A wooden beam 8″ × 12″ cross section is supported at the ends and subjected to a uniformly distributed load, including its own weight, of 600 lbs. per ft. Span = 16 ft. Find the following quantities, for points on the neutral axis and at 2″, 4″, and 6″ from the neutral axis at each of the following sections, viz., (1) Just to right of left hand support, (2) $\frac{1}{4}$ span from left support, (3) $\frac{1}{2}$ span from left support, (4) At the middle of the span:

(a) The intensity of the normal stress on the cross section;

(b) The intensity of vertical shearing stress on the cross section;

(c) The angles made by the principal axes of stress with the axis of the beam;

(d) The principal stress intensities.

Problem 5.

From the results obtained in Problem (4), make a sketch showing the position of the principal axes of stress and the general form of the ellipse of stress for each of the points given and also for points symmetrically located on the opposite side of the neutral layer of the beam.

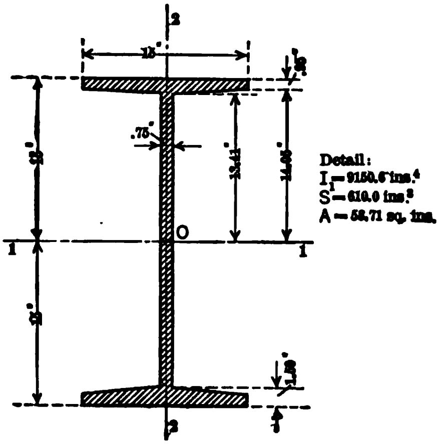


FIG. 116.

Problem 6.

A standard 30″ Bethlehem Girder Beam, 200 lbs. per ft. (Fig. 116), with a 30 ft. span, is supported at the ends and loaded uniformly. I_1 = 9150 (ins.)^4. Section modulus = 610 (ins.)^3. Area of cross section = 58.7 sq. ins. Total load including weight of beam = 200,000 lbs. Find the following quantities, for points on the neutral axis and at 6″, 13.4″ and 15″, from the neutral axis at each of the following sections, viz., (1) Just to the right of left support, (2) $\frac{1}{8}$ span from left hand support, (3) $\frac{1}{4}$ span from left hand support, (4) the middle of the span:

(a) The intensity of the normal stress on the cross section;

(b) The intensity of vertical shearing stress on the cross section;

(c) The angles that the principal axes of stress make with the axis of the beam;

(d) The principal stress intensities.

CHAPTER V.

THE DEFLECTION OF BEAMS.

94. Flexure of Beams. — In the preceding chapter the theory of bending, as applied to the determination of the stress, has been discussed without reference to the *stiffness*, or the ability of a beam to resist flexure. The importance of this property is well recognized in the design of floor beams and other members in structures and machines, where it is necessary to limit the amount of bending, or to limit the maximum deflection of a beam to a certain small fractional part of its length.

In the discussion of the theory of flexure, as applied to the determination of the deflection of beams, we shall, as in the previous chapter, use horizontal beams as illustrations although it will be clearly evident that the formulas derived will apply in any case where a beam is subjected to transverse loads, whatever its position may be.

The theory is a continuation of the theory for determining fiber stress (Arts. 66 and 69), and is based on the same assumptions and subject to the same limitations. The results obtained will be approximate so far as the assumptions made are inexact (Art. 69), but the amount of the error in any case, coming within the limitations imposed, will be slight.

95. Differential Equation of the Elastic Curve. — The straight line passing through the center of gravity of every cross section of a beam before it is bent is called the *axis of the beam*. After bending occurs this line takes the form of a continuous curve which is commonly known as the *elastic curve* of the beam. The curve will evidently be the same as the trace of the neutral layer (Art. 66) on the plane of loading. All straight lines, or fibers, parallel to the axis of the beam before bending, will become curves, parallel to the elastic curve, after bending takes place. The differential equation of the elastic curve, referred to rectangular coördinate axes, may be obtained as follows:

Let the sketch (Fig. 117) represent the form taken by a beam which is bent under the action of transverse loads, or by terminal couples, or by a combination of both.

Let OX and OV be a pair of rectangular coördinate axes, with the origin at one end of the beam and OX coinciding with the position of the axis of the beam before bending.

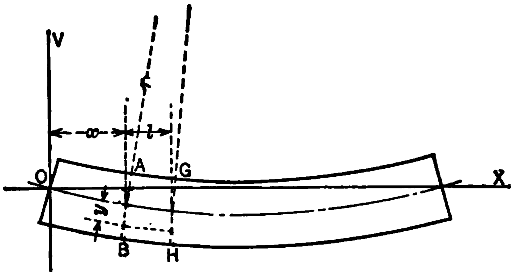


FIG. 117.

Let AB represent any cross section intersecting the elastic curve at a point whose coördinates are (x, v) and GH a cross section at such a small distance from AB that the portion of any fiber between the two sections can be considered circular in form.

At the cross section AB let M = the bending moment, I = the moment of inertia of the section about the neutral axis, f = the normal stress intensity at any point, at a distance y from the neutral axis, and e = the extension in the fiber intersecting the section at this point. Let r = the radius of curvature of the elastic curve between AB and GH and E = the modulus of elasticity of the material.

No extension in the neutral layer will be caused by the bending, since the normal stress intensity is zero at the neutral axis of every cross section. Hence, if we let l = the distance between the sections AB and GH before bending, the length of the portion of the elastic curve between the sections after bending will be equal to l and the length of any fiber at a distance y from the neutral layer will be equal to $l + el$. Therefore,

$$\frac{l + el}{l} = \frac{r + y}{r}$$

and

$$e = \frac{y}{r}. \quad \dots\dots\dots \quad (1)$$

From the third assumption of the theory (Art. 66) we obtain

$$e = \frac{f}{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and by equating (1) and (2)

$$\frac{1}{r} = \frac{f}{Ey}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives us the relation between the curvature of the elastic curve and the fiber stress at any section.

From the formula for fiber stress we obtain

$$\frac{f}{y} = \frac{M}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and substituting in equation (3),

$$\frac{1}{r} = \frac{M}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From the Differential Calculus,

$$\frac{1}{r} = \frac{\frac{d^2v}{dx^2}}{\left\{1 + \left(\frac{dv}{dx}\right)^2\right\}^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In ordinary cases $\frac{dv}{dx}$ is so small, when compared with unity, that higher powers than the first may be neglected, without appreciable error, hence

$$\frac{1}{r} = \frac{d^2v}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

By equating (5) and (7) we obtain

$$\frac{d^2v}{dx^2} = \frac{M}{EI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is the differential equation of the elastic curve from which the general equation for any case may be obtained, provided M and I can be expressed in integrable terms of the variable x.

It should be noted that the quantity M in equation (8) represents the value of the bending moment at any cross section and that the signs of bending moments adopted for horizontal beams (Art. 64) are in agreement with the sign of the derivative $\frac{d^2v}{dx^2}$, if the usual convention of signs of the horizontal and vertical coördinates

x and v is followed. Also, since the sign of the moment of resistance $\frac{fI}{y}$ is always opposite to the sign of the bending moment, equation (4) should be written

$$-\frac{f}{y}=\frac{M}{I},$$

if algebraic signs are to be used and M is taken to represent the bending moment. This equation is in accord with the system of signs, if y is positive when measured upward and a tension stress is called plus (Art. 66).

96. Slope and Deflection from the Elastic Curve. — If at any section, distant x from the origin (Fig. 117), we let i = the angle, in radians, which the tangent to the elastic curve makes with the horizontal, we have, when i is small throughout the length of the curve,

$$\frac{dv}{dx}=\tan i=i \text{ (very nearly).}$$

This quantity is called the angle of slope, or usually the *slope*, simply, of the elastic curve at the point x distant from the origin.

It is evident that one integration of equation (8) (Art. 95) will give the equation for the slope and a second integration will give the equation for the deflection at any point on the curve. Hence, for the slope

$$i=\frac{dv}{dx}=\int d\left(\frac{dv}{dx}\right)=\int\frac{M}{EI}\,dx, \quad . \quad . \quad . \quad . \quad (1)$$

and for the deflection,

$$v=\int i\,dx=\int\int\frac{M}{EI}\,dx\,dx. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

It should be observed that when the convention of signs adopted in the preceding article is followed, the value of v, as determined from the equation of the elastic curve, will be negative when the deflection is downward and positive when the deflection is upward; also that the sign of i will be in accord with the usual sign for a positive or negative angle.

If the beam is of *uniform section*, I will be constant and M must be expressed in terms of x in order to perform the integration.

If the beam is of *non-uniform* section, I will be variable and must be expressed in terms of x, as well as M.

Under the limitations imposed (Art. 63), the theory of flexure applies to beams of uniform cross section only, but it is customary to use the equations to a limited extent for determining the deflection of beams of varying section. Unless otherwise stated the deduction of formulas will be limited to cases where the cross sections are uniform.

In such cases equations (1) and (2) may be written

$$EIi = \int M\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$EIv = EI\int i\,dx = \int\int M\,dx\,dx. \quad . \quad (4)$$

If we substitute the following value of the bending moment, obtained from equation (8) (Art. 95),

$$M = EI\frac{d^2v}{dx^2}$$

in equations (1) and (2) (Art. 71), we obtain

$$S = \frac{dM}{dx} = EI\frac{d^3v}{dx^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$w = \frac{dS}{dx} = EI\frac{d^4v}{dx^4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

If w is constant, or a known integrable function of x, the general expressions for S, M, i and v at any cross section of a beam may evidently be found by one, two, three and four integrations of equation (6). If the integration is made without using limits, a constant must be added at each integration, each constant being determined from some condition of the problem.

Usually the first two integrations can be omitted and the equations for slope and deflection obtained by expressing M as a function of x and integrating equations (1) and (2).

In all cases in which the foregoing equations will apply, the bending will be slight and sufficient accuracy will be obtained if the length of the elastic curve and its projection on the OX axis are assumed to be equal. In other words the change in the distance between the end sections of a beam as well as that between any other two cross sections due to the curvature will be neglected.

97. General Formulas for Slope and Deflection. — Uniform Bending. — In this case the value of M is constant and, for beams of uniform cross section,

$$\frac{1}{r} = \frac{M}{EI} = \text{a constant}, \quad \ldots\ldots \quad (1)$$

and, therefore, the elastic curve is a circle.

If l = the length of the beam and the origin is taken at the left end, the integration of equation (3) (Art. 96) will give

$$EIi = M\int dx = Mx + c, \quad \ldots\ldots \quad (2)$$

where c is the constant of integration; and the integration of equation (4) (Art. 96) will give

$$EIv = EI\int i\,dx = \frac{Mx^2}{2} + cx + c_1, \quad \ldots\ldots \quad (3)$$

where c_1 is the second constant of integration. The values of the constants will be obtained for two cases.

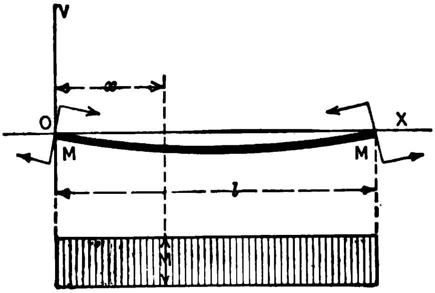


Fig. 118.

(a) *When the bending takes place so that the axis OX intersects the elastic curve at both ends of the beam* (Fig. 118). — In this case the deflection

$$v = 0 \quad \text{when} \quad x = 0$$

and also

$$v = 0 \quad \text{when} \quad x = l.$$

Applying the first condition to equation (3); we obtain

$$c_1 = 0$$

and applying the second condition to the same equation,

$$0 = \frac{Ml^2}{2} + cl$$

and hence

$$c = -\frac{Ml}{2}.$$

Substituting the values of c and c_1 in (2) and (3), we have for the general formulas for slope and deflection,

$$EIi = M\left(x - \frac{l}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$EIv = \frac{M}{2}(x^2 - lx). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

An inspection of equation (4) will show that the greatest slope occurs where $x = 0$, or $x = l$, and, if we represent its value where $x = 0$ by the symbol i_0, we shall have

$$i_0 = -\frac{Ml}{2EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Since $i = \frac{dv}{dx}$, the greatest deflection will evidently occur where $i = 0$, or when $x = \frac{l}{2}$, which is also evident from the fact that the curve is a circle; and, if we represent its value by the symbol v_0, we shall have

$$v_0 = -\frac{Ml^2}{8EI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the negative sign indicating that the deflection is downward when M is positive and upward when M is negative.

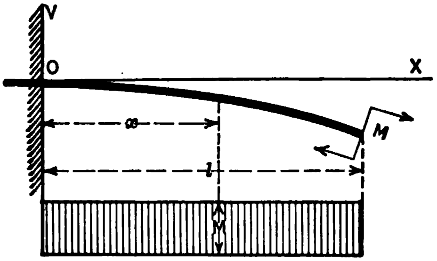


FIG. 119.

(b) *When the beam is fixed in direction at one end, as a cantilever beam subjected to uniform bending* (Fig. 119). — In this case the slope

$$i = 0 \quad \text{when} \quad x = 0$$

and hence, from equation (2),

$$c = 0;$$

also the deflection

$$v = 0 \quad \text{when} \quad x = 0$$

and hence, from equation (3),

$$c_1 = 0.$$

Substituting the values of c and c_1 in (2) and (3), we obtain

$$EIi = Mx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and

$$EIv = \frac{Mx^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

for the general equations for slope and deflection.

The maximum values evidently occur when $x = l$; and hence

$$i_0 = \frac{Ml}{EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$v_0 = \frac{Ml^2}{2\,EI} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In this case the deflection is evidently downward when M is negative and upward when M is positive.

98. General Formulas for Slope and Deflection. — Ordinary Bending. — When a beam is subjected to ordinary bending by transverse loads, the curvature of the elastic curve varies from point to point throughout its entire length except in cases where the loading is such that the bending moment over a portion of the beam is constant. In such cases the curvature will evidently be constant throughout the portion subjected to uniform bending and variable for the remainder of the length of the beam.

In this article the general formulas for slope and deflection for beams of uniform cross section, subjected to some of the simpler types of loading, will be deduced. The values of the greatest slope i_0 and of the greatest deflection v_0 will also be found. For simple loading the sections of the beam at which the slope and deflection have maximum values can be determined by inspection. When the loading is more complex, the condition that the slope is always zero at a maximum or minimum point on the elastic curve will enable us to determine the points of greatest deflection; and the condition that the bending moment is equal to zero at points on the elastic curve at which the slope is a maximum will enable us to determine the points of greatest slope.

In every case, if the load is distributed, W = the total load and w = the load per unit length. In each case the position of the loads and supporting forces, the length of the span l and other required dimensions, as well as the general form of the bending moment diagram, are indicated by the accompanying sketch. The effect of the weight of the material in the beam upon the slope and deflection is neglected.

(a) *Cantilever beam, single concentrated load at the free end* (Fig. 120). — The bending moment at any section, at a distance x from the origin, will be equal to

$$M = -W(l - x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Substituting this value in equation (3) (Art. 96) and integrating, we obtain

$$EIi = \int M\,dx = -W\int (l - x)\,dx = -W\left(lx - \frac{x^2}{2}\right) + c \quad . \quad . \quad (2)$$

and, by substituting in equation (4) (Art. 96) and integrating,

$$EIv = EI\int i\,dx = -W\int\left(lx - \frac{x^2}{2}\right)dx + c\int dx$$
$$= -W\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + cx + c_1. \qquad (3)$$

To obtain the constants c and c_1, observe that, when

$$x = 0, \quad i = 0 \text{ and } v = 0.$$

Therefore $c = 0$ and $c_1 = 0$ and the general formulas for the slope and deflection at any point in the elastic curve reduce to

$$EIi = -\frac{Wx}{2}(2l - x) \qquad (4)$$

and

$$EIv = -\frac{Wx^2}{6}(3l - x). \qquad (5)$$

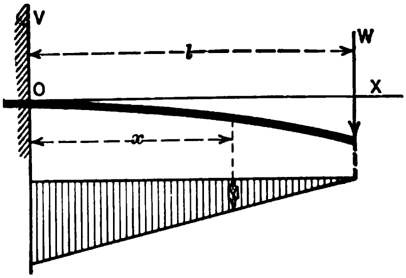


Fig. 120.

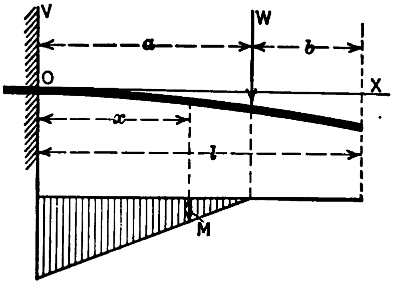


Fig. 121.

Evidently both the slope and deflection will be greatest at the free end of the beam, when $x = l$, and hence

$$i_0 = -\frac{Wl^2}{2EI} \qquad (6)$$

and

$$v_0 = -\frac{Wl^3}{3EI}. \qquad (7)$$

(b) *Cantilever beam, single concentrated load, not at the free end* (Fig. 121).—In this case the elastic curve of the portion of the beam between the fixed end and the load W will be represented by an equation of the same form as (5), in the preceding case, and the portion between the load and the free end will be a straight line, having the direction of the tangent at the point in the curve under the load W.

Therefore, by substituting a for l in (4) and (5), we obtain

$$EIi = -\frac{Wx}{2}(2a - x) \qquad (8)$$

and

$$EIv = -\frac{Wx^2}{6}(3a - x), \qquad (9)$$

the equations for the slope and deflection when $x < a$. When $x = a$

$$EIi = -\frac{Wa^2}{2} \quad \text{and} \quad EIv = -\frac{Wa^3}{3},$$

and hence the straight line between the load and the free end may be represented by the equation

$$EIv = -\frac{Wa^3}{3} - \frac{Wa^2}{2}(x - a) = -\frac{Wa^2}{6}(3x - a). \quad . \quad . \quad (10)$$

At the free end, evidently,

$$i_0 = -\frac{Wa^2}{2EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$v_0 = -\frac{Wa^2(3l - a)}{6EI} = -\frac{Wa^2(2a + 3b)}{6EI} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

(c) *Cantilever beam, load uniformly distributed over its entire length* (Fig. 122). — The total load $W = wl$ and the bending moment at any section, at a distance x from the origin, will be equal to

$$M = -\frac{w}{2}(l - x)^2 = -\frac{w}{2}(l^2 - 2lx + x^2). \quad . \quad . \quad . \quad . \quad (13)$$

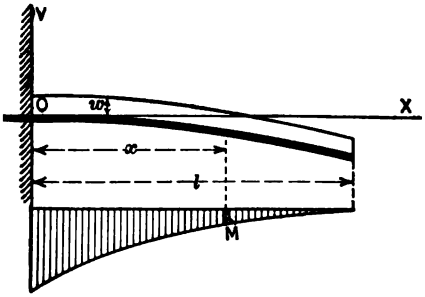


Fig. 122.

Proceeding as before:

$$EIi = -\frac{w}{2}\int\left(l^2 - 2lx + x^2\right)dx = -\frac{w}{2}\left(l^2x - lx^2 + \frac{x^3}{3}\right) \quad . \quad (14)$$

and

$$EIv = -\frac{w}{2}\int\left((l^2x - lx^2 + \frac{x^3}{3}\right)dx = -\frac{w}{2}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right), \quad . \quad (15)$$

the constants of integration being zero in each case.

The greatest slope and deflection will both occur at the free end when $x = l$, and hence

$$i_0 = -\frac{wl^3}{6EI} = -\frac{Wl^2}{6EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$v_0 = -\frac{wl^4}{8EI} = -\frac{Wl^3}{8EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

(d) *Simple beam, single concentrated load at center of span* (Fig. 123). — The supporting forces will each be equal to $\frac{W}{2}$ and the bending moment at any section, between the origin and the load, will be equal to

$$M = \frac{W}{2}x. \qquad (18)$$

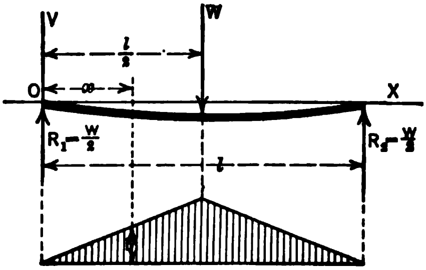


FIG. 123.

Hence, for values of x between 0 and $\frac{l}{2}$, the slope equation will be

$$EIi = \frac{W}{2}\int x\,dx = \frac{Wx^2}{4} + c \qquad (19)$$

and the deflection equation will be

$$EIv = \frac{W}{4}\int x^2\,dx + c\int dx = \frac{Wx^3}{12} + cx + c_1. \qquad (20)$$

To determine the constants c and c_1, observe that the elastic curve will be symmetrical with respect to the line of action of W and hence $i = 0$ when $x = \frac{l}{2}$, and therefore (19) gives

$$c = -\frac{Wl^2}{16};$$

also, since $v = 0$ when $x = 0$, equation (20) gives $c_1 = 0$. Substituting these values in (19) and (20) we obtain the general formulas for this case,

$$EIi = \frac{W}{4}\left(x^2 - \frac{l^2}{4}\right) \qquad (21)$$

and

$$EIv = \frac{W}{4}\left(\frac{x^3}{3} - \frac{l^2x}{4}\right). \qquad (22)$$

Since the elastic curve is symmetrical with respect to the middle point, equations (21) and (22) may be applied to determine the slope and deflection at any point, between the load W and the right hand support R_2, by letting $x =$ the distance of the point from R_2 instead of its distance from R_1.

The maximum deflection will evidently occur at the middle of the beam and the maximum slope at the support; hence, substituting $x = 0$ in (21), we obtain

$$i_0 = -\frac{Wl^2}{16EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and, substituting $x = \frac{l}{2}$ in (22),

$$v_0 = -\frac{Wl^3}{48EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Each half of the elastic curve will be of the same general form as the curve for the cantilever beam (Case a) and equations (23) and (24) may be easily obtained by substituting the proper values in (6) and (7).

(e) *Simple beam, load uniformly distributed over its entire length* (Fig. 124). — The total load $W = wl$. Hence the supporting forces $R_1 = R_2 = \frac{wl}{2}$ and the bending moment at any distance x from the origin is equal to

$$M = \frac{wl}{2}x - \frac{wx^2}{2} = \frac{w}{2}(lx - x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

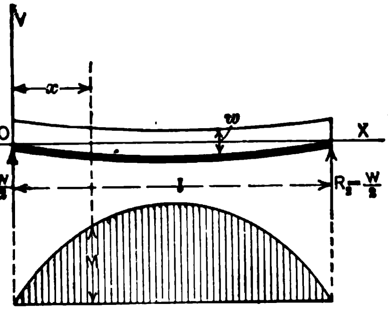


FIG. 124.

Hence, for all values of x between 0 and l,

$$EIi = \frac{w}{2}\int(lx - x^2)\,dx = \frac{w}{2}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and

$$EIv = \frac{w}{2}\int\left(\frac{lx^2}{2} - \frac{x^3}{3}\right)dx + c\int dx = \frac{w}{2}\left(\frac{lx^3}{6} - \frac{x^4}{12}\right) + cx + c_1. \quad . \quad (27)$$

To determine c and c_1, observe that from the symmetry of the elastic curve $i = 0$ when $x = \frac{l}{2}$ and hence, from (26),

$$c = -\frac{wl^3}{24};$$

and also that $v = 0$ when $x = 0$ and hence, from (27),

$$c_1 = 0.$$

Substituting these values in (26) and (27) the general equations for slope and deflection are

$$EIi = \frac{w}{24}(6\,lx^2 - 4\,x^3 - l^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and

$$EIv = \frac{w}{24}(2lx^3 - x^4 - l^3x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Evidently the magnitude of the slope will be greatest at the ends and the deflection will be greatest at the middle of the span.

Substituting $x = 0$ in (28), we obtain

$$i_0 = -\frac{wl^3}{24EI} = -\frac{Wl^2}{24EI}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

likewise, by substituting $x = l$,

$$i_0 = \frac{wl^3}{24EI} = \frac{Wl^2}{24EI},$$

the only difference being that of the sign.

Substituting $x = \frac{l}{2}$ in (29) we have

$$v_0 = -\frac{5wl^4}{384EI} = -\frac{5Wl^3}{384EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

(f) *Simple beam, single concentrated load not at the center* (Fig. 125).—The supporting forces will be equal to $R_1 = \frac{Wb}{l}$ and $R_2 = \frac{Wa}{l}$. Assume $a > b$. Then, for values of x from 0 to a, $M = \frac{Wb}{l}x$.

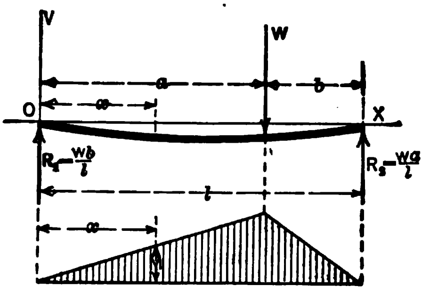


FIG. 125.

Substituting in the differential equation and integrating

$$EIi = \frac{Wbx^2}{2l} + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

and

$$EIv = \frac{Wbx^3}{6l} + cx + c_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

If we observe that $v = 0$ when $x = 0$, equation (33) gives $c_1 = 0$. The constant c, however, cannot be determined at this stage in the solution but, if we let $i_1 =$ the undetermined slope at the origin O, it is evident that $i = i_1$ when $x = 0$ and hence $c = EIi_1$.

Therefore, equations (32) and (33) may be written

$$EIi = \frac{Wbx^2}{2l} + EIi_1 \qquad (34)$$

and

$$EIv = \frac{Wbx^3}{6l} + EIi_1x. \qquad (35)$$

For values of x from a to l,

$$M = \frac{Wbx}{l} - W(x - a);$$

and, substituting in the differential equation and integrating,

$$EIi = \frac{Wbx^2}{2l} - \frac{W}{2}(x - a)^2 + c_2 \qquad (36)$$

and

$$EIv = \frac{Wbx^3}{6l} - \frac{W}{6}(x - a)^3 + c_2x + c_3. \qquad (37)$$

Since the elastic curve is continuous at the point $x = a$, we may determine the constants c_2 and c_3 from the conditions that when $x = a$ the values of the slope, given by equations (34) and (36), are identical and the values of the deflection, given by (35) and (37), are also identical.

Hence by substituting $x = a$ in (34) and (36) and equating and solving for c_2, we obtain

$$c_2 = EIi_1.$$

Similarly, by equating (35) and (37) we obtain $c_3 = 0$. Therefore, for values of x from a to l,

$$EIi = \frac{Wbx^2}{2l} - \frac{W(x - a)^2}{2} + EIi_1 \qquad (38)$$

and

$$EIv = \frac{Wbx^3}{6l} - \frac{W(x - a)^3}{6} + EIi_1x, \qquad (39)$$

which with (34) and (35) give a complete set of equations for the slope and deflection of the two sections of the elastic curve in terms of the unknown constant i_1.

Finally, since $v = 0$ when $x = l$, we obtain from equation (39), putting $l - a = b$,

$$i_1 = -\frac{Wb}{6lEI}(l^2 - b^2).$$

Substituting this value in equations (34), (35), (38) and (39), we obtain, for values of x from 0 to a,

$$EIi = \frac{Wb}{6l}[3x^2 - (l^2 - b^2)], \qquad (40)$$

$$EIv = \frac{Wb}{6l}[x^3 - (l^2 - b^2)x], \qquad (41)$$

and, for values of x from a to l,

$$EIi = \frac{Wb}{6l}\left[3x^2 - (l^2 - b^2) - \frac{3l}{b}(x - a)^2\right], \qquad (42)$$

$$EIv = \frac{Wb}{6l}\left[x^3 - (l^2 - b^2)x - \frac{l(x - a)^3}{b}\right]. \qquad (43)$$

Since $a > b$, the value of x for the point of greatest deflection will be found by placing the expression for EIi (equation 40) equal to zero and solving for x, which gives

$$x = \sqrt{\frac{l^2 - b^2}{3}}.$$

Substituting this value in (41) we obtain, for the greatest deflection,

$$v_0 = \frac{Wb}{6\,lEI}\left[\frac{(l^2 - b^2)^{\frac{3}{2}}}{3\sqrt{3}} - \frac{(l^2 - b^2)^{\frac{3}{2}}}{\sqrt{3}}\right] = -\frac{Wb}{9\sqrt{3}\,lEI}(l^2 - b^2)^{\frac{3}{2}}. \quad . \quad (44)$$

The slope at the left hand support, obtained by putting $x = 0$ in equation (40), is equal to

$$i_1 = -\frac{Wb}{6\,lEI}(l^2 - b^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and the slope at the right hand support, obtained by putting $x = l$ in equation (42), is equal to

$$i_2 = \frac{Wb}{6\,lEI}(2\,l^2 - 3\,bl + b^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

i_2 being evidently the greatest slope in the beam.

The deflection v_a of the beam at the point of application of the load may be found by substituting $x = a$ in equation (41), which gives

$$v_a = \frac{Wb}{6\,lEI}\left[a^3 - (l^2 - b^2)a\right] = -\frac{Wa^2}{3\,lEI}(l - a)^2 = -\frac{Wa^2b^2}{3\,lEI} \cdot \cdot \quad . \quad (47)$$

The deflection at the center of the beam may be found by substituting $x = \frac{l}{2}$ in equation (41), which gives

$$v_c = \frac{Wb}{6\,lEI}\left[\frac{l^3}{8} - (l^2 - b^2)\frac{l}{2}\right] = -\frac{Wb}{48\,EI}(3\,l^2 - 4\,b^2). \quad . \quad . \quad (48)$$

When the load W is applied at the middle of the span, that is when $b = \frac{l}{2}$, the value of the maximum deflection will evidently be greater than its value when the load is in any other position and equation (44) will reduce to the form of equation (24).

A mathematical proof of this may be had by differentiating (44) with respect to the variable (b), the value of b obtained by placing the derivative equal to zero being equal to $\frac{l}{2}$.

(g) *Simple beam, two equal concentrated loads, equidistant from the ends* (Fig. 126). — Let W = the magnitude of each load; then, for values of x from 0 to a,

$$M = Wx; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

and by integration the slope equation becomes

$$EIi = \frac{Wx^2}{2} + c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and the deflection equation

$$EIv = \frac{Wx^3}{6} + cx + c_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Since $v = 0$ when $x = 0$, it is evident from (51) that $c_1 = 0$ and, if we let $i_1 =$ the slope at the left hand support where $x = 0$, it is evident from (50) that $c = EIi_1$.

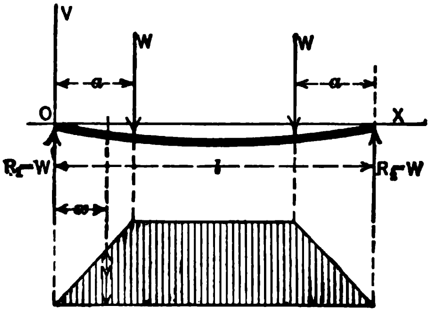


FIG. 126.

Hence

$$EIi = \frac{Wx^2}{2} + EIi_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (52)$$

and

$$EIv = \frac{Wx^3}{6} + EIi_1x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (53)$$

For values of x from a to $l - a$

$$M = Wa; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54)$$

and by integration the slope equation becomes

$$EIi = Wax + c_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55)$$

and the deflection equation

$$EIv = \frac{Wax^2}{2} + c_2x + c_3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (56)$$

Observe that, from the symmetry of the loading, $i = 0$ when $x = \frac{l}{2}$ and hence, from (55),

$$c_2 = -\frac{Wal}{2}.$$

Also, owing to the continuity of the elastic curve, when $x = a$ the value of i from (52) is identical with that obtained from (55) and hence

$$\frac{Wa^2}{2} + EIi_1 = Wa^2 - \frac{Wal}{2}$$

and

$$EIi_1 = -\frac{Wa}{2}(l - a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (57)$$

Similarly, when $x = a$, the value of v from (53) is identical with the value of v from (56) and, substituting the values of EIi_1 and c_2 and equating,

$$\frac{Wa^3}{6} - \frac{Wa^2}{2}(l - a) = \frac{Wa^3}{2} - \frac{Wa^2l}{2} + c_3$$

and hence

$$c_3 = \frac{Wa^3}{6}.$$

Therefore, the general equations for slope and deflection for values of x from 0 to a reduce to

$$EIi = \frac{W}{2}[x^2 - a(l - a)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

and

$$EIv = \frac{Wx}{6}[x^2 - 3a(l - a)], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and, for values of x from a to $(l - a)$,

$$EIi = \frac{Wa}{2}(2x - l) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

and

$$EIv = \frac{Wa}{6}[3x^2 - 3lx + a^2]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

The slope and deflection at any point on the curve, for values of x from $(l - a)$ to l, can evidently be found from equations (58) and (59) by taking the origin at the right hand support and calling values of x measured to the left from the support positive. The greatest deflection will evidently occur at the middle of the span and, substituting $x = \frac{l}{2}$ in (61) and reducing,

$$v_0 = -\frac{Wa}{24EI}[3l^2 - 4a^2]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

The greatest value of the slope will be obtained at the right and left supports, the magnitude being the same at each. At the left support we obtain from (57)

$$i_0 = -\frac{Wa}{2EI}(l - a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

The deflection under either of the loads can be found by putting $x = a$ in (59), giving the value

$$v_a = -\frac{Wa^2}{6EI}(3l - 4a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

If the value of the deflection at the middle of the span (equation 62) is compared with the value of deflection in the middle of the span for Case (f) (equation 48), the former value will be found to be double the latter.

(h) *Beam with overhanging ends, equal concentrated loads at ends* (Fig. 127). — Let a = the length of each of the overhanging ends and let each of the loads equal W. Take the origin at the left hand support. For values of x from 0 to l,

$$M_0 = -Wa, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

the portion of the beam between the supports being subjected to uniform bending, and hence, from (Art. 97), the general formulas for slope and deflection are

$$EIi = M_0\left(x - \frac{l}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

and

$$EIv = \frac{M_0}{2}(x^2 - lx). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

The greatest deflection in the portion of the beam between the supports will occur at the middle of the span and will be equal to

$$v_0 = -\frac{M_0 l^2}{8EI}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

and the greatest slope in this portion will occur at either support, the slope at the left support being equal to

$$i_0 = -\frac{M_0 l}{2EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

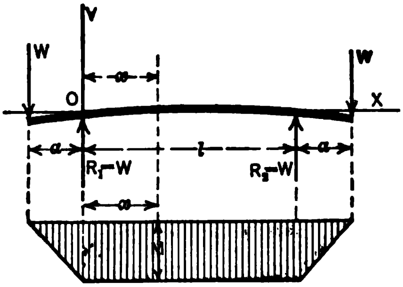


FIG. 127.

For points on the elastic curve between the left support and the left end of the beam, that is, for negative values of x from $-a$ to 0, the bending moment will be equal to

$$M = -W(a + x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

and by integration

$$EIi = -W\left(ax + \frac{x^2}{2}\right) + c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Since the slope at the support is equal to i_0 (equation 69),

$$c = EIi_0 = -\frac{M_0 l}{2} = \frac{Wal}{2},$$

and hence the general slope equation becomes

$$EIi = -\frac{W}{2}(2ax + x^2 - al). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Integrating again

$$EIv = -\frac{W}{2}\left(ax^2 + \frac{x^3}{3} - alx\right) + c_1, \quad . \quad . \quad . \quad . \quad . \quad (73)$$

where $c_1 = 0$, since $v = 0$ when $x = 0$.

The greatest slope and the greatest deflection in the overhanging end will both occur at the end of the beam where $x = -a$ and hence for this portion of the beam the greatest slope will be

$$i_0' = \frac{Wa}{2EI}(l + a) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

and the greatest deflection

$$v_0' = -\frac{Wa^2}{6EI}(3l + 2a) \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

(i) *Simple beam, load uniformly distributed over a portion of the span* (Fig. 128).—Let a = the distance from the left support and b = the distance from the right support to the center of the load, which is uniformly distributed over the distance d. Assume $a > b$ and note that $W = wd$.

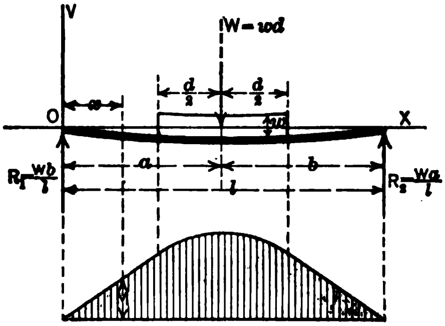


FIG. 128.

For values of x from 0 to $a - \frac{d}{2}$,

$$M = \frac{Wb}{l}x. \qquad (76)$$

Substituting in the differential equation and integrating

$$EIi = \frac{Wbx^2}{2l} + EIi_1, \qquad (77)$$

where i_1 = the slope at the left support and hence EIi_1 = the constant of integration.

Integrating again,

$$EIv = \frac{Wbx^3}{6l} + EIi_1x, \qquad (78)$$

the constant of integration being equal to zero, since $v = 0$ when $x = 0$.

For values of x from $a - \frac{d}{2}$ to $a + \frac{d}{2}$,

$$M = \frac{Wb}{l}x - \frac{w}{2}\left(x - a + \frac{d}{2}\right)^2 \qquad (79)$$

and hence by integration

$$EIi = \frac{Wbx^2}{2l} - \frac{w}{6}\left(x - a + \frac{d}{2}\right)^3 + c_3. \qquad (80)$$

To determine c_3 observe that when $x = a - \frac{d}{2}$ the values of the slope given by equations (77) and (80) are identical; and solving for c_3 we obtain

$$c_3 = EIi_1.$$

Substituting the value of c_3 in (80) and integrating

$$EIv = \frac{Wbx^3}{6l} - \frac{w}{24}\left(x - a + \frac{d}{2}\right)^4 + EIi_1x + c_4. \qquad (81)$$

Observing that the values of v given by (78) and (81) are identical when $x = a - \frac{d}{2}$ and solving for c_4, we obtain $c_4 = 0$.

For values of x from $a + \frac{d}{2}$ to l,

$$M = \frac{Wb}{l} x - W(x - a); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

and hence by integration

$$EIi = \frac{Wbx^2}{2l} - \frac{W}{2}(x - a)^2 + c_5. \quad . \quad . \quad . \quad . \quad . \quad (83)$$

Observing that the values of i given by (80) and (83) are identical when $x = a + \frac{d}{2}$, we obtain, by substituting the values of x and c_3, equating and cancelling equal terms,

$$-\frac{wd^3}{6} + EIi_1 = -\frac{Wd^2}{8} + c_5,$$

whence

$$c_5 = EIi_1 - \frac{Wd^2}{24}.$$

Substituting the value of c_5 in (83) and integrating

$$EIv = \frac{Wbx^3}{6l} - \frac{W}{6}(x - a)^3 - \frac{W\,d^2x}{24} + EIi_1x + c_6. \quad . \quad . \quad (84)$$

Observing that values of v given by (81) and (84) are identical when $x = a + \frac{d}{2}$ and substituting the values of x and c_3, equating and cancelling terms,

$$-\frac{wd^4}{24} = -\frac{Wd^3}{48} - \frac{W\,d^2a}{24} - \frac{Wd^3}{48} + c_6$$

and

$$c_6 = \frac{W\,d^2a}{24}.$$

To obtain the value of i_1, note that $v = 0$ when $x = l$ and hence from (84),

$$\begin{aligned} EIi_1 &= -\frac{Wbl}{6} + \frac{W}{6l}(l - a)^3 + \frac{Wd^2}{24} - \frac{W\,d^2a}{24\,l} \\ &= -\frac{Wb}{6l}(l^2 - b^2) + \frac{Wd^2}{24\,l}(l - a) = -\frac{Wb}{24\,l}[4(l^2 - b^2) - d^2]. \quad (85) \end{aligned}$$

Substituting the value of EIi_1 in the foregoing equations, the general equations for slope and deflection are as follows:

For values of x from 0 to $a - \frac{d}{2}$,

$$EIi = \frac{Wb}{24\,l}\{12\,x^2 - 4(l^2 - b^2) + d^2\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

$$EIv = \frac{Wbx}{24\,l}\{4\,x^2 - 4(l^2 - b^2) + d^2\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

For values of x from $a - \frac{d}{2}$ to $a + \frac{d}{2}$,

$$EIi = \frac{Wb}{24\,l}\{12\,x^2 - 4\,(l^2 - b^2) + d^2\} - \frac{W}{6\,d}\left(x - a + \frac{d}{2}\right)^3, \quad . \quad (88)$$

$$EIv = \frac{Wbx}{24\,l}\{4\,x^2 - 4\,(l^2 - b^2) + d^2\} - \frac{W}{24\,d}\left(x - a + \frac{d}{2}\right)^4 . \; . \quad (89)$$

For values of x from $a + \frac{d}{2}$ to l,

$$EIi = \frac{Wb}{24\,l}\{12\,x^2 - 4\,(l^2 - b^2) + d^2\} - \frac{W}{2}\,(x - a)^2 - \frac{Wd^2}{24}, \quad . \; . \; . \quad (90)$$

$$EIv = \frac{Wbx}{24\,l}\{4\,x^2 - 4\,(l^2 - b^2) + d^2\} - \frac{W}{6}\,(x - a)^3 - \frac{Wd^2}{24}\,(x - a). \quad (91)$$

The value of x for the point at which the greatest deflection occurs can be determined from the slope equations. Since $a > b$, to determine the greatest deflection it will be necessary to find the value of x for which the slope is zero by placing either equation (86) or (88) equal to zero. The one of these two equations which will give the solution will depend upon the distribution of the load. While ordinarily the equation to use can be determined from an inspection of the loading, in some cases it may be necessary to make a trial solution with one equation and if this fails to use the other. In either case, owing to the complexities of the equations, it is not worth while to attempt a solution in the algebraic form. The numerical values of the constants should be substituted first, after which a solution can easily be obtained. When the equation is of the third degree, as (88), a graphical solution, obtained by plotting values of EIi for different values of x, can be made to give the value of x for which $EIi = 0$ with sufficient accuracy for all ordinary cases.

When the load is considered as concentrated at a single point, $d = 0$ and equations (86), (87), (90) and (91) readily reduce to the forms in equations (40), (41), (42) and (43), Case (f).

Symmetrical loading.—When the loading of the beam is symmetrical with respect to the center of the span, $a = b$ and the greatest deflection will occur at the middle of the span, where $x = a = b = \frac{l}{2}$.

Substituting in (89) and reducing

$$v_0 = -\frac{W}{384\,EI}\,[8\,l^3 - 4\,ld^2 + d^3] . \; . \; . \; . \; . \; . \quad (92)$$

When the load is uniformly distributed over the entire span $d = l$ and equations (88), (89) and (92) will readily reduce to the form of equations (28), (29) and (31), Case (e).

99. Slope and Deflection, Loads Considered Simultaneously.

— In cases in which the load system is more complex than in those illustrated in Art. (98) the general method of procedure is the same as that illustrated in Case (i) of that article.

The beam is divided into as many parts as are necessary to obtain the general formulas for bending moments, throughout the length. These values are substituted in the differential equation of the elastic curve and two integrations for each division give the general formulas for slope and deflection.

The constants of integration are determined from the conditions of continuity of the elastic curve and other conditions, depending on the way in which the beam is supported. It is evident that the number of conditions must be the same as the number of unknown constants.

The greatest slope will be at one of the ends of the beam, or at a point of inflexion.

The greatest deflection will be at one end of the beam, or at a point of zero slope.

When the system of loading is complex the solution for any particular case can be simplified considerably by writing the values of the bending moment for the different divisions of the beam in terms of the numerical values of the loads and the variable x, as in the following case (Fig. 129) which is given as an illustration of the method of solving a numerical problem.

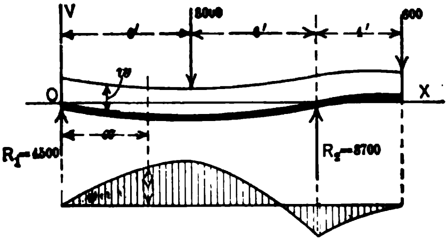


FIG. 129.

The beam is subjected to a uniformly distributed load of intensity $w = 600$ lbs. per ft. and the two concentrated loads of 3000 lbs. and 600 lbs. The supporting forces are $R_1 = 4500$ lbs. and $R_2 = 8700$ lbs.

For values of x from 0 to 6,

$$M = 4500\,x - 300\,x^2, \quad \ldots \ldots \ldots \quad (1)$$

$$EIi = 2250\,x^2 - 100\,x^3 + c, \quad \ldots \ldots \quad (2)$$

where $c = EIi_1$, i_1 being the slope at the left hand support.

$$EIv = 750\,x^3 - 25\,x^4 + EIi_1\,x + c_1, \quad \ldots \ldots \quad (3)$$

where the constant of integration $c_1 = 0$.

For values of x from 6 to 12,

$$M = 4500\,x - 300\,x^2 - 3000\,(x - 6), \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

$$EIi = 2250\,x^2 - 100\,x^3 - 1500\,(x - 6)^2 + c_2, \quad . \; . \; . \; . \quad (5)$$

where, from the condition of continuity when $x = 6$, $c_2 = EIi_1$.

$$EIv = 750\,x^3 - 25\,x^4 - 500\,(x - 6)^3 + EIi_1x + c_3, \quad . \; . \; . \quad (6)$$

where, from the condition of continuity when $x = 6$, $c_3 = 0$. When $x = 12$, $v = 0$ and hence, from (6),

$$EIi_1 = -\,55{,}800. \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

For values of x from 12 to 16,

$$M = -\,600\,(16 - x) - 300\,(16 - x)^2, \quad . \; . \; . \; . \; . \; . \quad (8)$$

$$EIi = 300\,(16 - x)^2 + 100\,(16 - x)^3 + c_4, \quad . \; . \; . \; . \quad (9)$$

where, from the condition of continuity when $x = 12$, $c_4 = 30{,}200$.

$$EIv = -\,100\,(16 - x)^3 - 25\,(16 - x)^4 + 30{,}200\,x + c_5, \; . \; . \quad (10)$$

where, since $v = 0$, when $x = 12$, $c_5 = -349{,}600$.

Putting (2) equal to zero and solving for x,

$$x = 5.78 \text{ ft.},$$

and hence, to find the greatest deflection between the supports, put $x = 5.78$ in (3) which gives

$$EIv_0 = 750 \times \overline{5.78}^3 - 25 \times \overline{5.78}^4 - 55{,}800 \times 5.78 = -\,205{,}600, \quad . \quad (11)$$

from which the value of v_0 can be easily found when E and I are known.

The value of EIi_1 for the left support is given by equation (7) and the value for the right support can be found by putting $x = 12$ in either (5) or (9), which will give

$$EIi_2 = 41{,}400. \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

For the right end of the beam, putting $x = 16$ in (9), we obtain

$$EIi' = 30{,}200 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

and putting $x = 16$ in (10),

$$EIv' = 133{,}600, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

which indicates that the deflection at the right hand end is upward but of less magnitude than the value of v_0 between the supports.

Note. — It should be observed that in the foregoing equations the linear unit is the foot and hence E represents the modulus of elasticity expressed in lbs./ft.^{2}, I the moment of inertia expressed in (ft.)^4 and v_0 the greatest deflection expressed in ft.

100. Slope and Deflection, Loads Considered Separately. — If the system of loads acting on a beam can be separated into one or more of the elementary systems given in Art. (98), the slope and deflection at any point may be computed by using the general formulas; and obtaining the slope and deflection at the point due to each elementary system separately and then adding together the results.

The method is based upon the same principle as that involved in determining shearing forces and bending moments by combining the results, obtained by considering each load as acting alone (Art. 75). According to that principle the bending moment at any section of a beam, subjected to any system of loading, may be represented by the expression

$$M = M_1 + M_2 + M_3 + \cdots, \qquad (1)$$

where M_1, M_2, M_3, etc., are the bending moments at the section due to each of the loads, or parts of the load system, acting on the beam.

Hence the general slope equation for any point in the beam may be written

$$\begin{aligned} EIi &= \int M\,dx = \int (M_1 + M_2 + M_3 + \cdots)\,dx \\ &= \int M_1\,dx + \int M_2\,dx + \int M_3\,dx + \cdots \\ &= EIi_1 + EIi_2 + EIi_3 + \cdots, \qquad (2) \end{aligned}$$

where i_1, i_2, i_3, etc., evidently represent the values of the slope at the point due to each part of the load system acting separately.

Similarly,

$$\begin{aligned} EIv &= EI\int i\,dx = EI\int (i_1 + i_2 + i_3 + \cdots)\,dx \\ &= EI\int i_1\,dx + EI\int i_2\,dx + EI\int i_3\,dx + \cdots \\ &= EIv_1 + EIv_2 + EIv_3 + \cdots, \qquad (3) \end{aligned}$$

where v_1, v_2, v_3, etc., represent values of the deflection at the point due to each part of the load system acting separately.

As a simple illustration, the deflection at the free end of a cantilever beam, due to a load W_1, concentrated at the end, and a uniformly distributed load W_2, may be found by adding (algebraically) the deflections due to each load separately, as given by the formulas (7) and (17) (Art. 98), giving for the total deflection

$$v_0 = -\frac{W_1 l^3}{3EI} - \frac{W_2 l^3}{8EI}. \qquad (4)$$

Similarly, the deflection at any point at a distance x from the fixed end can be found by adding (5) and (15) (Art. 98), giving

$$v = -\frac{W_1 x^2}{6EI}(3l - x) - \frac{W_2}{2lEI}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right). \qquad (5)$$

The slopes due to the combined loading may be found in a similar manner, by combining (6) with (16) and (4) with (14) (Art. 98), giving for the slope at the free end

$$i_0 = -\frac{W_1 l^2}{2EI} - \frac{W_2 l^2}{6EI}, \quad \ldots \ldots \quad (6)$$

and for the slope at any section

$$i = -\frac{W_1 x}{2EI}(2l - x) - \frac{W_2}{2lEI}\left(l^2 x - lx^2 + \frac{x^3}{3}\right). \quad . \quad . \quad (7)$$

101. Built-in Beams, or Beams with Fixed Ends. — Built-in, or beams with "fixed ends," so called, are beams that are held in such a manner that the slope at each support is equal to zero. Such a beam is rarely met with in practice but cases occur which approximate these conditions.

If a section is taken through such a beam just inside of either support the resultant of the stress on the section will consist of a couple, which is the resultant of the normal stress, and a vertical force, which is the resultant of the shearing stress; and the fundamental difference between the force system, acting on such a beam, and that on a simple beam loaded in the same manner, is in the terminal couples M_1 and M_2, acting as indicated (Fig. 130).

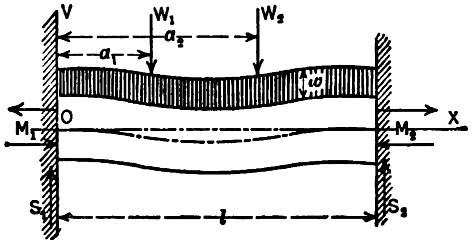


FIG. 130.

When the supports are at or near the same level, the effect of these couples will be to reduce the magnitude of the greatest bending moment and the greatest deflection below the values which would exist if the beam were supported as a simple beam. Hence *a beam with the ends securely fastened will have greater strength and stiffness than if it is supported as a simple beam.*

When the loading is symmetrical with respect to the middle of the span the couples M_1 and M_2 are of equal magnitude and opposite sign and the shearing forces at the ends are equal and of the same magnitude as if the beam were supported as a simple beam. When the loading is not symmetrical, both the bending moments and shearing forces at the ends are unequal; and the values of the shearing forces are not the same as in a simple beam loaded in the same manner.

The value of the shearing force at any section, at a distance x from the origin, will evidently be represented by the expression

$$S = S_1 - \sum W - \int w\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the bending moment at the section, by the expression

$$M = M_1 + S_1 x - \sum W(x - a) - \int\int w\,dx\,dx \text{ (Art. 73)}.. \quad (2)$$

To determine the values of S_1 and M_1 for any particular case it will in general be necessary to deduce the formulas for the slope and deflection in terms of the value of M, as expressed in equation (2), by integration of the differential equation of the elastic curve (Art. 96).

In a few simple cases a solution can be made by the addition of the slope and deflection equations as indicated (Art. 100).

The following cases are given as illustrations of the methods which may be used. In each case the supports are on the same level and M_1 and M_2 represent the couples and S_1 and S_2 the resultant shearing stresses acting on the end sections.

(a) *Beam fixed at the ends, single concentrated load at the center of the span* (Fig. 131). — From the symmetry $M_1 = -M_2$ and, taking moments about the right support,

$$S_1 l - \frac{Wl}{2} + M_1 - M_2 = 0$$

and hence

$$S_1 = \frac{W}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The load system may be divided into two parts:

(1) Including the load W and the vertical end reactions;

(2) Including the terminal couples M_1 and M_2.

The slope at O due to the load W and vertical reactions alone would be

$$i_1' = -\frac{Wl^2}{16\,EI} \text{ (Art. 98)},$$

and the slope at O due to the uniform bending, produced by the terminal couples, would be

$$i_1'' = -\frac{M_1 l}{2\,EI} \text{ (Art. 97)}.$$

Adding together (Art. 100) we have, since the slope under the combined loading is zero,

$$0 = -\frac{M_1 l}{2EI} - \frac{Wl^2}{16EI},$$

whence

$$M_1 = -\frac{Wl}{8}. \quad \ldots \ldots \quad (4)$$

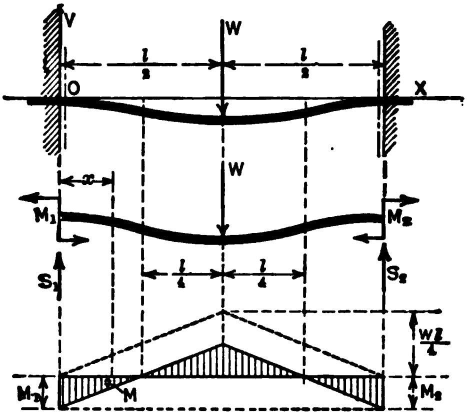


FIG. 131.

The general equations for the slope and deflection may be found in a similar manner, by adding the expressions already deduced for the two parts of the load system; or, by integration, from the differential equation of the elastic curve as follows:

For values of x from 0 to $\frac{l}{2}$,

$$S = \frac{W}{2} \quad \ldots \ldots \quad (5)$$

and

$$M = M_1 + S_1 x = -\frac{Wl}{8} + \frac{Wx}{2} = -\frac{W}{8}(l - 4x); \quad \ldots \quad (6)$$

and by integration

$$EIi = -\frac{W}{8}(lx - 2x^2) \quad \ldots \ldots \quad (7)$$

and

$$EIv = -\frac{W}{8}\left(\frac{lx^2}{2} - \frac{2x^3}{3}\right), \quad \ldots \ldots \quad (8)$$

both constants of integration being equal to zero.

Owing to the symmetry, these equations are evidently all that are required to find the slope and deflection throughout the length of the beam.

When $x = \frac{l}{2}$, we obtain from (6)

$$M' = \frac{Wl}{8}, \quad \ldots \ldots \quad (9)$$

the greatest positive value of the bending moment, and from (8)

$$v_0 = -\frac{Wl^3}{192\,EI}, \quad \ldots\ldots\ldots \quad (10)$$

the greatest deflection in the beam.

To determine the greatest slope note that $M = 0$ when $x = \frac{l}{4}$ and, by substituting in (7), we obtain

$$i_0 = -\frac{Wl^2}{64\,EI}. \quad \ldots\ldots\ldots \quad (11)$$

From the symmetry there are evidently two points of inflexion, at a distance $\frac{l}{4}$ on either side of the middle of the span.

(b) *Beam fixed at the ends, load uniformly distributed over entire span* (Fig. 132). — Let $W = wl$ equal the total load on the beam.

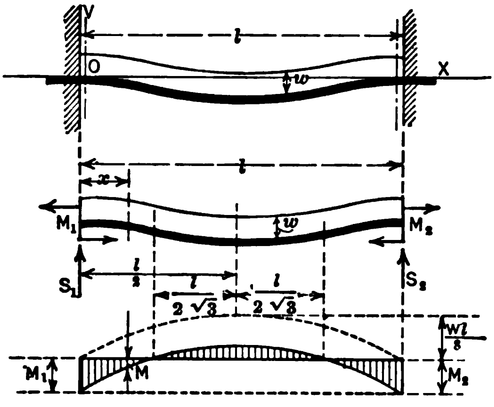


FIG. 132.

Then $M_1 = - M_2$ and

$$S_1 l - \frac{Wl}{2} + M_1 - M_2 = 0$$

and

$$S_1 = \frac{W}{2}. \quad \ldots\ldots\ldots \quad (12)$$

Dividing the load system into two parts, (1) the uniformly distributed load and vertical reactions and (2) the terminal couples M_1 and M_2, the slope at O due to (1) acting alone would be

$$i_1' = -\frac{Wl^2}{24\,EI} \text{ (Art. 98)}$$

and that due to (2) alone would be

$$i_1'' = -\frac{M_1 l}{2\,EI}.$$

Adding together and solving for M_1,

$$M_1 = -\frac{Wl}{12}. \quad \ldots\ldots\ldots \quad (13)$$

Proceeding as before: for values of x for the entire span,

$$S = \frac{W}{2} - wx = \frac{w}{2}(l - 2x), \quad \ldots \ldots \quad (14)$$

$$M = M_1 + S_1 x - \frac{wx^2}{2} = -\frac{Wl}{12} + \frac{W}{2}x - \frac{wx^2}{2}$$

$$= -\frac{w}{12}(l^2 - 6lx + 6x^2), \quad \ldots \ldots \quad (15)$$

$$EIi = -\frac{w}{12}(l^2x - 3lx^2 + 2x^3) \quad \ldots \ldots \quad (16)$$

and

$$EIv = -\frac{w}{12}\left(\frac{l^2x^2}{2} - lx^3 + \frac{x^4}{2}\right), \quad \ldots \ldots \quad (17)$$

both constants of integration being equal to zero.

When $x = \frac{l}{2}$, we obtain for the greatest positive bending moment

$$M' = \frac{wl^2}{24} = \frac{Wl}{24} \quad \ldots \ldots \quad (18)$$

and for the greatest deflection

$$v_0 = -\frac{wl^4}{384EI} = -\frac{Wl^3}{384EI}. \quad \ldots \ldots \quad (19)$$

Placing (15) equal to zero, we obtain for the points of inflexion

$$x = \frac{l}{2} \pm \frac{l}{2\sqrt{3}} = \frac{l}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right), \quad \ldots \ldots \quad (20)$$

showing that the points of inflexion are at a distance

$$\frac{l}{2\sqrt{3}} = 0.2887\,l$$

from the middle of the span.

At the left hand point of inflexion the value of the slope, obtained by substituting the above value of x in (16), will be equal to

$$i_0 = -\frac{wl^3}{72\sqrt{3}EI} = -\frac{Wl^2}{72\sqrt{3}EI} \quad \ldots \ldots \quad (21)$$

(c) *Beam fixed at the ends, two equal concentrated loads, equidistant from the ends* (Fig. 133). — Let W = the magnitude of each load and a = the distance between the load and the end of the beam. Since the loading is symmetrical $M_1 = -M_2$ and, evidently,

$$S_1 = W. \quad \ldots \ldots \quad (22)$$

Dividing the load system into two parts, (1) the loads W and the vertical reactions and (2) the terminal couples M_1 and M_2, the slope at O due to (1) acting alone would be

$$i_1' = -\frac{Wa}{2EI}(l - a) \text{ (Art. 98)}$$

and that due to (2) alone would be

$$i_1'' = -\frac{M_1 l}{2EI}.$$

Adding together and solving for M_1,

$$M_1 = -\frac{Wa}{l}(l-a). \quad \ldots \ldots \quad (23)$$

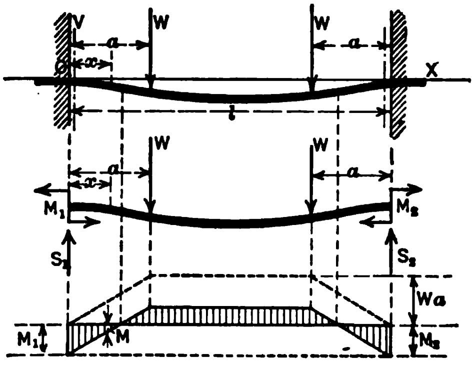


FIG. 133.

Proceeding as before: for values of x from 0 to a,

$$S = W, \quad \ldots \ldots \quad (24)$$

$$M = M_1 + S_1 x = Wx - \frac{Wa}{l}(l-a), \quad \ldots \ldots \quad (25)$$

$$EIi = \frac{Wx^2}{2} - \frac{Wax}{l}(l-a) \quad \ldots \ldots \quad (26)$$

and

$$EIv = \frac{Wx^3}{6} - \frac{Wax^2}{2l}(l-a), \quad \ldots \ldots \quad (27)$$

both constants of integration being equal to zero.

For values of x from a to $l-a$,

$$S = 0, \quad \ldots \ldots \quad (28)$$

$$M' = M_1 + S_1 x - W(x-a) = \frac{Wa^2}{l}, \quad \ldots \ldots \quad (29)$$

which is evidently the greatest positive bending moment,

$$EIi = \frac{Wa^2x}{l} - \frac{Wa^2}{2}, \quad \ldots \ldots \quad (30)$$

where the constant $-\frac{Wa^2}{2}$ is easily determined from the condition of continuity at $x = a$, and

$$EIv = \frac{Wa^2x^2}{2l} - \frac{Wa^2x}{2} + \frac{Wa^3}{6}, \quad \ldots \ldots \quad (31)$$

where the constant $\frac{Wa^3}{6}$ is readily determined from the continuity.

The slope and deflection at any point between $x = l - a$ and $x = l$ can evidently be obtained from (26) and (27) by taking the origin at the right hand end of the beam. The greatest deflection will occur when $x = \frac{l}{2}$ and substituting in (31) and reducing

$$v_0 = -\frac{Wa^2}{24EI}(3l - 4a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

The points of inflexion will be located between the fixed ends and the loads W. The distance of these points from the supports can be easily obtained by putting (25) equal to zero.

(d) ***Beam fixed at ends, single concentrated load not at center*** (Fig. 134). — In this case the terminal couples are not equal and both S_1 and M_1 must be determined from the properties of the elastic curve. Assume $a > b$.

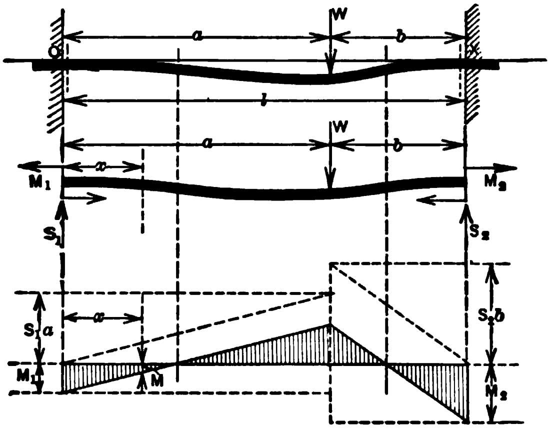


Fig. 134.

For values of x from 0 to a,

$$S = S_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

$$M = M_1 + S_1x_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

$$EIi = M_1x + \frac{S_1x^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

$$EIv = \frac{M_1x^2}{2} + \frac{S_1x^3}{6}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (36)$$

both constants being equal to zero, since $i = 0$ and $v = 0$ when $x = 0$.

For values of x from a to l,

$$S = S_1 - W, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (37)$$

$$M = M_1 + S_1x - W(x - a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (38)$$

$$EIi = M_1x + \frac{S_1x^2}{2} - \frac{W}{2}(x - a)^2 + c_3, \quad . \quad . \quad . \quad . \quad . \qquad (39)$$

where, from the condition of continuity when $x = a$, it is evident that $c_3 = 0$,

$$EIv = \frac{M_1x^2}{2} + \frac{S_1x^3}{6} - \frac{W}{6}(x - a)^3 + c_4, \quad . \quad . \quad . \quad . \quad . \qquad (40)$$

where, from the condition of continuity, $c_4 = 0$ also.

The values of M_1 and S_1 may now be determined from the conditions $i = 0$ and $v = 0$ when $x = l$. Making the substitutions in (39) and (40),

$$2\,M_1 l + S_1 l^2 - Wb^2 = 0,$$
$$3\,M_1 l^2 + S_1 l^3 - Wb^3 = 0.$$

Solving for M_1 and S_1,

$$M_1 = -\frac{Wab^2}{l^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

$$S_1 = \frac{Wb^2}{l^3}(l + 2\,a) = \frac{Wb^2}{l^3}(b + 3\,a). \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Substituting the values of S_1 and M_1 in (33), (34), (35) and (36) and reducing, we have for the general formulas, for values of x from 0 to a,

$$S = \frac{Wb^2}{l^3}(l + 2\,a), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$M = \frac{Wb^2}{l^3}[(l + 2\,a)\,x - al], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$EIi = \frac{Wb^2x}{2\,l^3}[(l + 2\,a)\,x - 2\,al], \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

$$EIv = \frac{Wb^2x^2}{6\,l^3}[(l + 2\,a)\,x - 3\,al]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The greatest positive bending moment will be obtained when $x = a$ and, substituting in (44), its value is found to be equal to

$$M' = \frac{2\,Wa^2b^2}{l^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Placing (45) equal to zero, we obtain, for the point of greatest deflection,

$$x = \frac{2\,al}{l + 2\,a}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

and substituting in (46) we obtain, for the greatest deflection,

$$v_0 = -\frac{2\,Wa^3b^2}{3\,(l + 2\,a)^2\,EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The deflection at the load W, where $x = a$, is equal to

$$v_a = -\frac{Wa^3b^3}{3\,l^3EI}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and, when $x = \frac{l}{2}$, we have for the deflection at the middle of the span,

$$v_c = -\frac{Wb^2}{48\,EI}(4\,a - l). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

Putting (44) equal to zero, we obtain for the point of inflexion

$$x = \frac{al}{l + 2\,a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

and, substituting in (45), the slope at this point is equal to

$$i' = -\frac{Wa^2b^2}{2\,l(l + 2\,a)\,EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

With the exception of equations (48), (49) and (51), all of the equations will evidently apply to the portion of the beam between the load and the right support, provided the origin is taken at the right end, with x positive to the left and the dimensions a and b reversed.

For example, the expression for the bending moment at the right end will be

$$M_2 = -\frac{Wa^2b}{l^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (54)$$

and that for the shearing force at the right end,

$$S_2 = \frac{Wa^2}{l^3}(a + 3b) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (55)$$

102. Determination of Supporting Forces for Beams which are Statically Indeterminate. — The method employed in Art. (101), by means of which the equations for slope and deflection are made to furnish the conditions, in addition to the laws of equilibrium, which are needed to determine shearing forces and bending moments in beams fixed at the ends, may be extended to apply to other statically indeterminate cases.

By use of the formulas for the cases in Art. (98) the method may be used to determine the bending moments and shearing forces in any beam subjected to concentrated and uniformly distributed loads. The beam may be supported at any number of points and one or both ends may be either supported or fixed. The method is not convenient to use, however, except in a few simple cases, such as the following:

(a) *Beam fixed at one end and supported at the other and loaded uniformly* (Fig. 135). — Let $W = wl$ equal the total load and assume that the supports are on the same level. Consider the beam as a cantilever subjected to the two load systems: (1) the uniformly distributed load $W = wl$, (2) the right hand supporting force R_2.

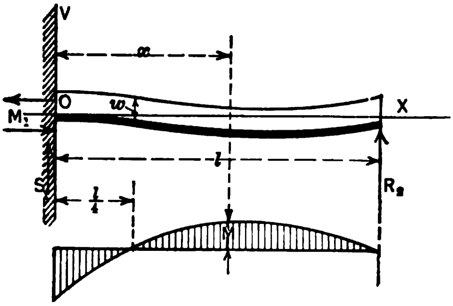


Fig. 135.

The deflection at the right end, if the load W were acting alone, would be

$$v_2' = -\frac{Wl^3}{8EI}$$

and the deflection, if the force R_2 were acting alone, would be

$$v_2' = \frac{R_2l^3}{3EI}.$$

As the supports are on the same level, we obtain, by adding,

$$0 = -\frac{Wl^3}{8EI} + \frac{R_2l^3}{3EI},$$

whence

$$R_2 = \frac{3W}{8}. \quad \ldots\ldots\ldots\ldots \quad (1)$$

Therefore the bending moment at the fixed end will be equal to

$$M_1 = R_2l - \frac{wl^2}{2} = \frac{3Wl}{8} - \frac{Wl}{2} = -\frac{Wl}{8}. \quad \ldots\ldots \quad (2)$$

and the shearing force at the fixed end will be equal to

$$S_1 = W - \frac{3W}{8} = \frac{5W}{8}. \quad \ldots\ldots\ldots \quad (3)$$

Hence, for any section at a distance x to the right of the origin,

$$S = \frac{5W}{8} - wx = \frac{w}{8}(5l - 8x), \quad \ldots\ldots\ldots \quad (4)$$

$$M = -\frac{Wl}{8} + \frac{5W}{8}x - \frac{wx^2}{2} = \frac{w}{8}(5lx - 4x^2 - l^2), \quad \ldots \quad (5)$$

$$EIi = \frac{w}{8}\left(\frac{5lx^2}{2} - \frac{4x^3}{3} - l^2x\right) \quad \ldots\ldots\ldots\ldots \quad (6)$$

and

$$EIv = \frac{w}{8}\left(\frac{5lx^3}{6} - \frac{x^4}{3} - \frac{l^2x^2}{2}\right), \quad \ldots\ldots\ldots\ldots \quad (7)$$

both constants of integration being equal to zero.

The shearing force will be equal to zero when

$$x = \frac{5}{8}l \quad \ldots\ldots\ldots\ldots \quad (8)$$

and the greatest positive bending moment will be equal to

$$M' = \frac{9}{128}wl^2 = \frac{9}{128}Wl. \quad \ldots\ldots\ldots \quad (9)$$

The slope will equal zero when

$$x = 0.58\,l \text{ (nearly)} \quad \ldots\ldots\ldots \quad (10)$$

and the greatest deflection will be equal to

$$v_0 = -\frac{0.0054\,wl^4}{EI} = -\frac{0.0054\,Wl^3}{EI} \text{ (nearly)}. \quad \ldots \quad (11)$$

For the point of inflexion

$$x = \frac{l}{4} \quad \ldots\ldots\ldots\ldots \quad (12)$$

and the slope at this point will be equal to

$$i_0 = -\frac{11\,wl^3}{768\,EI} = -\frac{11\,Wl^2}{768\,EI}. \qquad (13)$$

The greatest slope will occur at the right end, where $x = l$, and will be equal to

$$i_1 = \frac{wl^3}{48\,EI} = \frac{Wl^2}{48\,EI}. \qquad (14)$$

(b) *Beam supported at three points on the same level, with the two spans equal, and subjected to a uniformly distributed load* (Fig. 136). — Evidently the slope of the beam over the middle support will be equal to zero and the elastic curve will be symmetrical with respect to the vertical axis at the middle support; and, moreover, the curve for each span will be of the same form as the elastic curve of the beam in Case (a).

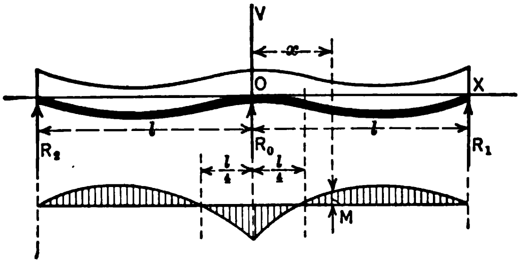


FIG. 136.

Hence, if we let l = the length of each span and $W = wl$ equal the total load in each span, all of the formulas for Case (a) will apply directly to the right hand span; and, for all points in the left hand span, the values of bending moments, slopes and deflections can be determined from symmetry.

The supporting forces at the ends will evidently be equal to

$$R_1 = R_2 = \frac{3\,wl}{8}. \qquad (15)$$

and the support at the center will be equal to

$$R_0 = \frac{5\,wl}{4}. \qquad (16)$$

Another solution of the problem may be made by treating the beam as having a single span $2\,l$ and subjected to the load system consisting of: (1) a uniformly distributed load of intensity w and (2) an unknown concentrated load R_0 at the middle of the span, acting in the direction opposite to the distributed load and of sufficient magnitude to make the deflection at its point of application equal to zero. Applying the formulas (Art. 98), the deflection at the center of the span, due to the uniformly distributed load acting alone, would be equal to

$$v_0' = -\frac{5\,(2\,wl)\,(2\,l)^3}{384\,EI} = -\frac{5\,wl^4}{24\,EI} \qquad (17)$$

and that due to the concentrated force R_0 acting alone would be equal to

$$v_0'' = \frac{R_0 l^3}{6EI} \cdot \qquad (18)$$

Since the resultant deflection at the middle support is equal to zero,

$$v_0' + v_0'' = \frac{R_0 l^3}{6EI} - \frac{5\,wl^4}{24\,EI} = 0, \qquad (19)$$

and hence

$$R_0 = \frac{5}{4} wl,$$

as before. The rest of the solution would be the same as that for Case (a).

The latter method of solving the problem can evidently be applied when the middle support is above or below the level of the other two, by placing (19) equal to the difference in level of the supports, observing that the difference in level must be positive when the middle support is higher than the other two and negative when lower.

(c) *Beam fixed at one end and supported at the other, subjected to a uniformly distributed and a concentrated load* (Fig. 137). — A numerical solution of this case will be given, assuming that $w = 1000$ lbs. per ft. and that the supports are on the same level.

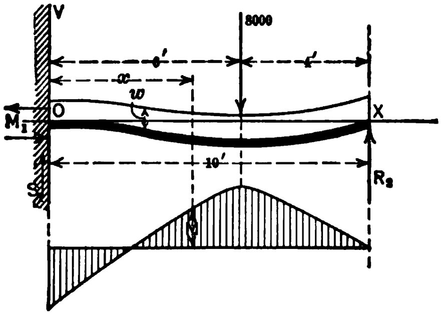


FIG. 137.

Treat the beam as a cantilever and apply the condition that the total deflection at the right support, due to (1) the loads on the beam and (2) the supporting force R_2, is equal to zero. Then, by substituting in formulas (12), (17) and (7) (Art. 98) and adding the deflections, we obtain

$$-\frac{8000 \times 36\,(12 + 12)}{6EI} - \frac{10{,}000 \times 1000}{8EI} + \frac{R_2 \times 1000}{3EI} = 0.$$

$$\therefore R_2 = 7206 \text{ lbs.}$$

Observing that, according to the usual system of signs, the shearing force at the left of the support R_2 is negative, and letting S_1 = the shearing force at the right of the fixed support,

$$S_1 - 8000 - 10{,}000 = -7206$$

and hence

$$S_1 = 10{,}794 \text{ lbs.}$$

Following again the usual system of signs, the bending moment at the fixed support will be equal to

$$M_1 = 7206 \times 10 - 8000 \times 6 - 10{,}000 \times 5 = -25{,}940 \text{ ft. lbs.}$$

The general equations, for values of x from 0 to 6, will be

$$S = 10{,}794 - 1000\,x, \quad \ldots \ldots \ldots \quad (1)$$

$$M = -25{,}940 + 10{,}794\,x - 500\,x^2, \quad \ldots \ldots \quad (2)$$

$$EIi = -25{,}940\,x + 5397\,x^2 - \frac{500\,x^3}{3} \quad \ldots \ldots \quad (3)$$

and

$$EIv = -12{,}970\,x^2 + 1799\,x^3 - \frac{125\,x^4}{3} \ldots \ldots \quad (4)$$

The shearing force is evidently zero under the concentrated load and hence the greatest positive bending moment will be equal to

$$M' = -25{,}940 + 10{,}794 \times 6 - 500 \times 36 = 20{,}820 \text{ ft. lbs.}$$

Placing (3) equal to zero and solving for x, we obtain

$$x = 5.9 \text{ ft.},$$

and, substituting in (4), the value of EIv at the point of greatest deflection becomes equal to

$$EIv_0 = -132{,}000,$$

from which the value of v_0 for any given values of E and I may be determined.

Placing (2) equal to zero, the value of x for the point of inflexion is found to be

$$x = 2.8 \text{ ft.}$$

To obtain the greatest slope it would be necessary to deduce the general equations for values of x from 6 to 10, the greatest value of the slope being evidently obtained when $x = 10$. If the solution of (3) had given a value of $x > 6$ it would also have been necessary to use these equations to determine the greatest deflection.

Reference should be made to the note (Art. 99) in regard to the units for the values of E, I and v_0 in the preceding equations.

103. Deflection of Beams of Non-Uniform Cross Section. Beams of Uniform Strength.

— If the principles of the theory of bending are assumed to apply to beams in which the cross sections vary, the equations of the elastic curve may be obtained by integrating the differential equation (Art. 96), provided the value of $\frac{M}{I}$ can be expressed in integrable terms of x.

The following are given as illustrations of the application of the theory in the case of beams of uniform strength and of rectangular cross section (Art. 85), the same limitations being imposed as in the article referred to. The formulas obtained can be used to give approximate results only, inasmuch as for nearly all systems of loading it is impossible to construct a beam of exact uniform strength throughout its entire length.

(a) *Cantilever beam of uniform depth subjected to a concentrated load at the free end.* — Let W = the load, l = the length of the beam, I_0 = the moment of inertia and M_0 = the bending moment at the cross section at the fixed end. Fig. (120) may be taken to illustrate the beam and its bending moment diagram.

For a beam of uniform strength

$$f = \frac{Mc}{I} = \text{a constant.}$$

Hence in this case, since c = a constant, the value at any cross section of the quantity

$$\frac{M}{I} = \frac{M_0}{I_0} = \frac{-Wl}{I_0} = \text{a constant.} \quad \dots\dots \quad (1)$$

and, therefore, the equation of the elastic curve may be written

$$\frac{d^2v}{dx^2} = \frac{M}{EI} = \frac{-Wl}{EI_0}. \quad \dots\dots \quad (2)$$

Integrating

$$i = -\frac{Wlx}{EI_0} \quad \dots\dots \quad (3)$$

and

$$v = -\frac{Wlx^2}{2\,EI_0}, \quad \dots\dots \quad (4)$$

both constants being equal to zero.

The greatest slope and the greatest deflection will occur when $x = l$ and hence

$$i_0 = -\frac{Wl^2}{EI_0} \quad \dots\dots \quad (5)$$

and

$$v_0 = -\frac{Wl^3}{2\,EI_0}. \quad \dots\dots \quad (6)$$

(b) *Cantilever beam of uniform breadth, subjected to a concentrated load at the free end* (Fig. 120). — Following the notation adopted in the preceding case, we shall have

$$\frac{Mc}{I} = \frac{M_0c_0}{I_0};$$

and, if we let b = the breadth of the beam and h_0 = the depth of the cross section at the fixed end and h = the depth at any section at a distance x from the fixed end, this equation will reduce to

$$\frac{-6\,W(l-x)}{bh^2} = \frac{-6\,Wl}{bh_0^2}.$$

Solving for the value of the depth h at any point, we obtain

$$h = h_0\sqrt{\frac{l-x}{l}};$$

and hence at any cross section the value of

$$\frac{M}{I} = \frac{12\,M}{bh^3} = -\frac{12\,W(l-x)\,l^{\frac{3}{2}}}{bh_0^3\,(l-x)^{\frac{3}{2}}} = -\frac{Wl^{\frac{3}{2}}}{I_0\,(l-x)^{\frac{1}{2}}}. \quad \dots \quad (7)$$

Substituting in the differential equation,

$$\frac{d^2v}{dx^2} = \frac{M}{EI} = -\frac{Wl^{\frac{3}{2}}}{EI_0}(l-x)^{-\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and integrating

$$i = \frac{2\,Wl^{\frac{3}{2}}}{EI_0}(l-x)^{\frac{1}{2}} - \frac{2\,Wl^2}{EI_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where the constant of integration is equal to $-\frac{2\,Wl^2}{EI_0}$.

Integrating again

$$v = -\frac{4\,Wl^{\frac{3}{2}}}{3\,EI_0}(l-x)^{\frac{3}{2}} - \frac{2\,Wl^2x}{EI_0} + \frac{4\,Wl^3}{3\,EI_0}, \quad . \quad . \quad . \quad . \quad (10)$$

the constant of integration being equal to $\frac{4\,Wl^3}{3\,EI_0}$.

Substituting $x = l$ in (9) and (10) and reducing, the equations for the greatest slope and deflection become

$$i_0 = -\frac{2\,Wl^2}{EI_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and

$$v_0 = -\frac{2\,Wl^3}{3\,EI_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

(c) *Beam of uniform depth, supported at the ends and subjected to a single load concentrated at the middle of the span* (Fig. 123). — Let W = the load and l = the length of the span. The half of the beam on either side of the center will evidently be loaded under the same conditions as the cantilever beam in Case (a) and hence the width of the cross section will vary uniformly from zero at the support to a maximum at the center of the beam.

The following equations for the greatest slope and deflection can, therefore, be obtained by substituting $\frac{W}{2}$ for W and $\frac{l}{2}$ for l in equations (5) and (6), and making proper allowance for the signs, whence

$$i_0 = -\frac{Wl^2}{8\,EI_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$v_0 = -\frac{Wl^3}{32\,EI_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where I_0 evidently represents the moment of inertia of the cross section at the middle of the beam.

(d) *Beam of uniform breadth, supported at the ends and subjected to a single load concentrated at the middle of the span* (Fig. 123). — The two halves of the beam will be of the same form as the cantilever beam Case (b) and the equations for the greatest slope and deflection can be obtained from (11) and (12) by substituting $\frac{W}{2}$ for W and $\frac{l}{2}$ for l in the same manner as in the preceding case.

Hence

$$i_0 = -\frac{Wl^2}{4EI_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and

$$v_0 = -\frac{Wl^3}{24EI_0},$$

where I_0 = the moment of inertia of the middle cross section as before.

With a few exceptions the formulas for slope and deflections for rectangular beams of uniform strength can be determined for all simple systems of loading by the methods indicated in the preceding cases.

104. Greatest Deflection in Terms of the Greatest Bending Moment or the Greatest Fiber Stress. — The formula for the greatest deflection of a beam, subjected to any system of loading, can always be expressed in terms of the greatest bending moment by substituting for the load W in the formula its value in terms of M_0. Likewise by expressing W in terms of the greatest outside fiber stress, determined from the formula

$$f = \frac{M_0c}{I},$$

the formula for the greatest deflection can be expressed in terms of f.

For an illustration take the case of a simple beam, of length l, subjected to a uniformly distributed load W (Fig. 124). The greatest bending moment in this case is equal to

$$M_0 = \frac{Wl}{8},$$

and hence

$$W = \frac{8M_0}{l}$$

and the load, expressed in terms of the greatest fiber stress, is equal to

$$W = \frac{8fI}{lc}.$$

Substituting these values in the formula for the greatest deflection and disregarding signs we have

$$v_0 = \frac{5Wl^3}{384EI} = \frac{5M_0l^2}{48EI} = \frac{5fl^2}{48Ec}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

When the cross section is symmetrical with respect to the neutral axis, if we let h = the depth of the beam, equation (1) may be written

$$v_0 = \frac{5fl^2}{24Eh}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As another illustration we may take the case of the simple beam symmetrically loaded with two concentrated loads W (Fig. 126). In this case $M_0 = Wa$ and hence each load

$$W = \frac{M_0}{a};$$

and each load, expressed in terms of the greatest fiber stress, will be equal to

$$W = \frac{fI}{ac}.$$

Substituting the value in the formula for the greatest deflection, disregarding signs and reducing, we have

$$v_0 = \frac{M_0}{24EI}(3l^2 - 4a^2) = \frac{f}{24Ec}(3l^2 - 4a^2); \quad . \quad . \quad (3)$$

and for a cross section of depth h, symmetrical with respect to the neutral axis,

$$v_0 = \frac{f}{12Eh}(3l^2 - 4a^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Similar expressions can easily be obtained for all of the simple systems of loading.

105. Allowable Deflection of Floor Beams. — To provide against too great a deflection of the beams supporting a floor, when subjected to the full load that may be put upon them, it is customary to limit the greatest allowable deflection to some small fractional part of the span, such as

$$v_0 = \frac{l}{400}, \quad v_0 = \frac{l}{360}, \quad \text{or} \quad v_0 = kl, \quad . \quad . \quad . \quad (1)$$

where k = any small fractional part of the span.

In any particular case the size of the beam required to satisfy this condition may be found by solving the formula for the greatest deflection for the value of I. For example, if we limit to kl the greatest deflection of a beam supported at the ends and subjected to a uniformly distributed load W (Fig. 124), by substituting this value for v_0 in equation (31) (Art. 98) we shall obtain

$$kl = \frac{5\,Wl^3}{384\,EI}$$

and solving for I,

$$I = \frac{5\,Wl^2}{384\,Ek}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Having the value of I, the size of the beam of any cross section and material, required to satisfy the condition, may be readily found from a table of moments of inertia.

If the size of the beam, determined in this manner, is compared with that obtained from the formula

$$\frac{I}{c} = \frac{M_0}{f} = \frac{Wl}{8f}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

we shall find that, when the length of the span is greater than a certain definite amount, the size of the beam determined from equation (2) will be greater than that obtained from (3) and vice versa when the length of the span is less than the amount.

The length of span, for which both conditions will be satisfied simultaneously, can be obtained by combining (2) and (3) and solving for l, which will give

$$l = \frac{48\,Ekc}{5f}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When the cross section is symmetrical with respect to the neutral axis, the value obtained by substituting $v_0 = kl$ in formula (2) (Art. 104) for the deflection in terms of the greatest fiber stress, of the beam supported at the ends and subjected to a uniformly distributed load, is,

$$kl = \frac{5\,fl^2}{24\,Eh};$$

and solving for h,

$$h = \frac{5\,fl}{24\,Ek} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

A similar expression for the value of c when the cross section is not symmetrical with respect to the neutral axis may be obtained by substituting $v_0 = kl$, in equation (1) (Art. 104):

$$c = \frac{5\,fl}{48\,Ek}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Similar expressions for determining the depth of the beam in terms of the greatest fiber stress and the constant k for the other systems of loading discussed in the preceding articles may readily be obtained.

It is evident from the preceding that it is always possible to design the cross section of a beam, of any material subjected to any given system of loading, in such a manner that the greatest deflection and the greatest fiber stress shall have any predetermined values.

It is not always practicable to do this, but the determination of the value of h is frequently an aid in making the choice of the cross section for a given case. When, for example, the depth of the cross section of a beam supported at the ends and loaded uniformly is equal to the value given by equation (5), the required moment of inertia of the section may be determined, either from the formula for the section modulus in terms of the greatest fiber stress, or from the formula for the moment of inertia in terms of the greatest deflection. If, however, the depth h is less than the value given by (5), the required value of I must be determined from the deflection formula and, if h is greater than the value given by (5), the value of I must be determined from the fiber stress formula.

106. Summary — Formulas for Bending Moments and Deflections. — The table on page (224) comprises a summary of the formulas, for the more common cases discussed in Articles (74), (97), (98), (101) and (105), the values given for each case being the following:

(1) The greatest bending moment M_0 in terms of W.

(2) The greatest deflection v_0: (a) in terms of W; (b) in terms of the greatest bending moment M_0; (c) in terms of the greatest fiber stress f and the depth h of a symmetrical cross section.

(3) The depth h of a cross section, symmetrical with respect to the neutral axis, required to satisfy the two conditions: (a) the greatest deflection shall equal kl and (b) the greatest fiber stress shall equal f.

The numbers in the second column refer to the figures showing sketches of the loading and bending moment diagrams for each case. The algebraic signs are omitted.

Beams of uniform section.	Fig.	Values of M_0.	Values of v_0.			Values of λ.
			(a)	(b)	(c)	
Cantilever beam, single concentrated load at free end.	(120)	Wl	$\frac{Wl^3}{3\,EI}$	$\frac{M_0l^2}{3\,EI}$	$\frac{2\,fl^2}{3\,Eh}$	$\frac{2\,fl}{3\,Ek}$
Cantilever beam, uniformly distributed load.	(122)	$\frac{Wl}{2}$	$\frac{Wl^3}{8EI}$	$\frac{M_0l^2}{4\,EI}$	$\frac{fl^2}{2\,Eh}$	$\frac{fl}{2\,Ek}$
Cantilever beam, uniform bending moment M_0.	(119)	M_0	$\frac{M_0l^2}{2\,EI}$	$\frac{M_0l^2}{2\,EI}$	$\frac{fl^2}{Eh}$	$\frac{fl}{Ek}$
Simple beam, single concentrated load at middle of span.	(123)	$\frac{Wl}{4}$	$\frac{Wl^3}{48\,EI}$	$\frac{M_0l^2}{12\,EI}$	$\frac{fl^2}{6\,Eh}$	$\frac{fl}{6\,Ek}$
Simple beam, uniformly distributed load.	(124)	$\frac{Wl}{8}$	$\frac{5\,Wl^3}{384\,EI}$	$\frac{5\,M_0l^2}{48\,EI}$	$\frac{5\,fl^2}{24\,Eh}$	$\frac{5\,fl}{24\,Ek}$
Simple beam, single concentrated load at distance a from support, $a > b$.	(125)	$\frac{Wab}{l}$	$\frac{Wb\,(l^2-b^2)^{\frac{3}{2}}}{9\sqrt{3}\,lEI}$	$\frac{M_0\,(l^2-b^2)^{\frac{3}{2}}}{9\sqrt{3}\,aEI}$	$\frac{2\,f\,(l^2-b^2)^{\frac{3}{2}}}{9\sqrt{3}\,aEh}$	$\frac{2\,f\,(l^2-b^2)^{\frac{3}{2}}}{9\sqrt{3}\,aEk}$
Simple beam, two concentrated loads W at equal distances a from supports.	(126)	Wa	$\frac{Wa\,(3\,l^2-4\,a^2)}{24\,EI}$	$\frac{M_0\,(3\,l^2-4\,a^2)}{24\,EI}$	$\frac{f\,(3\,l^2-4\,a^2)}{12\,Eh}$	$\frac{f\,(3\,l^2-4\,a^2)}{12\,lEk}$
Beam supported at ends, subjected to uniform bending.	(118)	M_0	$\frac{M_0l^2}{8\,EI}$	$\frac{M_0l^2}{8\,EI}$	$\frac{fl^2}{4\,Eh}$	$\frac{fl}{4\,Ek}$
Beam fixed at the ends, single concentrated load at middle of span.	(131)	$\frac{Wl}{8}$	$\frac{Wl^3}{192\,EI}$	$\frac{M_0l^2}{24\,EI}$	$\frac{fl^2}{12\,Eh}$	$\frac{fl}{12\,Ek}$
Beam fixed at the ends, uniformly distributed load.	(132)	$\frac{Wl}{12}$	$\frac{Wl^3}{384\,EI}$	$\frac{M_0l^2}{32\,EI}$	$\frac{fl^2}{16\,Eh}$	$\frac{fl}{16\,Ek}$
Beam fixed at the ends, two concentrated loads W at equal distances a from supports.	(133)	$\frac{Wa\,(l-a)}{l}$	$\frac{Wa^2\,(3\,l-4\,a)}{24\,EI}$	$\frac{M_0al\,(3\,l-4\,a)}{24\,(l-a)\,EI}$	$\frac{fal\,(3\,l-4\,a)}{12\,(l-a)\,Eh}$	$\frac{fa\,(3\,l-4\,a)}{12\,(l-a)\,Ek}$

107. Slope and Deflection from the Bending Moment Diagram. — If we refer to the general equation for the slope of the elastic curve (Art. 96) for any beam of uniform section, it will be evident that the difference between the product EIi_1 for any point on the curve, distant x_1 from the origin, and the product EIi_0, of EI and the slope i_0 of the curve at the origin, will be represented by the area of the portion of the bending moment diagram between the origin and the ordinate at the point x_1.

As an illustration let the curve OnX (Fig. 138) represent the

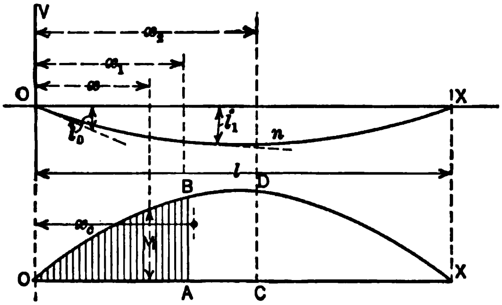


FIG. 138.

elastic curve of a beam, referred to the axes OX and OV, and let OBX represent the bending moment diagram for the same beam. If we let i_1 = the slope at any point, distant x_1 from O, the general equation may be written, using limits of integration,

$$\int_{x=0}^{x=x_1} M\,dx = EI\int_{i=i_0}^{i=i_1} d\left(\frac{dv}{dx}\right) = EI\,(i_1 - i_0), \quad . \quad . \quad (1)$$

and hence the value of the difference of the products EIi_1 and EIi_0 is represented by the area OAB, under the bending moment curve between the origin O and the ordinate AB, at the distance x_1 from O.

This furnishes another method of finding the value of i_1 when the constant i_0 can be found. If we let x_2 = the coördinate of the point on the curve at which the slope is zero, it is evident that for this point equation (1) reduces to

$$\int_{x=0}^{x=x_2} M\,dx = -\,EIi_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and therefore, when the value of x_2 is known, by inspection or otherwise, the value of EIi_0 will be represented by the area OCD, between the origin and the ordinate CD, at the distance x_2 from O.

The general equation for the deflection at any point in the elastic curve may evidently be written

$$EI\int dv = EI\int i\,dx.$$

Substituting for EIi in this equation its general value of EIi_1, determined from equation (1), and introducing limits we have

$$EI\int_{v=0}^{v=v_1} dv = EIi_0\int_{x=0}^{x=x_1} dx + \int_{x=0}^{x=x_1}\int_{x=0}^{x=x} M\,dx\,dx, \quad . \quad (3)$$

which for any case reduces to

$$EIv_1 = EIi_0x_1 + \int_{x=0}^{x=x_1}\int_{x=0}^{x=x} M\,dx\,dx = EIi_0x_1 + Y, \quad . \quad (4)$$

where v_1 = the deflection at any point, distant x_1 from O, and

$$Y = \int_{x=0}^{x=x_1}\int_{x=0}^{x=x} M\,dx\,dx$$

is evidently equal to the moment of the area OAB about the ordinate AB at a distance x_1 from O.

When the beam is a cantilever, if the origin is taken at the fixed end, $i_0 = 0$ and (4) reduces to

$$EIv_1 = Y; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and the greatest deflection will occur at the free end, where Y = the moment of the entire area of the bending moment diagram about the ordinate through the free end.

If the beam is a simple beam, or a beam fixed at the ends, the greatest deflection will occur when $x = x_2$ and in this case the greatest deflection can be found from the equation

$$EIv_0 = EIi_0x_2 + Y_{(x_1=x_2)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

the subscript of Y denoting the limit of integration. If we let x_c = the distance of the ordinate through the center of gravity of the area OCD (Fig. 138) from the origin, it is evident that equation (6) may be written

$$\begin{aligned} EIv_0 &= -\text{ (area } OCD)\,x_2 + \text{(area } OCD)\,(x_2 - x_c) \\ &= -\text{ (area } OCD)\,x_c; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7) \end{aligned}$$

that is, the greatest deflection is represented by the moment about the axis OV of the area under the bending moment curve, between the origin and the ordinate at the point of greatest deflection.

When the loading is symmetrical $x_2 = \frac{l}{2}$ and it is evidently unnecessary to find the value of EIi_0, in order to determine the value of EIv_0.

When the loading is unsymmetrical the value of EIi_0 may be found by putting (4) equal to zero when $x_1 = l$, whence

$$EIi_0 = \frac{Y_{(x_1=l)}}{l}; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

that is, EIi_0 is equal to the moment of the area under the entire bending moment diagram, about the ordinate through the right hand support, divided by the length of the span. Having the value of EIi_0, the values of EIi_1 and EIv_1 for any point can be found from (1) and (4). The value x_2, for the point of greatest deflection, may be obtained from the plot of equation (2) and the value of EIv_0 from (6).

In the application of the foregoing method, due allowance must be made for the scale of the diagram and for the signs, the area of a diagram representing positive bending moments being considered positive and the area representing negative bending moments, negative.

As simple illustrations of the application of this method we may take the following:

(a) *The cantilever beam with a concentrated load at the free end*, given in Case (a) (Art. 98). — Following the notation used in that case, the area of the bending moment diagram will be equal to

$$-Wl \times \frac{l}{2} = -\frac{Wl^2}{2}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

Since $EIi_0 = 0$ in this case, it is evident from (1) that the above quantity is equal to the value of EIi_1 at the free end.

The distance of the center of gravity of the area from the free end will be equal to $\frac{2l}{3}$ and hence

$$Y = -\frac{Wl^3}{3};$$

and, from (5), the value of the greatest deflection

$$v_0 = -\frac{Wl^3}{3EI}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

(b) *Simple beam subjected to a uniformly distributed load*, given in Case (e) (Art. 98). — The greatest deflection is at the center of the span and the bending moment curve is a parabola. The area of the half segment under the curve between the origin and the middle ordinate will be equal to

$$\frac{2}{3} \times \frac{Wl}{8} \times \frac{l}{2} = \frac{Wl^2}{24}; \quad \dots\dots\dots\dots \quad (11)$$

and hence, from (2),

$$i_0 = -\frac{Wl^2}{24EI}. \quad \dots\dots\dots\dots \quad (12)$$

The distance of the center of gravity of the half segment from the origin

$$x_c = \frac{l}{2} \times \frac{5}{8} = \frac{5l}{16},$$

and hence, from (7),

$$v_0 = -\frac{1}{EI} \times \frac{Wl^2}{24} \times \frac{5l}{16} = -\frac{5Wl^3}{384EI}. \quad \dots\dots \quad (13)$$

(c) *Beam fixed at the ends with a concentrated load at the middle of the span* (Case a, Art. 101). — In this case $i_0 = 0$ and equation (6) reduces to

$$EIv_0 = Y_{\left(x_1=\frac{l}{2}\right)}. \quad \dots\dots\dots\dots \quad (14)$$

Taking account of signs,

$$Y_{\left(x_1=\frac{l}{2}\right)} = \frac{Wl}{8} \times \frac{l}{8} \times \frac{l}{12} - \frac{Wl}{8} \times \frac{l}{8} \times \frac{5l}{12} = -\frac{Wl^3}{192};$$

and hence

$$v_0 = -\frac{Wl^3}{192EI}. \quad \dots\dots\dots\dots \quad (15)$$

For any symmetrically loaded beam, fixed at the ends, it follows from (1) and (2) that the total area of the bending moment diagram, taking account of signs, is equal to zero; and the bending moment diagram differs from that for a simple beam subjected to the same load by the constant ordinate M_1 = the bending moment at either fixed end, as illustrated in Cases *a*, *b* and *c* (Art. 101).

Therefore, *the bending moment at the support of any symmetrically loaded beam, fixed at the ends, is equal to the average value of the bending moments throughout the span of a simple beam subjected to the same load.*

The foregoing method is of the most value in computing deflections in cases in which the load on a beam is irregularly distributed in such a manner that the value of M cannot be expressed as an integrable function of x. In such cases a load diagram can be constructed, from which a bending moment diagram can be derived by graphical integration, as indicated in Art. (72). Then, by dividing the bending moment diagram into narrow strips, of width Δx, the areas and moments of areas required to determine the values of EIi_0 and EIv_0 can be easily obtained.

The following illustrations of the application of this method in two simple cases are given.

(d) *Cantilever beam of varying cross section with a concentrated load at the free end.* — The beam (Fig. 138a) is of rectangular section, of uniform width and uniformly varying depth, and is subjected to a concentrated load of 2000 lbs. at the free end. Assuming that the material is homogeneous and neglecting the weight of the beam, the form of the elastic curve and the deflection at any point can be found as follows:

Assuming the origin at O, divide the span into small sections of length Δx and draw the bending moment diagram AB. Since the value of I is variable, the slope equation (1) can be expressed in the approximate form

$$Ei = \Sigma \frac{M}{I} \Delta x + c; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and the deflection equation (4) can be written in the approximate form

$$Ev = \Sigma Ei\Delta x + c_1 = \Sigma \left[\Sigma \frac{M}{I} \Delta x \right] \Delta x + c\Sigma \Delta x + c_1 . \quad . \quad . \quad (17)$$

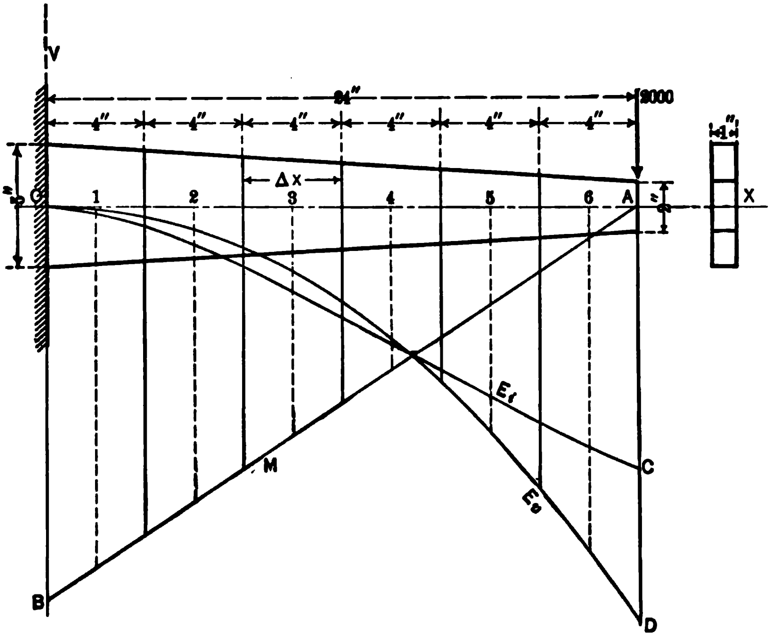


FIG. 138a.

In this case the values of both c and c_1 are evidently zero. The increments of Ei in (16) can be found by multiplying the mean value of $\frac{M}{I}$ for each increment of length by Δx; and, by adding successively, the values of Ei for the successive values of x can be obtained. Plotting these values, we have the curve OC, the ordinates being negative since M is negative. The increments of Ev in (17) can be found by multiplying the mean value of Ei for each increment of length by Δx; and, by adding successively, the values of Ev for the successive values of x can be obtained. Plotting these values we have the curve OD. The results of the computation are shown in the following table.

Increment.	M (in. lbs.).	I (ins.)^4.	$\frac{M}{I}\Delta x$.	Ei.	$Ei\Delta x$.	Ev.
1	−44000	8.93	−19700	−19700	−39000	−39000
2	−36000	6.40	−22500	−42200	−124000	−163000
3	−28000	4.39	−25500	−67700	−220000	−383000
4	−20000	2.86	−28000	−95700	−327000	−710000
5	−12000	1.73	−27700	−123400	−438000	−1148000
6	−4000	0.95	−16900	−140300	−527000	−1675000

Assuming that $E = 28{,}000{,}000$ lbs. per sq. in. we have, for the values of the greatest slope and deflection,

$$i_0 = -\frac{140{,}300}{28{,}000{,}000} = -0.0050 \text{ rads.}, \quad v_0 = -\frac{1{,}675{,}000}{28{,}000{,}000} = -0.060''.$$

The ordinates of the true elastic curve may evidently be obtained by dividing the ordinates of the curve OD by the value of E.

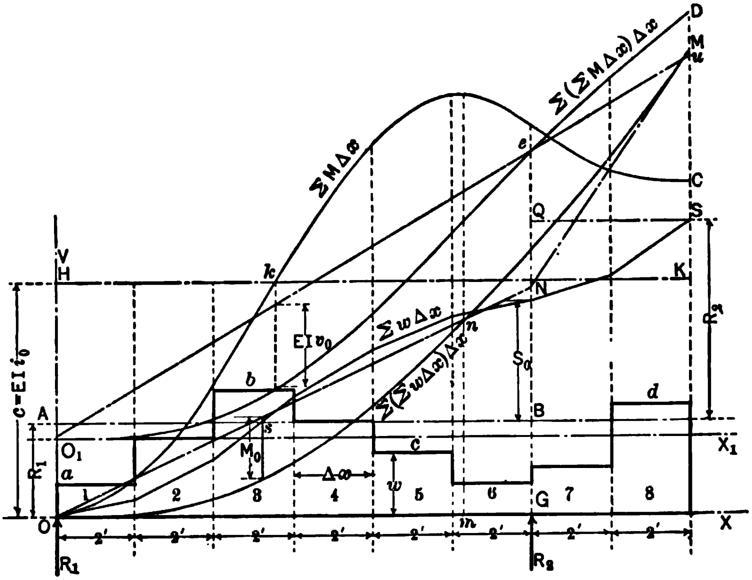


FIG. 138b.

(e) *Beam of uniform cross section subjected to a varying load.* — Let the broken line *abcd* represent the variation in the load intensity w and R_1 and R_2 equal the supporting forces (Fig. 138b).

Writing equation (4) (Art. 72) in the form of an approximate integral, we have, for values of x from 0 to 12,

$$S = R_1 - \Sigma w\Delta x, \quad \ldots \ldots \quad (18)$$

and, for values of x from 12 to 16,

$$S = R_1 + R_2 - \Sigma w\Delta x. \quad \ldots \ldots \quad (19)$$

Writing (6) (Art. 72) in the form of the approximate integral, we have, for values of x from 0 to 12,

$$M = \Sigma S\Delta x = \Sigma\,(R_1 - \Sigma w\Delta x)\,\Delta x = R_1x - \Sigma\,(\Sigma w\Delta x)\,\Delta x, \quad \ldots \quad (20)$$

and, for values of x from 12 to 16,

$$M = R_1x + R_2\,(x - 12) - \Sigma\,(\Sigma w\Delta x)\,\Delta x. \quad \ldots \ldots \quad (21)$$

Since I is constant, equations (1) and (4) may be written

$$EIi = \Sigma M\Delta x + c, \quad \ldots \ldots \quad (22)$$

$$EIv = \Sigma\,(\Sigma M\Delta x)\,\Delta x + cx + c_1. \quad \ldots \ldots \quad (23)$$

Dividing the length into equal increments, $\Delta x = 2$ ft., and tabulating results as in Case (d), we find the products of the load intensity w and the increments Δx, given in the third column of the table, and the successive values of $\Sigma w\Delta x$, given in the fourth column. Plotting the values of $\Sigma w\Delta x$ on the axis OX as a base, we obtain the broken line OS. By adding successively the values of $(\Sigma w\Delta x\ \Delta x$, represented by the increments of the area under the line OS, we obtain the values of the summation $\Sigma\,(\Sigma w\Delta x)\Delta x$, given in the sixth column of the table. Plotting these values as ordinates on the base OX will give the curve OM.

To obtain the supporting forces, apply the conditions that when $x = 16$, the values of S and M, given by (19) and (21), are equal to zero; then, by substituting in (19) and (21) the values of $\Sigma w\Delta x$ and $\Sigma\,(\Sigma w\Delta x)\,\Delta x$, when $x = 16$, we have

$$R_1 + R_2 - 14{,}800 = 0$$

and

$$16\,R_1 + 4\,R_2 - 116{,}400 = 0;$$

and, by solving simultaneously,

$$R_1 = 4770 \text{ lbs.}, \qquad R_2 = 10{,}030 \text{ lbs.}$$

If horizontal lines AB and QS are drawn at distances representing R_1 and $(R_1 + R_2)$, respectively, from OX, the vertical intercepts between these lines and the line OS will evidently represent values of S.

The sections of zero shear will be located at the ordinates through s and Q, the values of x for these points being

$$x = 5.23 \text{ ft. and } x = 12 \text{ ft.}$$

The greatest shearing force, located at the section to the left of the support R_2, will be equal to

$$S_0 = -6030 \text{ lbs.}$$

When $x = 12$, $R_1x = 57{,}200$ and, if this value is laid off as an ordinate GN, at $x = 12$, and the straight lines ON and NM are drawn, the ordinates of ON will evidently represent values of R_1x, for values of x from 0 to 12, and the ordinates of NM, values of $[R_1x + R_2\,(x - 12)]$, for values of x from 12 to 16. Hence values of M will be represented by the vertical intercepts between the broken line ONM and the curve OM.

The greatest bending moment will evidently be located under the point *s* on the shearing force diagram and will be equal to

$$M_0 = 15{,}900 \text{ ft. lbs.}$$

The ordinate through the point *n* on the bending moment diagram will locate the point of inflexion *m*, for which

$$x = 10.3 \text{ ft.}$$

The increments $M\Delta x$, required in equation (22), will be represented by the increments of the area between the line *ONM* and the curve *OM*. By adding these increments successively, the values of $\Sigma M\Delta x$, given in the eighth column of the table, are obtained and, plotting these values as ordinates on the base *OX*, will give the curve *OC*.

Increment.	w lbs. per ft.	$w\Delta x$.	$\Sigma w\Delta x$.	$(\Sigma w\Delta x)\Delta x$.	$\Sigma(\Sigma w\Delta x)\Delta x$.	$M\Delta x$.	$\Sigma M\Delta x$.	$(\Sigma M\Delta x)\Delta x$	$\Sigma(\Sigma M\Delta x)\Delta x$.
1	400	800	800	800	800	9,200	9,200	8,000	8,000
2	1000	2000	2,800	3,600	4,400	24,400	33,600	40,000	48,000
3	1600	3200	6,000	8,800	13,200	31,600	65,200	99,000	147,000
4	1200	2400	8,400	14,400	27,600	27,200	92,400	159,000	306,000
5	800	1600	10,000	18,400	46,000	13,000	105,400	201,000	507,000
6	400	800	10,800	20,800	66,800	−7,600	97,800	206,000	713,000
7	600	1200	12,000	22,800	89,600	−11,800	86,000	184,000	897,000
8	1400	2800	14,800	26,800	116,400	−1,400	84,600	170,000	1,067,000

By adding successively the increments $(\Sigma M\Delta x)\Delta x$, of the area under the curve *OC*, the values of the ordinates $\Sigma(\Sigma M\Delta x)\Delta x$ of the curve *OD*, given in the tenth column, are obtained. To avoid confusion, the curve *OD* is plotted with the ordinates measured from the base line O_1X_1. Observing that $v = 0$, when $x = 0$ and also when $x = 12$, and substituting the values of $\Sigma(\Sigma M\Delta x)\Delta x$ for these values of x in equation (23), we obtain

$$c_1 = 0, \quad 713{,}000 + 12c = 0, \quad \text{and} \quad c = -59{,}400.$$

If a horizontal line *HK* is drawn at a distance representing *c* from *OX*, the vertical intercepts between this line and the curve *OC* will evidently represent values of EIi. It is evident that the greatest slope occurs at the left-hand support, where

$$EIi_0 = c = -59{,}400.$$

The slope is zero, evidently, at the foot of the ordinate through *k*.

If a straight line O_1u is drawn through the origin and the point of intersection *e*, of the curve O_1D with the ordinate through the right-hand support, the ordinates of O_1u will represent values of cx in equation (23); and hence the vertical intercepts between the straight line O_1u and the curve O_1D will represent values of EIv. The greatest value of EIv is evidently represented by the intercept on the ordinate passing through the point *k*, which is equal to

$$EIv_0 = 210{,}000,$$

the value of x for the point *k* being equal to

$$x = 5.6 \text{ ft.}$$

In constructing the diagrams the scales for the ordinates should be so chosen that the increments of the areas under the different curves can be determined with sufficient accuracy to give results with the precision required.

108. Deflection of a Simple Beam under Two Concentrated Loads. — It has been shown (Art. 100) that the deflection at any point in a beam, subjected to a system of transverse loads, is equal to the sum of the deflections at the point, which would be produced by each load acting separately. The application of this principle in the case of a simple beam subjected to two concentrated loads (Fig. 139), applied at any two points in the span, will lead to the following relation between the components of the deflections due to the loads.

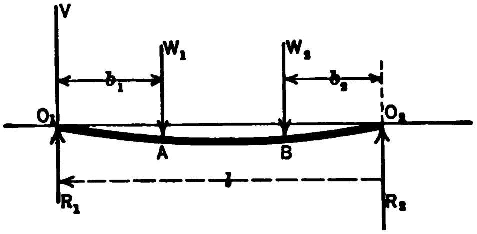


FIG. 139.

Let W_1 and W_2 equal the magnitudes of the loads applied at the points A and B, at distances b_1 and b_2 from the supports at O_1 and O_2, respectively. The deflection v_a at the point A, due to the load W_2 acting alone, could be found by taking O_1 as the origin and substituting b_1 for x and b_2 for b in equation (41) (Art. 98), which would give

$$v_a = \frac{W_2 b_2}{6\,lEI}\left[b_1^3 - (l^2 - b_2^2)\,b_1\right] = \frac{W_2 b_2 b_1}{6\,lEI}\left[b_1^2 + b_2^2 - l^2\right]. \quad (1)$$

Similarly, the deflection v_b at the point B, due to the load W_1 acting alone, could be found by taking O_2 as the origin and substituting b_2 for x and b_1 for b which would give

$$v_b = \frac{W_1 b_1}{6\,lEI}\left[b_2^3 - (l^2 - b_1^2)\,b_2\right] = \frac{W_1 b_1 b_2}{6\,lEI}\left[b_2^2 + b_1^2 - l^2\right]. \quad (2)$$

Therefore,

$$\frac{v_a}{v_b} = \frac{W_2}{W_1}; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

and, when $W_1 = W_2$, $v_a = v_b$; that is, *when equal concentrated loads are applied at any two points A and B in a simple beam, the deflection at A, due to the load at B, is equal to the deflection at B, due to the load at A.* It is evident from Art. (100) that the relation between the components of the deflection due to two concentrated loads will hold true when loads in addition to the two are applied at the same or at other points in the beam.

The following will serve as an illustration of the application of this principle. The deflection at a point A in a beam due to a load of 8000 lbs. acting at another point B is found by calculation, or by measurement, to be 0.16″. If an additional load of 6000 lbs. is applied at A the increase in the deflection under the load at B will be equal to 0.12″; that is, the increase in deflection at B is the same as would be the increase in the deflection at A if 6000 lbs. were added to the load of 8000 lbs. at B.

109. Resilience due to Bending. — The resilience due to bending is the potential energy due to the strain in a member caused by the action of a system of transverse forces, or couples, which produce flexure. If the material is perfectly elastic, the resilience will be equal to the work done during the distortion of the beam by a *gradual application* of the system of forces.

The term is commonly employed to designate the strain energy due to longitudinal extensions only, the effect of small strains due to shear being neglected, as has been done in the derivation of deflection formulas by the ordinary theory of flexure.

The resilience of a beam of elastic material subjected to a system of concentrated loads may, therefore, be determined by computing the half sum of the products of the loads and deflections at the points of application of the loads, the deflection at each load being determined from the usual formulas.

For example, in the case of a beam supported at the ends and loaded at the center with a single concentrated load W (Fig. 123), the resilience will be

$$R = \frac{Wv_0}{2} = \frac{W^2l^3}{96\,EI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where W = the load and v_o = the deflection at the center of the beam.

When the loads are distributed the resilience may be represented by the expression

$$R = \frac{1}{2}\int wv\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where w = the load intensity and v = the deflection at any point, at a distance x from the origin.

For the general case, applying to any system of concentrated or distributed loads, the expression for resilience may be deduced as follows:

Let $ONXK$ (Fig. 140) represent the elastic curve of a beam supported at two points and loaded with a given system of loads. Let AB be any cross section at a distance x from the origin, and GH a cross section at a distance dx from AB. After bending occurs, let r = the radius of curvature of NT, the portion of the elastic curve between the cross sections AB and GH, and M = the bending moment at section AB. The angle between AB and GH will evidently be equal to di, the difference in the slopes at N and T. If the forces producing the flexure are *gradually applied*, the work done in straining the portion of the beam between the sections AB and GH will be equal to

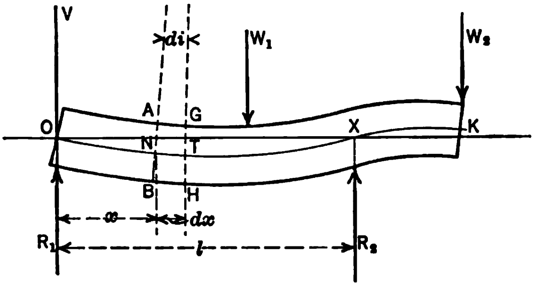


FIG. 140.

$$\frac{M}{2}di = \frac{M}{2}\frac{ds}{r} = \frac{M^2\,ds}{2\,EI} = \frac{M^2\,dx}{2\,EI} \text{ (very nearly),}$$

and hence the resilience of the entire beam may be represented by the expression

$$R = \int\frac{M}{2}di = \int\frac{M^2\,dx}{2\,EI} \text{ (very nearly).} \quad . \quad . \quad . \quad (3)$$

The following illustrations of the application of the formula in a few simple cases are given.

(a) *Cantilever beam, load uniformly distributed* (Fig. 122). — Let $W = wl$ equal the total load. In this case the bending moment at any section, at a distance x from the fixed end, is equal to

$$M = \frac{w}{2}(l - x)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting in (3) we have

$$R = \frac{w^2}{8EI}\int_0^l (l-x)^4\,dx = -\frac{w^2}{8EI}\,\frac{(l-x)^5}{5}\Bigg|_0^l = \frac{w^2l^5}{40EI} = \frac{W^2l^3}{40EI} = \frac{1}{5}Wv_0, \quad (5)$$

where v_0 = the greatest deflection.

(b) *Simple beam, load uniformly distributed* (Fig. 124). — Let $W = wl$ equal the total load. The bending moment at any section, distant x from the support, is equal to

$$M = \frac{w}{2}(lx - x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Substituting in (3) we have

$$R = \frac{w^2}{8EI}\int_0^l (lx - x^2)^2\,dx = \frac{w^2l^5}{240EI} = \frac{W^2l^3}{240EI} = \frac{8}{25}Wv_0, \quad . \quad . \quad (7)$$

where v_0 = the greatest deflection.

(c) *Beam subjected to uniform bending* (Fig. 118). — Let M_0 = the bending moment.

Then

$$R = \frac{M_0^2}{2EI}\int_0^l dx = \frac{M_0^2l}{2EI} = \frac{4M_0v_0}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where v_0 = the deflection at the middle of the span. The resilience in this case is evidently equal to the product of $\frac{M_0}{2}$, and the difference $\frac{M_0l}{EI}$, between the angles of slope at the ends (Art. 97).

The same method may be employed when the loads are concentrated but in simple cases of this kind the resilience can be more easily determined by computing the sum of the products of the half loads and the deflections at the loads, as shown in the following cases.

(d) *Simple beam, single concentrated load W not on the center of the span* (Fig. 125). — The deflection at the load is equal to

$$v = \frac{Wa^2b^2}{3lEI}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and hence the resilience

$$R = \frac{W}{2}v = \frac{W^2a^2b^2}{6lEI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

(e) *Simple beam with two concentrated loads W at equal distances from the ends* (Fig. 126). — The deflection at either load is equal to

$$v_1 = \frac{Wa^2}{6EI}(3l - 4a); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and hence the resilience

$$R = 2 \times \frac{Wv}{2} = \frac{W^2a^2}{6EI}(3l - 4a). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Resilience in terms of the greatest fiber stress. — The resilience may also be expressed in each case in terms of the greatest fiber stress

f and the volume V of the beam. For example, in the case of the simple beam with a concentrated load at the middle of the span the load will be equal to

$$W = \frac{4fI}{lc} = \frac{4fA\rho^2}{lc} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and, by substituting in (1),

$$R = \frac{W^2l^3}{96\,EI} = \frac{f^2A\rho^2l}{6\,Ec^2} = \frac{f^2\rho^2}{6\,Ec^2}V. \quad . \quad . \quad . \quad . \quad (14)$$

When the cross section is symmetrical with respect to the neutral axis, $c = \frac{h}{2}$ and

$$R = \frac{2\,f^2\rho^2}{3\,Eh^2}V. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If the section is rectangular, $\rho = \frac{h}{\sqrt{12}}$, and

$$R = \frac{f^2V}{18\,E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Similar expressions for the resilience in terms of the greatest fiber stress and volume can be determined for any of the preceding cases. The following is a summary of formulas for a few cases when the cross sections are rectangular. These may be compared with the expression for resilience under uniform tension or compression (Art. 15).

(a) *Cantilever beam, load uniformly distributed:*

$$R = \frac{f^2V}{30\,E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

(b) *Simple beam load uniformly distributed:*

$$R = \frac{4\,f^2V}{45\,E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

(c) *Beam subjected to uniform bending:*

$$R = \frac{f^2V}{6\,E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

(d) *Simple beam with concentrated load, not at the center of the span:*

$$R = \frac{f^2V}{18\,E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

(e) *Simple beam with two concentrated loads at equal distances from the ends:*

$$R = \frac{f^2}{18\,E}(V - 2\,V_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where V_1 = the volume of the portion of the beam between the sections under the concentrated loads.

110. Deflection Determined from Resilience — General Equation. — If the resilience of a beam has been calculated from the bending moments by equation (3) (Art. 109), the deflection may be determined from the resilience. When the beam is subjected to a single concentrated load the process is very simple, being the reverse of the method of determining the resilience from the half sum of the products of loads and deflections.

As an illustration take the case of the rectangular beam with a single concentrated load not at the center.

From equation (16) (Art. 109) we have

$$\frac{Wv}{2} = \frac{f^2V}{18E}.$$

Hence

$$v = \frac{f^2V}{9WE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and, using the usual notation, this expression reduces to

$$v = \left(\frac{Wabh}{2lI}\right)^2 \frac{Al}{9WE} = \frac{Wa^2b^2}{3lEI}, \quad . \quad . \quad . \quad . \quad (2)$$

which is the same as equation (47) (Art. 98).

For the general case an expression for the deflection at any point may be deduced as follows: Let M = the value of the bending moment at any cross section AB of a beam (Fig. 141), due to any system of loads acting on the beam. Let v_1 = the deflection at any point in the elastic curve, at a distance x_1 from the origin.

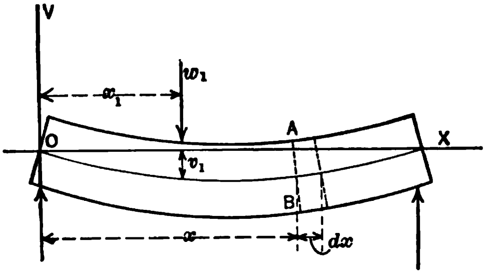


FIG. 141.

To find v_1, assume that a very small load w_1 is gradually applied at the point x_1, in addition to the loads acting on the beam, and let m = the increase in the bending moment at the section AB, due to the load w_1.

The general equation of the elastic curve, if the load w_1 were acting alone, would be

$$\frac{d^2v}{dx^2} = \frac{di}{dx} = \frac{m}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Hence, for a point on the elastic curve at a distance x from O,

$$di = \frac{m}{EI}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

The addition to the total strain energy of the beam, due to the addition of the load w_1, would be equal to

$$w_1\left(v_1 + \frac{v_1'}{2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

where v_1' = the increment in v_1, due to the additional load w_1. Expressed in terms of the bending moments the additional strain energy would be equal to

$$\int\left(M + \frac{m}{2}\right)di \text{ (Art. 109).}$$

Substituting the value of di from (4) and combining with (5), we have

$$w_1\left(v_1 + \frac{v_1'}{2}\right) = \int \frac{\left(Mm + \frac{m^2}{2}\right)}{EI}dx. \quad . \quad . \quad . \quad . \qquad (6)$$

This equation will hold true, however small the magnitude of w_1 may be and, as w_1 decreases, $\left(v_1 + \frac{v_1'}{2}\right)$ approaches the value v_1 and $\left(Mm + \frac{m^2}{2}\right)$ approaches the value Mm.

Hence, as the value of w_1 approaches zero, equation (6) becomes

$$w_1v_1 = \int \frac{Mm}{EI}dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

which reduces to

$$v_1 = \int \frac{Mm}{w_1EI}dx = \int \frac{Mm_1}{EI}dx, \quad . \quad . \quad . \qquad (8)$$

where $m_1 = \frac{m}{w_1}$ *will evidently be equal to the increment in the bending moment when* w_1 = *unity*. By the use of either (7) or (8) the deflection at any point in a beam subjected to any load system may be found.

As an illustration of the application of the formula we may take the case of the simple beam, with a single concentrated load not at the center (Fig. 125), for which the deflection at the center of the span is to be determined.

The unit load will be placed at the center of the span and, following the notation in Case (f) (Art. 98) and noting that all bending moments are positive and that $a > b$, the value of Mm_1, for values of x from 0 to $\frac{l}{2}$ will be equal to

$$Mm_1 = \frac{Wbx}{l}\left(\frac{x}{2}\right) = \frac{Wbx^2}{2l} \quad . \; . \; . \; . \; . \quad (9)$$

and, for values of x from $\frac{l}{2}$ to a,

$$Mm_1 = \frac{Wbx}{l}\left[\frac{x}{2} - \left(x - \frac{l}{2}\right)\right] = \frac{Wbx\,(l - x)}{2l}. \quad . \quad (10)$$

For sections between the load W and the right hand support, if the origin is taken at the right end of the beam and x is measured toward the left,

$$Mm_1 = \frac{Wax}{l}\left(\frac{x}{2}\right) = \frac{Wax^2}{2l}. \quad . \; . \; . \; . \; . \quad (11)$$

Substituting in (8) we have

$$EIv_1 = \frac{Wb}{2l}\int_0^{\frac{l}{2}} x^2\,dx + \frac{Wb}{2l}\int_{\frac{l}{2}}^{a} x\,(l - x)\,dx + \frac{Wa}{2l}\int_0^{b} x^2\,dx$$

$$= \frac{W}{2l}\left[\frac{bl^3}{24} + \frac{bla^2}{2} - \frac{ba^3}{3} - \frac{bl^3}{8} + \frac{bl^3}{24} + \frac{ab^3}{3}\right], \quad . \; . \; . \; . \quad (12)$$

which reduces to

$$v_1 = \frac{Wb}{48\,EI}\,(3\,l^2 - 4\,b^2). \quad . \; . \; . \; . \; . \quad (13)$$

This result agrees with equation (48) (Art. 98).

The method can be used to determine the deflection of beams of varying cross section, the value of I in this case being variable. If the cross section varies constantly and I can be expressed as an integrable function of x, the value of v_1 can be readily obtained by substituting $\frac{Mm_1}{I} = \phi\,(x)$ in (8) and integrating as before. In other cases where the cross section varies, the deflection may be found by dividing the span into sections and considering I constant for each section, taking for any one section the average value of I through that portion of the span. This will in some cases give approximate results; the approximation depending on the number of sections into which the span is divided.

As an illustration we may take the case of a girder supported at the ends and subjected to a uniformly distributed load $W = wl$, with the outside cover plates on both flanges extending over the middle portion only (Fig. 142). To find the greatest deflection the unit load will be placed at the center of the span and, since both halves of the beam are under the same state of stress, it will be necessary to apply equation (8) to one-half of the span only and multiply the result by two.

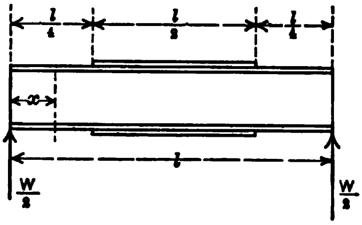


Fig. 142.

Let I_1 = the moment of inertia of the cross section throughout the end quarter of the span and I_2 = the moment of inertia throughout the middle portion of the span.

For values of x from 0 to $\frac{l}{4}$,

$$\frac{Mm_1}{I} = \frac{w}{2I_1}(lx - x^2)\frac{x}{2} = \frac{w}{4I_1}(lx^2 - x^3), \quad . \; . \quad (14)$$

and, for values of x from $\frac{l}{4}$ to $\frac{l}{2}$,

$$\frac{Mm_1}{I} = \frac{w}{2I_2}(lx - x^2)\frac{x}{2} = \frac{w}{4I_2}(lx^2 - x^3). \quad . \; . \quad (15)$$

Substituting in (8) and multiplying by 2 we have

$$v_1 = 2\frac{w}{4E}\left[\frac{1}{I_1}\int_0^{\frac{l}{4}}(lx^2 - x^3)\,dx + \frac{1}{I_2}\int_{\frac{l}{4}}^{\frac{l}{2}}(lx^2 - x^3)\,dx\right]$$

$$= \frac{w}{2E}\left[\frac{13\,l^4}{3072\,I_1} + \frac{67\,l^4}{3072\,I_2}\right] = \frac{wl^4}{6144\,E}\left[\frac{13}{I_1} + \frac{67}{I_2}\right]$$

$$= \frac{Wl^3}{6144\,E}\left[\frac{13}{I_1} + \frac{67}{I_2}\right]. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

If the cross section were uniform throughout the span, we would have $I_2 = I_1 = I$, and equation (16) would reduce to

$$v_1 = \frac{5\,Wl^3}{384\,EI}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

111. Principle of Least Work as Applied to Beams. — This principle of Mechanics as applied in the case of elastic bodies may be stated as follows: *The deformation of any elastic body under the* *action of a balanced system of external forces will be such that the work done in causing the deformation, or the resilience of the body, is a minimum.* This is based on the principle of the conservation of energy, whereby it may easily be shown that in order to have stable equilibrium the strain energy produced by the action of the external force system must be a minimum. Applied in the case of the beam, the principle can evidently be represented by the expression

$$R = \int \frac{M^2\,dx}{2EI} = \text{a minimum} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This principle can be employed, in conjunction with the static conditions of equilibrium, to determine the reactions at the supports of a beam in a statically indeterminate case. As an illustration we may take the following:

(a) ***Beam supported at three points and subjected to a uniformly distributed load*** (Fig. 143). — Let w = the load per unit of length. The bending moment at any point of the span l_1, at a distance x from A, will be equal to

$$M = F_1x - \frac{wx^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and for the span l_2, if we take the origin at B,

$$M = F_2x - \frac{wx^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

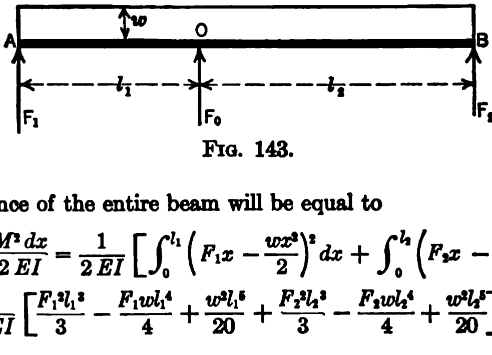


FIG. 143.

The resilience of the entire beam will be equal to

$$R = \int \frac{M^2\,dx}{2EI} = \frac{1}{2EI}\left[\int_0^{l_1}\left(F_1x - \frac{wx^2}{2}\right)^2 dx + \int_0^{l_2}\left(F_2x - \frac{wx^2}{2}\right)^2 dx\right]$$

$$= \frac{1}{2EI}\left[\frac{F_1^2l_1^3}{3} - \frac{F_1wl_1^4}{4} + \frac{w^2l_1^5}{20} + \frac{F_2^2l_2^3}{3} - \frac{F_2wl_2^4}{4} + \frac{w^2l_2^5}{20}\right] \quad . \quad . \quad . \quad . \quad (3)$$

Taking moments about O.

$$F_1l_1 - \frac{wl_1^2}{2} = F_2l_2 - \frac{wl_2^2}{2},$$

and hence

$$F_2l_2 = F_1l_1 - \frac{w}{2}(l_1^2 - l_2^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Substituting the value of F_2l_2 in (3) and reducing, we have

$$R = \frac{1}{2\,EI}\left[\frac{F_1^2l_1^3}{3} - \frac{F_1wl_1^4}{4} + \frac{l_2}{3}\left\{F_1l_1 - \frac{w}{2}(l_1^2 - l_2^2)\right\}^2\right.$$
$$\left. - \frac{wl_2^3}{4}\left\{F_1l_1 - \frac{w}{2}(l_1^2 - l_2^2)\right\} + \frac{w^2}{20}(l_1^5 + l_2^5)\right], \quad . \; . \; . \quad (5)$$

which gives the value of R in terms of the supporting force F_1.

In order that R shall have the minimum possible value, the magnitude of F_1 must be such that

$$\frac{dR}{dF_1} = 0.$$

Differentiating

$$\frac{dR}{dF_1} = \frac{1}{2\,EI}\left[\frac{2\,F_1l_1^3}{3} - \frac{wl_1^4}{4} + \frac{2\,l_2}{3}\left\{F_1l_1 - \frac{w}{2}(l_1^2 - l_2^2)\right\}l_1 - \frac{wl_2^3}{4}l_1\right]. \; . \; . \quad (6)$$

Hence

$$\frac{2\,F_1l_1^2}{3}(l_1 + l_2) - \frac{wl_1}{3}\left[\frac{3\,l_1^3}{4} + l_2(l_1^2 - l_2^2) + \frac{3\,l_2^3}{4}\right]$$
$$= 2\,F_1l_1 - \frac{w}{4}(3\,l_1^2 + l_1l_2 - l_2^2) = 0:$$

and therefore

$$F_1 = \frac{w}{8\,l_1}(3\,l_1^2 + l_1l_2 - l_2^2). \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

By substituting in (4),

$$F_2 = \frac{w}{8\,l_2}(3\,l_1^2 + l_1l_2 - l_2^2) - \frac{w}{2\,l_2}(l_1^2 - l_2^2) = \frac{w}{8\,l_2}(3\,l_2^2 + l_2l_1 - l_1^2). \; . \quad (8)$$

Hence for the middle support,

$$F_0 = w(l_1 + l_2) - \frac{w}{8\,l_1}(3\,l_1^2 + l_1l_2 - l_2^2) - \frac{w}{8\,l_2}(3\,l_2^2 + l_2l_1 - l_1^2)$$
$$= \frac{w}{8}\left[l_1\left(4 + \frac{l_1}{l_2}\right) + l_2\left(4 + \frac{l_2}{l_1}\right)\right]. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

When the spans are equal, if we let

$$l = l_1 = l_2,$$
$$F_1 = F_2 = \frac{3\,wl}{8} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

and

$$F_0 = \frac{5\,wl}{4}, \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

which agrees with the results in Case (b) (Art. 102).

112. Shearing Resilience. — When an elastic body undergoes a shearing strain a certain amount of strain energy is stored up, the same as when the body is subjected to tension, or compression (Art. 15). This energy is called the resilience due to shear, or the *shearing resilience*, and, if the body is perfectly elastic, will be equal to the work done during a gradual application of the forces producing the shear. For example, if we imagine the prism *abcd* (Fig. 144) to be distorted in simple shear by a force W gradually applied, the total displacement of W being equal to v_s while the other forces required to maintain equilibrium remain stationary, the work done will be equal to

$$\frac{Wv_s}{2}.$$

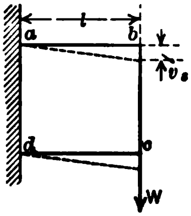


Fig. 144.

Let A = the area of the cross section, l = the length and V = the volume of the prism. Let s = the intensity of the shearing stress on the cross section and G = the modulus of rigidity of the material. Then, if the shearing stress is assumed to be uniform throughout the prism, the shearing strain at any point in directions parallel to *ab* and *ad* will be equal to

$$\gamma = \frac{v_s}{l} = \frac{s}{G} \quad \text{(Art. 7).}$$

Hence

$$v_s = \frac{sl}{G} = \frac{Wl}{AG}; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and the shearing resilience will be equal to

$$R_s = \frac{Wv_s}{2} = \frac{W^2l}{2AG} = \frac{s^2Al}{2G} = \frac{1}{2}\frac{s^2}{G}V, \quad . \; . \; . \; . \quad (2)$$

which is similar to the expression for the resilience in tension (Art. 15).

If V = unity, equation (2) reduces to the expression for the shearing resilience per unit of volume, or the modulus of resilience in shear,

$$R_{s_u} = \frac{1}{2}\frac{s^2}{G}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

Hence, when the shearing strain is uniform throughout a body, its total resilience in shear is equal to

$$R_s = R_{s_u}V. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

113. Deflection Due to Shearing. — Thus far in the discussion of the theory of flexure, no account has been taken of the distortions due to the shearing stresses, which exist in beams subjected to ordinary bending. The equation of the elastic curve was deduced from the relations existing between the normal stresses and the longitudinal strains at the various cross sections and is, therefore, the true equation of the curve, formed by the central axis of a beam, only in the case of simple, or uniform, bending.

In a beam subjected to ordinary bending there is, in addition to the deflection computed from the ordinary equation of the elastic curve, a deflection due to shearing; but, since the shearing stresses in a beam are small as compared with the fiber stresses, the deflection due to shearing is so small that it can be neglected in ordinary calculations. In order to form an idea of its magnitude, we may deduce the expressions for the deflections due to shearing alone in a few simple cases.

Approximate results may be obtained by assuming that the shearing stress on a cross section of a beam is uniformly distributed, or more exact results can be had by taking into account the variation in the shearing stress intensity on the cross section, as given in Art. (89). In using the latter method, however, it is to be remembered that the calculations are based on the assumptions of the common beam theory which, while giving the deflection due to bending quite accurately, will not in all cases give the deflection due to shearing with the same degree of accuracy.

(a) *Cantilever beam of rectangular cross section with load W at the free end* (Fig. 120). — In this case the shearing force is constant for every cross section of the beam and, if we assume that the shearing stress on each cross section is uniform, the deflection due to shearing can be obtained directly from equation (1) (Art. 112), which will give

$$v_s = \frac{Wl}{AG} = \frac{WV}{A^2G} \text{ (approx.)}, \quad \ldots\ldots\ldots \quad (1)$$

where A = the area of the cross section of the beam.

If we assume the cross section to be rectangular and the distribution of the shearing stress to be that given in Art. (89), we may consider the beam to be made up of a series of horizontal layers, of thickness dy, and compute the shearing resilience of each layer from equation (2) (Art. 112) and add together as follows:

Let b = the breadth and h = the depth of the section. For a layer at any distance y from the neutral layer we have for the intensity of the shearing stress

$$s = \frac{3W}{2bh^3}(h^2 - 4y^2); \quad \ldots\ldots\ldots\ldots \quad (2)$$

and hence, for the resilience of the layer,

$$dR_s = \frac{s^2lA}{2G} = \frac{9W^2l}{8bh^6G}(h^2 - 4y^2)^2\,dy.$$

Integrating,

$$R_s = \frac{9\,W^2 l}{8\,bh^6 G}\int_{-\frac{h}{2}}^{\frac{h}{2}}(h^4 - 8\,h^2y^2 + 16\,y^4)\,dy = \frac{3}{5}\,\frac{W^2 l}{bhG} = \frac{3}{5}\,\frac{W^2 l}{AG}. \qquad (3)$$

Hence

$$v_s = \frac{6\,Wl}{5\,AG}, \quad \ldots\ldots\ldots\ldots \quad (4)$$

which is 20 per cent higher than the value given by (1).

Adding the value of v_s in (4) to the deflection due to bending, as given by equation (7) (Art. 98), and neglecting signs, we have, for the total deflection at the free end of the beam due to bending and shear,

$$v_0' = v_0 + v_s = \frac{Wl^3}{3\,EI} + \frac{6\,Wl}{5\,AG} = \frac{4\,Wl^3}{Ebh^3} + \frac{6\,Wl}{5\,Gbh} = \frac{4\,Wl^3}{Ebh^3}\left(1 + \frac{3\,Eh^2}{10\,Gl^2}\right)$$
$$= \frac{Wl^3}{3\,EI}\left(1 + \frac{3\,Eh^2}{10\,Gl^2}\right). \quad \ldots\ldots\ldots\ldots \quad (5)$$

If $G = \frac{2}{5}E$ (Art. 7), equation (5) reduces to

$$v_0' = \frac{Wl^3}{3\,EI}\left[1 + \frac{3}{4}\left(\frac{h}{l}\right)^2\right]. \quad \ldots\ldots\ldots \quad (6)$$

It is evident from (6) that in this case the deflection due to shearing is small compared with the deflection due to bending, the ratio between the two being equal to

$$\frac{v_s}{v_0} = \frac{3}{4}\left(\frac{h}{l}\right)^2. \quad \ldots\ldots\ldots\ldots \quad (7)$$

When the ratio of length to depth is 10 : 1 the ratio of the deflections reduces to

$$\frac{v_s}{v_0} = \frac{3}{400} = 0.0075; \quad \ldots\ldots\ldots \quad (8)$$

that is, v_s is about 0.75 per cent of v_0. If the deflection due to shearing were assumed to be that given by equation (1), the ratio would be

$$\frac{v_s}{v_0} = \frac{5}{6} \times \frac{3}{400} = \frac{1}{160} = 0.00625. \quad \ldots\ldots\ldots \quad (9)$$

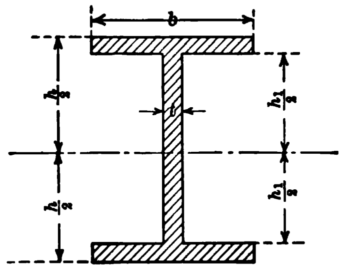


FIG. 145.

(b) *Cantilever beam, with a load W at the free end, having a cross section of the dimensions given* (Fig. 145). — In this case an approximate solution may be made by assuming that the entire shearing stress on any cross section is uniformly distributed over the web (Arts. 90–91). If we let l = the span and W = the load, as before, and assume the dimensions of the section as shown (Fig. 145), the shearing resilience of the beam will be equal to

$$R_s = \frac{W^2 l}{2\,AG} = \frac{W^2 l}{2\,th_1 G}; \quad \ldots\ldots\ldots \quad (10)$$

and hence

$$v_s = \frac{Wl}{th_1 G}. \quad \ldots\ldots\ldots\ldots \quad (11)$$

If the distribution of the shearing stress over the section is to be taken into account the algebraic equations will become complex, but a comparison may be made for a specific case in which the approximate dimensions of the I-section may be taken as follows:

$$h = 10''; \; h_1 = 9''; \quad b = 5''; \; t = \tfrac{1}{2}''.$$

For this cross section

$$A = 9.5 \text{ sq. ins.}, \qquad I = 143.3 \text{ (ins.)}^4.$$

For any point in the flange

$$Q = \tfrac{5}{2}\,(5^2 - y^2) = \tfrac{5}{2}\,(25 - y^2),$$

$$s = \frac{5\,W\,(25 - y^2)}{2 \times 5 \times 143.3} = \frac{W\,(25 - y^2)}{286.6}$$

and

$$s^2 = \frac{W^2}{82{,}140}\,(625 - 50\,y^2 + y^4).$$

For any point in the web

$$Q = \tfrac{5}{2} \times 4.75 + \tfrac{1}{4}\,(\overline{4.5^2} - y^2) = \tfrac{1}{4}\,(67.75 - y^2),$$

$$s = \frac{W\,(67.75 - y^2)}{4 \times \frac{1}{2} \times 143.3} = \frac{W\,(67.75 - y^2)}{286.6}$$

and $$s^2 = \frac{W^2}{82{,}140}\,(4590 - 135.5\,y^2 + y^4).$$

Hence the total shearing resilience of the beam will be equal to

$$R_s = 2\left[\frac{5\,W^2l}{2 \times 82{,}140\,G}\int_{4.5}^{5}(625 - 50\,y^2 + y^4)\,dy + \frac{W^2l}{4 \times 82{,}140\,G}\int_{0}^{4.5}(4590 - 135.5\,y^2 + y^4)\,dy\right] = \frac{W^2l}{G}\,(0.00024 + 0.10292) = 0.1032\,\frac{W^2l}{G}. \quad (12)$$

and hence

$$v_s = 0.206\,\frac{Wl}{G}. \quad (13)$$

It is evident from (12) that the shearing strain energy in the flanges is very small compared with that in the web and could be omitted in the calculation.

The value of the deflection calculated from (11) would be

$$v_s = 0.222\,\frac{Wl}{G}. \quad (14)$$

Hence the value of v_s given by (13) is about 7.2 per cent lower than that given by the approximate solution made by assuming the total shearing stress to be uniformly distributed over the web.

The greatest deflection due to bending, in this case, would be equal to

$$v_0 = \frac{Wl^3}{3\,EI} = \frac{Wl^3}{429.9\,E} = 0.00233\,\frac{Wl^3}{E};$$

and hence the total deflection due to shearing and bending,

$$v_0' = v_0 + v_s = 0.00233\,\frac{Wl^3}{E} + 0.206\,\frac{Wl}{G}. \quad (15)$$

If $G = \frac{2}{5}E$,

$$v_0' = \frac{Wl}{E}(0.00233\,l^2 + 0.515). \qquad (16)$$

Hence the ratio between the deflection due to shearing and that due to bending will be

$$\frac{v_s}{v_0} = \frac{0.515}{0.00233\,l^2} = \frac{221}{l^2} \qquad (17)$$

and, when the ratio of the length of the beam to its depth is 10 : 1, $l = 100''$, and equation (17) becomes

$$\frac{v_s}{v_0} = 0.022; \qquad (18)$$

that is, the deflection due to shearing is in this case about 2.2 per cent of that due to bending.

(c) *Simple beam with concentrated load W at the center of the span* (Fig. 123). — Let W = the load and l = the length of the span. In this case since each half of the beam will be under the same state of stress as the cantilever, considered in Cases (a) and (b), and the deflection due to shearing may be calculated by substituting $\frac{l}{2}$ for l and $\frac{W}{2}$ for W in the equations for v_s.

Rectangular section. — If the cross section is rectangular, by making these substitutions in equation (4) and reducing, we obtain

$$v_s = \frac{3\,Wl}{10\,AG}, \qquad (19)$$

and the total deflection due to bending and shearing will be equal to

$$v_0' = \frac{Wl^3}{48\,EI} + \frac{3\,Wl}{10\,AG} = \frac{Wl^3}{48\,EI}\left[1 + \frac{6}{5}\,\frac{E}{G}\left(\frac{h}{l}\right)^2\right]; \qquad (20)$$

and, if $G = \frac{2}{5}E$,

$$v_0' = \frac{Wl^3}{48\,EI}\left[1 + 3\left(\frac{h}{l}\right)^2\right]. \qquad (21)$$

The ratio of the deflections will be

$$\frac{v_s}{v_0} = 3\left(\frac{h}{l}\right)^2, \qquad (22)$$

and when $l : h = 10 : 1$, the deflection due to shearing will be about 3 per cent of that due to bending.

I-Section (Fig. 145). — When the cross section is of the approximate dimensions given for a 10″ I-section (Case b), we have by substituting $\frac{l}{2}$ for l and $\frac{W}{2}$ for W in equation (13) and reducing,

$$v_s = 0.0515\,\frac{Wl}{G} \qquad (23)$$

and the total deflection due to bending and shearing

$$v_0' = 0.000145\,\frac{Wl^3}{E} + 0.0515\,\frac{Wl}{G}; \qquad (24)$$

and, if $G = \frac{2}{5}E$,

$$v_0' = \frac{Wl}{E}(0.000145\,l^2 + 0.129). \qquad (25)$$

The ratio of the deflections will be

$$\frac{v_s}{v_0} = \frac{0.129}{0.000145\, l^2} = \frac{890}{l^2}; \quad \ldots\ldots\ldots \quad (26)$$

and, when $l : h = 10 : 1$, the deflection due to shearing will be about 8.9 per cent of the deflection due to bending.

It is evident that, in all the foregoing cases, as the ratio of length to depth increases, the ratio of the deflection due to shearing and that due to bending decreases.

In making calculations of the deflections of beams with I-sections, and average spans, however, some allowance should be made for shearing. This is usually done by using values of E, in the ordinary formulas for deflection due to bending, which are somewhat smaller than the actual tensile modulus of elasticity, the modified values of E being based on the results obtained by actual measurements of the total deflection of beams under different systems of loading. It is evident that the values of E, calculated from such measurements by means of the ordinary deflection formulas, will vary, not only with the shape of the cross section, but also with the ratio of the depth of the section to the length of the beam.

114. Transverse Curvature. — No mention has been made, so far, of any change in the shape of the cross sections of a beam subjected to flexure. That a slight distortion does take place, however, is evident from the fact that the longitudinal strains in the different layers are always accompanied by lateral strains unless the latter are prevented by constraining forces (Art. 5).

If we assume the lateral movement in the different layers to be free, each layer contracting, or expanding, independently of the others, in the same manner in which the longitudinal contraction or expansion is assumed to take place (Art. 66), the layers which are in compression will expand laterally and those in tension will contract.

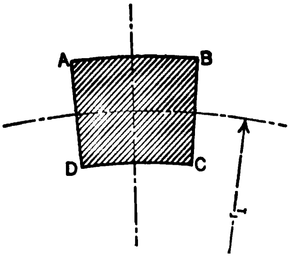


FIG. 146.

If a beam of rectangular cross section is bent so that it is concave upwards the section will take the general form shown in (Fig. 146). Since the longitudinal strains are uniformly varying, the lateral strains will be likewise uniformly varying; and the lateral strain in any layer will be $\frac{1}{m} \times$ the longitudinal strain in the layer. Hence the horizontal axis through the center of the section will become slightly curved and, if the longitudinal curvature at the section is equal to $\frac{1}{r} = \frac{e}{y}$, the curvature of the horizontal axis will be equal to $\frac{1}{r_1} = \frac{e}{my}$.

Hence $$\frac{1}{r_1} = \frac{1}{mr}, \quad \text{or} \quad r_1 = mr.$$

The assumption that the lateral expansion or contraction is free, while nearly correct when the depth of the cross section is greater than the breadth, is not correct in the case of a wide flat section. In such a case the lateral strains except at the edges will be resisted by shearing stresses between the layers.

115. Limitations of the Theory of Flexure. — All of the results, obtained in the discussion of the theory of bending in this and the preceding chapter, have been based on the primary assumption of the theory of simple bending; viz., that plane cross sections remain plane after bending (Art. 66). This assumption was first made by Bernouilli and is frequently called Bernouilli's Assumption.

A more exact analysis of the stresses and strains due to flexure by means of the principles of the Theory of Elasticity was first made by St. Venant, who took as a basis the assumption that the layers, or fibers, are free to expand or contract laterally but did not assume that plane cross sections remain plane.

The results of his investigation show that Bernouilli's assumption is correct when the shearing force is constant, but in other cases a plane cross section does not remain exactly plane after bending takes place.

The results obtained by the common theory, however, when compared with those given by the more complex theory of St. Venant are found to be substantially in agreement as far as fiber stresses and deflections due to bending are concerned, the largest difference being found in the shearing stresses and strains.

For practical purposes, the common theory, when used within the limitations imposed (Art. 63), gives results with as much accuracy as the conditions, regarding the distribution of the loads, the dimensions and homogeneity of the beams with which the engineer ordinarily has to deal, will warrant.

116. Problems — Deflection of Beams. —

Problem 1.

A standard 15″ I-beam, 42 lbs. per ft., is subjected to a system of loads such that the radius of curvature of the neutral layer at a given section is 1200 ft. Find the magnitude of the outside fiber stress at the section. $E = 30{,}000{,}000$ lbs. per sq. in.

Problem 2.

Find the diameter of the smallest pulley upon which a steel band saw, $\frac{1}{16}''$ thick, may be run, provided the allowable fiber stress due to bending is 25,000 lbs. per sq. in. $E = 30{,}000{,}000$ lbs. per sq. in.

Problem 3.

A standard 12″ I-beam, 31.5 lbs. per ft., is supported at the ends and subjected to a single concentrated load of 10,000 lbs. applied at the center. Find the greatest deflection and the deflection at a section 4 ft. from the left end. Span = 16 ft. $E = 28{,}000{,}000$ lbs. per sq. in. $I = 215.8$ (ins.)^4.

Note. — Find the deflection due to the concentrated load and that due to the weight of beam acting separately and add together (Art. 100).

Problem 4.

Solve Problem (3), assuming that the load of 10,000 lbs. is uniformly distributed over the entire length of the beam, instead of being concentrated at the center.

Problem 5.

A wooden beam 6″ × 12″ cross section and 12 ft. long is fixed at one end and subjected to a single concentrated load of 1200 lbs. applied at the free end. Find the greatest deflection. $E = 1{,}000{,}000$ lbs. per sq. in.

Problem 6.

Solve Problem (5), assuming that the load of 1200 lbs. is uniformly distributed over the entire length, instead of concentrated at the free end.

Problem 7.

Solve Problem (5), assuming that the load of 1200 lbs. is applied at a point 8 ft. from the fixed end, instead of at the free end.

Problem 8.

Find the greatest deflection of a standard 8″ I-beam, 18 lbs. per ft., due to a single concentrated load of 4000 lbs., acting as shown (Fig. 147). $I = 56.9$ (ins.)^4. $E = 28{,}000{,}000$ lbs. per sq. in. Also find the deflection under the load and at the middle of the span.

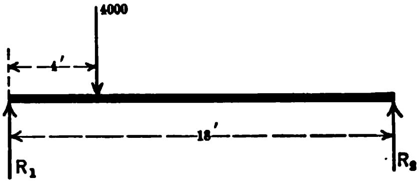


Fig. 147.

Problems 9–13.

Note. — In Problems (9–13) inclusive, deduce the general formulas for slope and deflection and find the greatest slope and greatest deflection. Express these values in terms of E, I and the constants. When the necessary dimensions are given, find the greatest deflection in inches.

Problem 9.

Cantilever beam, uniformly varying load, as shown (Fig. 148).

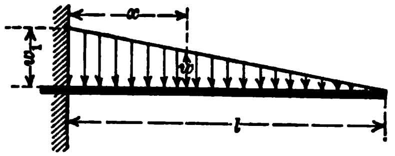


Fig. 148.

Problem 10.

Simple beam, uniformly varying load, as shown (Fig. 149).

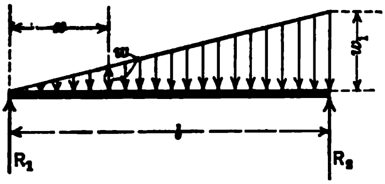


Fig. 149.

Problem 11.

Simple beam, uniformly varying load, as shown (Fig. 150).

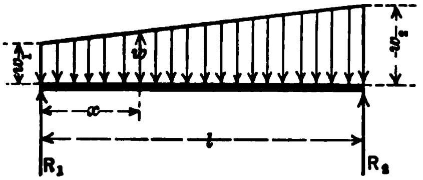


Fig. 150.

Problem 12.

A wooden beam 10″ × 12″ cross section, subjected to a load of 16,000 lbs. uniformly distributed over its entire length (Fig. 151). $E = 1,500,000$ lbs. per sq. in.

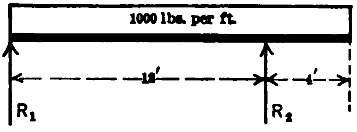


Fig. 151.

Problem 13.

A standard 6″ I-beam, 12.25 lbs. per ft., subjected to concentrated loads of 2000 lbs. and 1000 lbs., as shown (Fig. 152). $E = 28,000,000$ lbs. per sq. in.

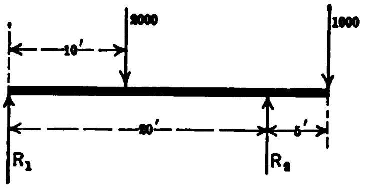


Fig. 152.

Problem 14.

A simple beam is subjected to a total load of 12,000 lbs. uniformly distributed over a portion of the span, as shown (Fig. 153). Using the general formulas

(Case i, Art. 98), write the general equations for the slope and deflection for this particular case, neglecting the weight of the beam. Also find the value of EIv_0.

Problem 15.

A floor is supported on standard 15″ I-beams, 42 lbs. per ft., having a 20-ft. span. If the total load, including the weight of the floor and beams, is 200 lbs. per sq. ft. of floor area, determine the spacing of the beams required to fulfil the condition that the greatest deflection is not to exceed $\frac{1}{360}$ of the span. $I = 442$ (ins.)^4. $E = 28{,}000{,}000$ lbs. per sq. in.

What is the magnitude of the greatest outside fiber stress in this case?

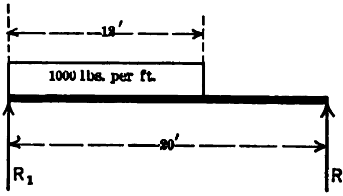


Fig. 153.

Problem 16.

Find the necessary moment of inertia and select a suitable I-section for a beam to support the loads shown (Fig. 154), provided the greatest allowable deflection is 0.5″. $E = 28{,}000{,}000$ lbs. per sq. in. Use the method indicated in Art. (100), dividing the load system into three parts.

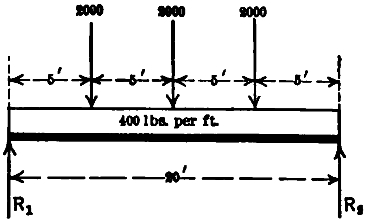


Fig. 154.

Problem 17.

An 8″ steel I-beam, 18 lbs. per ft., is fixed at the ends and subjected to a total uniformly distributed load of 9600 lbs. (including the weight of the beam). The span is 12 ft. Find the greatest deflection and the magnitude of the greatest outside fiber stress. $I = 57$ (ins.)^4. $E = 28{,}000{,}000$ lbs. per sq. in.

Problem 18.

Solve Problem (17), assuming that the load of 9600 lbs. is concentrated at the center of the span instead of being uniformly distributed over the entire length of the beam. Neglect the weight of the beam.

Problem 19.

A 12″ steel I-beam, 31.5 lbs. per ft., having a span of 12 ft., is fixed at the ends and subjected to a single concentrated load of 4000 lbs., acting at a distance of 8 ft. from one end. Find the deflection at the center and under the concentrated load. $I = 216$ (ins.)^4. $E = 28{,}000{,}000$ lbs. per sq. in. Find the greatest outside fiber stress.

Problem 20.

Solve Problem (19), assuming that the load of 4000 lbs. is divided equally into two parts and applied at points 4 ft. from the ends.

Problems 21-24.

Note. — In Problems (21-24), inclusive, determine the magnitudes of the supporting forces, and find the maximum shearing force and maximum bending moment. The supports are assumed to be on the same level in each case.

Problem 21.

Beam fixed at one end and subjected to a uniformly distributed load of 1000 lbs. per ft., as shown (Fig. 155).

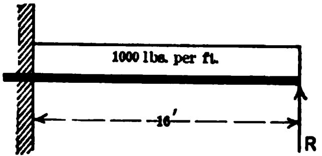


Fig. 155.

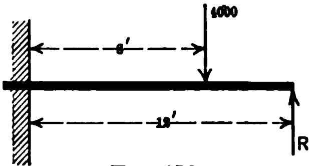


Fig. 156.

Problem 22.

Beam fixed at one end and subjected to a single concentrated load, as shown (Fig. 156).

Problem 23.

Beam fixed at one end and subjected to a single concentrated load, as shown (Fig. 157).

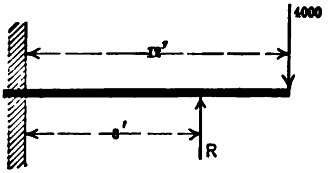


Fig. 157.

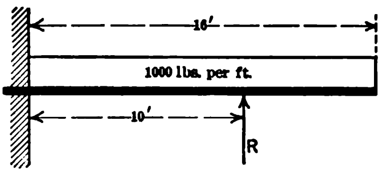


Fig. 158.

Problem 24.

Beam fixed at one end and subjected to a uniformly distributed load, as shown (Fig. 158).

Problem 25.

A wooden beam 10" × 12" cross section is supported at three points at equal distances apart, and is subjected to a total uniformly distributed load of

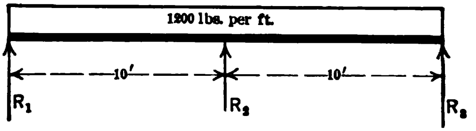


Fig. 159.

24,000 lbs. (including the weight of the beam) (Fig. 159). Find the magnitudes of the supporting forces R_1, R_2 and R_3: (a) When the supports are on the same level; (b) When the middle support is ¼" below the level of the end supports. $E = 1{,}200{,}000$ lbs. per sq. in.

Problem 26.

Find the greatest outside fiber stress in the beam given in Problem (25): (a) When the supports are on the same level; (b) When the middle support is ¼″ below the level of the end supports.

Problem 27.

Find the deflection at the middle point in each span of the beam given in Problem (25): (a) When the supports are on the same level; (b) When the middle support is ¼″ below the level of the end supports. Use the method given in Art. (100).

Problem 28.

Find the difference in level between the middle support and the two end supports of the beam in Problem (25), when the two greatest positive bending moments are equal to the greatest negative bending moment.

Problem 29.

Find the difference in level between the middle support and the two end supports of the beam in Problem (25), in order that the end reactions shall be equal to zero.

Problem 30.

A wooden beam 6″ × 12″ cross section is supported at the ends and subjected to a single concentrated load at the center of the span. Find the length of beam required to satisfy both of the following conditions: (a) The greatest deflection is $\frac{1}{360}$ of the span; (b) The greatest outside fiber stress is 1000 lbs. per sq. in.; (c) $E = 1{,}200{,}000$ lbs. per sq. in.

Problem 31.

Solve Problem (30), assuming the load to be uniformly distributed over the entire length of the beam, instead of concentrated at the center.

Problem 32.

A round bar is fixed at one end subjected as a cantilever beam to a single concentrated load at the free end. Find the ratio of length to diameter required to satisfy the conditions: (a) The greatest deflection is equal to $\frac{l}{400}$; (b) The greatest outside fiber stress is equal to 20,000 lbs. per sq. in.; (c) $E = 28{,}000{,}000$ lbs. per sq. in.

Problem 33.

Find the values of the diameter and length, if the load on the beam in Problem (32) is 20,000 lbs.

Problem 34.

A standard 24″ steel I-beam, 80 lbs. per ft., is supported at the ends and loaded uniformly throughout its entire length. If the span is 30 ft., find the greatest deflection if the greatest allowable outside fiber stress is 16,000 lbs. per sq. in. $E = 28{,}000{,}000$ lbs. per sq. in.

Problem 35.

A wooden beam of rectangular cross section, 12 ft. long and supported at the ends, is subjected to a single concentrated load at the center of the span. Find the depth of beam required to satisfy the following conditions: (a) The greatest deflection is $\frac{1}{360}$ of the span; (b) The greatest outside fiber stress is 1000 lbs. per sq. in. If the ratio of breadth to depth is 1 : 2, what load will be required to fulfil these conditions? $E = 1{,}200{,}000$ lbs. per sq. in.

Problem 36.

Solve Problem (35), assuming the load to be uniformly distributed over the entire length of the span instead of concentrated at the center.

Problem 37.

A steel I-beam, 20 ft. span, is supported at the ends and subjected to a single concentrated load at the center. Find the depth of beam required to satisfy both of the following conditions: (a) The greatest deflection is $\frac{1}{360}$ of the span; (b) The greatest outside fiber stress is 16,000 lbs. per sq. in. What is the size of the standard beam which will most nearly satisfy the conditions? What load would be required? $E = 28{,}000{,}000$ lbs. per sq. in.

Problem 38.

Solve Problem (37), assuming the load to be uniformly distributed over the entire length of the beam, instead of concentrated at the center.

Problem 39.

Solve Problem (37), assuming that the beam is fixed at the ends.

Problem 40.

Solve Problem (38), assuming that the beam is fixed at the ends.

Problem 41.

Determine the reaction R (Fig. 157) for the beam given in Problem (23) by the method of least work (Art. 111).

Problem 42.

Determine the reaction R (Fig. 158) for the beam given in Problem (24) by the method of least work.

Problem 43.

Two 15″ steel I-beams, 42 lbs. per ft., together support a total uniformly distributed load of 12,800 lbs. The upper beam AB rests on three rollers carried by the lower beam CD, and the lower beam rests on supports as indicated (Fig. 160). Find the reaction at the center roller and the maximum fiber stress in each beam. $E = 30{,}000{,}000$ lbs. per sq. in. $I = 442$ (ins.)^4. Neglect the weight of the beams.

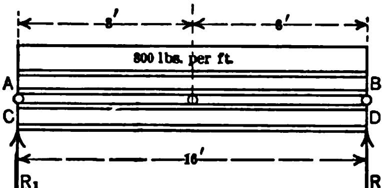


FIG. 160.

Note. — Let R = the reaction at the center roller and apply the principle of least work.

Problem 44.

Two square bars of equal length l are fixed at one end as indicated (Fig. 161). The lower bar is supported at the free end and the upper bar carries a concentrated load W at the free end and rests on a roller at A. Assuming that the supports of the lower bar are on the same level and neglecting the weight of the bars find the reaction on the roller and the supporting force F.

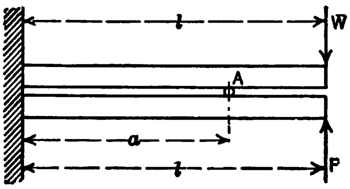


Fig. 161.

Problem 45.

Find the deflection at the center of the beam given in Problem (8), using the general formula for deflection (Art. 110).

Problem 46.

Find the deflection of the beam given in Problem (16) under the center load of 2000 lbs., using the general formula for deflection.

Problem 47.

Solve Problem (14), by using the general formula for deflection.

Problem 48.

Determine by graphical integration the greatest deflection of a built-up girder, represented in Fig. (103), under the greatest allowable load W, concentrated at the middle of the span. The dimensions of the girder are as follows: span = 30 ft.; depth of cross section at center = 33.5″; web plate — 30″ × $\frac{3}{8}$″; 4 flange angles — 5″ × 3″ × $\frac{1}{2}$″; 2 flange plates — 12″ × $\frac{1}{2}$″ × 24 ft. long; 2 flange plates — 12″ × $\frac{1}{2}$″ × 16 ft. long; 2 flange plates — 12″ × $\frac{1}{2}$″ × 8 ft. long. E = 28,000,000 lbs. per sq. in., f = 12,000 lbs. per sq. in.

Problem 49.

Determine by graphical integration the greatest deflection of the tapered shaft (Fig. 105) due to the load of 400 lbs. acting at the center of the span, neglecting the weight of the shaft. E = 30,000,000 lbs. per sq. in.

Problem 50.

Determine by graphical integration the greatest deflection of the tapered shaft (Fig. 105), due to its own weight only, assuming w = 0.28 lb. per cu. in. and E = 30,000,000 lbs. per sq. in.

CHAPTER VI.

CONTINUOUS BEAMS.

117. Continuous Beams. — A beam, or girder, which is supported at more than two points and is continuous through two or more spans is called a continuous beam, or a continuous girder. An illustration of the simplest type of a continuous beam has already been given in the case of the beam supported at three points (Art. 102). Evidently, a beam may be continuous over any number of supports and the supports may, or may not, be on the same level; and in addition, such a beam may be fixed at the ends, as in the case of the built-in beam, or the ends may be free to turn as in the case of the simple beam supported at two points. The supporting forces cannot be determined from the laws of equilibrium of Statics alone, but only when conditions in addition to these laws can be applied, as was done in the abovementioned case of the beam supported at three points.

The elastic curve will be continuous throughout the length of the beam, and the conditions resulting from continuity at the supports, as well as at concentrated loads, will be found sufficient, when employed in conjunction with the static conditions of equilibrium, to determine the supporting forces for any beam of this type. The assumptions of the common beam theory will evidently apply equally as well as in the case of the beam supported at two points. Hence the differential equation of the elastic curve will be the same in either case; and the integration of this equation under the conditions of continuity at the supports will furnish the additional conditions required in the calculation of the reactions at the supports.

When the beam is of uniform cross section and material and the loads are concentrated, or uniformly distributed, the determination of the supporting forces is comparatively simple. When the loading is irregular and the cross section varies, an approximate solution is ordinarily all that can be made.

In the following discussion, therefore, only beams of uniform section, subjected to load systems comprised of concentrated and uniformly distributed loads, will be considered.

118. The Theorem of Three Moments. — By means of the conditions due to the continuity of a continuous beam over any intermediate support, an algebraic relation between the bending moment at that support and the bending moments at the adjacent supports on either side can be determined. This relation is commonly known as the *three moment equation* and the process of its derivation is known as the *theorem of three moments.* The derivation of the three moment equation will be given here for some of the common systems of loading, the beams being assumed to be horizontal and the loads vertical in each case.

Case I. — A Single Concentrated Load in Each Span. — Let B, O and A (Fig. 162) be any three consecutive supports of a beam which is continuous over three or more supports. Take the origin at O and the horizontal axis XOX through the intersection of the neutral layer of the beam and the cross section over the support O.

Let M_0 = the bending moment at O,
M_a = the bending moment at A,
M_b = the bending moment at B,
S_0 = the shearing force at a section adjacent to and at the right side of the support O,
S_{-0} = the shearing force at a section adjacent to and at the left side of the support O,
i_0 = the slope at O, the angle being measured between the tangent and the axis OX to the right of O,
i_{-0} = the slope at O, the angle being measured between the tangent and the axis OX to the left of O,
v_a = the difference in level between the supports A and O,
v_b = the difference in level between the supports B and O,
l_1 = the length of the span OA,
l_2 = the length of the span OB,
W_1 = a concentrated load at any point in the span O–A, at distances c_1 from O and d_1 from A,
W_2 = a concentrated load at any point in the span O–B, at distances c_2 from O and d_2 from B,
M_1 = the bending moment, i_1 = the slope and v_1 = the deflection under the load W_1,

M_2 = the bending moment, i_2 = the slope and v_2 = the deflection under the load W_2,

S = the shearing force, M = the bending moment, i = the slope and v = the deflection at any cross section at a distance x from O.

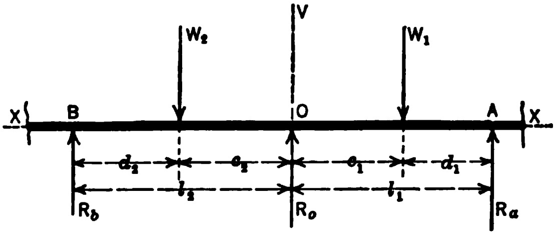


FIG. 162.

Consider first the span O–A, taking x plus when measured to the right of the origin and using the convention regarding signs which has been followed in the theory of ordinary bending.

Then for values of x from 0 to c_1,

$$S = S_0, \qquad (1)$$

$$M = M_0 + S_0x \text{ (Art. 73)}, \qquad (2)$$

$$EIi = \int M\,dx = M_0x + \frac{S_0x^2}{2} + c, \qquad (3)$$

where $c = EIi_0$, since $i = i_0$ when $x = 0$,

$$EIv = EI\int i\,dx = EIi_0x + \frac{M_0x^2}{2} + \frac{S_0x^3}{6} + c', \qquad (4)$$

where $c' = 0$, since $v = 0$ when $x = 0$.

For values of x from c_1 to l_1,

$$S = S_0 - W_1, \qquad (5)$$

$$M = M_0 + S_0x - W_1(x - c_1) \text{ (Art. 73)}, \qquad (6)$$

$$EIi = \int M\,dx = M_0x + \frac{S_0x^2}{2} - \frac{W_1}{2}(x - c_1)^2 + c'', \qquad (7)$$

where $c'' = EIi_0$, since from the continuity the value of EIi given by (7) is equal to the value given by (3) when $x = c_1$,

$$EIv = EI\int i\,dx = EIi_0x + \frac{M_0x^2}{2} + \frac{S_0x^3}{6} - \frac{W_1}{6}(x - c_1)^3 + c''', \qquad (8)$$

where $c''' = 0$, since the value of EIv given by (8) is equal to the value given by (4) when $x = c_1$.

Substituting $x = l_1$ in (6) we obtain

$$M_a = M_0 + S_0l_1 - W_1(l_1 - c_1) = M_0 + S_0l_1 - W_1d_1, \quad . \quad (9)$$

and hence

$$S_0 = \frac{M_a - M_0}{l_1} + \frac{W_1d_1}{l_1}. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

By substituting $x = l_1$ in (8),

$$EIv_a = EIi_0l_1 + \frac{M_0l_1^2}{2} + \frac{S_0l_1^3}{6} - \frac{W_1}{6}(l_1 - c_1)^3; \quad . \quad . \quad . \quad (11)$$

and eliminating S_0 between (11) and (10) and reducing,

$$EIv_a = EIi_0l_1 + \frac{M_0l_1^2}{3} + \frac{M_al_1^2}{6} + \frac{W_1d_1}{6}(l_1^2 - d_1^2); \quad . \quad . \quad (12)$$

and solving for EIi_0,

$$EIi_0 = \frac{EIv_a}{l_1} - \frac{M_0l_1}{3} - \frac{M_al_1}{6} - \frac{W_1d_1}{6\,l_1}(l_1^2 - d_1^2). \quad . \quad . \quad . \quad (13)$$

If we treat the span O–B in the same manner, taking x positive when measured to the left and reversing the signs which have been previously used in designating the directions of shearing forces and slopes, the expressions obtained for S_{-0} and EIi_{-0} will be analogous to (10) and (13) and the equations will take the forms

$$S_{-0} = \frac{M_b - M_0}{l_2} + \frac{W_2d_2}{l_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$EIi_{-0} = \frac{EIv_b}{l_2} - \frac{M_0l_2}{3} - \frac{M_bl_2}{6} - \frac{W_2d_2}{6\,l_2}(l_2^2 - d_2^2). \quad . \quad . \quad (15)$$

Adding (13) and (15) and observing that, since the signs of the slope given by the two equations are opposite, the condition of continuity at O will give $i_0 = -i_{-0}$, we shall have

$$0 = \frac{EIv_a}{l_1} + \frac{EIv_b}{l_2} - \frac{M_0}{3}(l_1 + l_2) - \frac{M_al_1}{6} - \frac{M_bl_2}{6}$$

$$- \frac{W_1d_1}{6\,l_1}(l_1^2 - d_1^2) - \frac{W_2d_2}{6\,l_2}(l_2^2 - d_2^2), \quad . \quad (16)$$

which is a form of the three moment equation for this case.

It should be observed that values of v_a and v_b will be positive when the supports A and B are higher than the support O, and negative if A and B are lower than O; also that equations (14) and (15) will give positive values of the shear and slope, respectively, when the shear and the slope are opposite in direction to those which have been previously called positive.

In the case of a continuous beam, therefore, the *rule for the sign of the shearing force*, as determined from the algebraic equations, will be: *the shearing force at a section is positive when the part of the beam between the section and the origin tends to slide upwards by the part on the other side of the section, and negative when the tendency to slide is the reverse; and the slope is positive when the tangent slopes upward and away from the origin.* Having adopted this system of signs for shearing forces and slopes, the signs of bending moments and deflections, as determined by the algebraic equations, evidently will be the same as those previously adopted, whether the span is taken to the right or the left of the origin. In plotting shearing force diagrams for continuous beams, however, it will be convenient to follow the convention of signs adopted for simple beams.

When the supports are on the same level, the three moment equation (16) reduces to the form

$$M_a l_1 + 2\,M_0\,(l_1 + l_2) + M_b l_2 = -\frac{W_1 d_1}{l_1}\,(l_1^2 - d_1^2) - \frac{W_2 d_2}{l_2}\,(l_2^2 - d_2^2). \quad . \quad . \quad . \quad . \quad (17)$$

When the loads are applied at the middle of each span, equation (17) reduces to

$$M_a l_1 + 2\,M_0\,(l_1 + l_2) + M_b l_2 = -\frac{3}{8}\,(W_1 l_1^2 + W_2 l_2^2). \quad (18)$$

When the spans are equal, if we let $l = l_1 = l_2$, equation (17) reduces to

$$(M_a + 4\,M_0 + M_b)\,l^2 = -\,W_1 d_1\,(l^2 - d_1^2) - W_2 d_2\,(l^2 - d_2^2), \quad (19)$$

and, if in this case the loads are applied at the middle of each span,

$$(M_a + 4\,M_0 + M_b) = -\frac{3}{8}\,l\,(W_1 + W_2). \quad . \quad . \quad . \quad (20)$$

Also, when the supports are on the same level and the load is at the middle of the span, equation (13) will reduce to

$$EIi_0 = -\frac{M_0 l_1}{3} - \frac{M_a l_1}{6} - \frac{W_1 l_1^2}{16}; \quad . \quad . \quad . \quad . \quad (21)$$

and equation (10) will reduce to

$$S_0 = \frac{M_a - M_0}{l_1} + \frac{W_1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Case II. — Any Number of Concentrated Loads in Each Span. — Using a notation similar to that in Case (I), let l_1 = the length of the span *O–A* and denote the loads in the span by the symbols

W_1', W_1'', W_1''', etc., the distances of the loads from O by the symbols c_1', c_1'', c_1''', etc., and their distances from A by the symbols d_1', d_1'', d_1''', etc.

The expressions for the shearing force and the slope, at the right of the support O, can be found for each load acting separately; and the values of these quantities due to the combination of all the loads in the span may then be found by adding together, by the method indicated in (Art. 100).

Let i_0', M_0', S_0', represent the values of the slope, bending moment and shearing force, respectively, at the support O; and M_a' and v_a' the bending moment and deflection at the support A if the load W' were the only load in the span O–A; and, similarly, let i_0'', M_0'', S_0'', M_a'' and v_a'' represent the respective values of these quantities if the load W'' were the only load in the span; and i_0''', M_0''', S_0''', M_a''' and v_a''' the respective values due to the load W''' alone; etc.

Then from equation (13)

$$EIi_0' = \frac{EIv_a'}{l_1} - \frac{M_0'l_1}{3} - \frac{M_a'l_1}{6} - \frac{W_1'd_1'}{6\,l_1}(l_1^2 - d_1'^2),$$

$$EIi_0'' = \frac{EIv_a''}{l_1} - \frac{M_0''l_1}{3} - \frac{M_a''l_1}{6} - \frac{W_1''d_1''}{6\,l_1}(l_1^2 - d_1''^2),$$

$$EIi_0''' = \frac{EIv_a'''}{l_1} - \frac{M_0'''l_1}{3} - \frac{M_a'''l_1}{6} - \frac{W_1'''d_1'''}{6\,l_1}(l_1^2 - d_1'''^2), \text{ etc.}$$

By adding these equations and substituting

$$\begin{aligned} i_0 &= i_0' + i_0'' + i_0''' + \cdots, \\ v_a &= v_a' + v_a'' + v_a''' + \cdots, \\ M_0 &= M_0' + M_0'' + M_0''' + \cdots, \\ M_a &= M_a' + M_a'' + M_a''' + \cdots, \end{aligned}$$

we obtain for the value of EIi_0, when all the loads act together,

$$EIi_0 = \frac{EIv_a}{l_1} - \frac{M_0l_1}{3} - \frac{M_al_1}{6} - \frac{\Sigma W_1d_1\,(l_1^2 - d_1^2)}{6\,l_1}, \quad . \quad (23)$$

where

$$\begin{aligned} \Sigma W_1d_1\,(l_1^2 - d_1^2) = {} & W'd_1'\,(l_1^2 - d_1'^2) + W_1''d_1''\,(l_1^2 - d_1''^2) \\ & + W_1'''d_1'''\,(l_1^2 - d_1'''^2) + \text{etc.} \end{aligned}$$

In the same manner, a similar expression for the value of EIi_{-0}, in terms of the loads acting in the span O–B, can be obtained, viz.,

$$EIi_{-0} = \frac{EIv_b}{l_2} - \frac{M_0l_2}{3} - \frac{M_bl_2}{6} - \frac{\Sigma W_2d_2\,(l_2^2 - d_2^2)}{6\,l_2}. \quad . \quad (24)$$

By adding (23) and (24), the three moment equation for any system of concentrated loads is obtained, viz.,

$$0 = \frac{EIv_a}{l_1} + \frac{EIv_b}{l_2} - \frac{M_0}{3}(l_1 + l_2) - \frac{M_a l_1}{6} - \frac{M_b l_2}{6} - \frac{\Sigma W_1 d_1 (l_1^2 - d_1^2)}{6\, l_1} - \frac{\Sigma W_2 d_2 (l_2^2 - d_2^2)}{6\, l_2}. \quad (25)$$

Proceeding in the same manner, the expressions for the shearing force at the right of O, due to each load in the span O–A acting alone, would be

$$S_0' = \frac{M_a' - M_0'}{l_1} + \frac{W_1' d_1'}{l_1},$$

$$S_0'' = \frac{M_a'' - M_0''}{l_1} + \frac{W_1'' d_1''}{l_1}, \text{ etc.}$$

Letting $S_0 = S_0' + S_0'' +$ etc., equal the shearing force at the section at the right of O, due to the combined action of the loads in the span $O - A$, and adding together as before,

$$S_0 = \frac{M_a - M_0}{l_1} + \frac{\Sigma W_1 d_1}{l_1}, \quad \ldots \ldots \quad (26)$$

where $\quad \Sigma W_1 d_1 = W_1' d_1' + W_1'' d_1'' + W_1''' d_1''' +$ etc.

Similarly,

$$S_{-0} = \frac{M_b - M_0}{l_2} + \frac{\Sigma W_2 d_2}{l_2}. \quad \ldots \ldots \quad (27)$$

When the supports are on the same level, the three moment equation (25) may be written

$$M_a l_1 + 2\, M_0 (l_1 + l_2) + M_b l_2 = -\frac{\Sigma W_1 d_1 (l_1^2 - d_1^2)}{l_1} - \frac{\Sigma W_2 d_2 (l_2^2 - d_2^2)}{l_2}, \quad . \quad (28)$$

and if, in addition, the spans are equal and we let $l = l_1 = l_2$, equation (28) reduces to

$$(M_a + 4\, M_0 + M_b)\, l^2 = -\Sigma W_1 d_1 (l^2 - d_1^2) - \Sigma W_2 d_2 (l^2 - d_2^2). \quad (29)$$

The formulas for this case will evidently apply when any of the loads act upwards, instead of downwards, by simply giving a negative sign to the numerical value of any upward load, when substituting in the algebraic equations.

Case III. — *Load Uniformly Distributed over Each Span* (Fig. 163).

Let w_1 = the load intensity in the span O–A and w_2 = the load intensity in the span O–B. Then, using the same notation as in Case (I) we have for all values of x in the span O–A

$$S = S_0 - w_1 x, \qquad (30)$$

$$M = M_0 + S_0 x - \frac{w_1 x^2}{2} \text{ (Art. 73)}, \qquad (31)$$

$$EIi = \int M\,dx = M_0 x + \frac{S_0 x^2}{2} - \frac{w_1 x^3}{6} + c, \qquad (32)$$

where $c = EIi_0$, since $i = i_0$ when $x = 0$,

$$EIv = EI\int i\,dx = EIi_0 x + \frac{M_0 x^2}{2} + \frac{S_0 x^3}{6} - \frac{w_1 x^4}{24} + c', \qquad (33)$$

where $c' = 0$, since $v = 0$ when $x = 0$.

Substituting $x = l_1$ in (31) we obtain

$$M_a = M_0 + S_0 l_1 - \frac{w_1 l_1^2}{2}, \qquad (34)$$

and hence

$$S_0 = \frac{M_a - M_0}{l_1} + \frac{w_1 l_1}{2}; \qquad (35)$$

and by substituting $x = l_1$ in (33),

$$EIv_a = EIi_0 l_1 + \frac{M_0 l_1^2}{2} + \frac{S_0 l_1^3}{6} - \frac{w_1 l_1^4}{24}. \qquad (36)$$

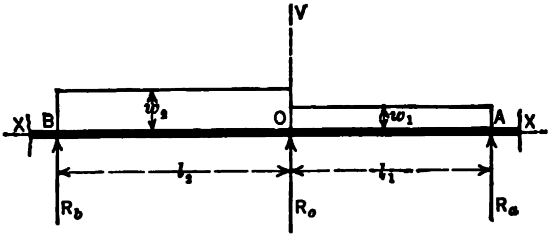


FIG. 163.

Eliminating S_0 between (36) and (35) and reducing and solving for EIi_0,

$$EIi_0 = \frac{EIv_a}{l_1} - \frac{M_0 l_1}{3} - \frac{M_a l_1}{6} - \frac{w_1 l_1^3}{24}. \qquad (37)$$

Treating the span O–B in the same manner and reversing the signs of shearing forces and slopes, as in Case (I), we obtain from the analogy with (35) and (37),

$$S_{-0} = \frac{M_b - M_0}{l_2} + \frac{w_2 l_2}{2}, \quad . \; . \; . \; . \; . \; . \; . \; . \tag{38}$$

$$EIi_{-0} = \frac{EIv_b}{l_2} - \frac{M_0 l_2}{3} - \frac{M_b l_2}{6} - \frac{w_2 l_2^3}{24}; \quad . \; . \; . \tag{39}$$

and by adding (37) and (39) we obtain

$$0 = \frac{EIv_a}{l_1} + \frac{EIv_b}{l_2} - \frac{M_0}{3}(l_1 + l_2) - \frac{M_a l_1}{6} - \frac{M_b l_2}{6} - \frac{w_1 l_1^3}{24} - \frac{w_2 l_2^3}{24}, \tag{40}$$

which is the three moment equation for this case.

When the supports are on the same level, equation (40) may be written

$$M_a l_1 + 2M_0(l_1 + l_2) + M_b l_2 = -\frac{w_1 l_1^3}{4} - \frac{w_2 l_2^3}{4}; \quad . \tag{41}$$

and equation (37) reduces to

$$EIi_0 = -\frac{M_0 l_1}{3} - \frac{M_a l_1}{6} - \frac{w_1 l_1^3}{24}. \quad . \; . \; . \; . \; . \tag{42}$$

When the spans are equal, if we let $l = l_1 = l_2$, equation (41) reduces to

$$M_a + 4M_0 + M_b = -\frac{l^2}{4}(w_1 + w_2). \quad . \; . \; . \; . \tag{43}$$

Case IV. — Concentrated Loads and Uniformly Distributed Loads over the Entire Length of Each Span. — By the same method of reasoning as that employed in Case (II), the following expressions for the three moment equation, and for the slope and shearing force at the origin, for the span O–A, for any system of concentrated loads and loads uniformly distributed over the entire length of each span may be obtained:

$$\begin{aligned} 0 = {} & \frac{EIv_a}{l_1} + \frac{EIv_b}{l_2} - \frac{M_0}{3}(l_1 + l_2) - \frac{M_a l_1}{6} - \frac{M_b l_2}{6} \\ & - \frac{\Sigma W_1 d_1 (l_1^2 - d_1^2)}{6 l_1} - \frac{w_1 l_1^3}{24} - \frac{\Sigma W_2 d_2 (l_2^2 - d_2^2)}{6 l_2} - \frac{w_2 l_2^3}{24}, \end{aligned} \tag{44}$$

$$EIi_0 = \frac{EIv_a}{l_1} - \frac{M_0 l_1}{3} - \frac{M_a l_1}{6} - \frac{\Sigma W_1 d_1 (l_1^2 - d_1^2)}{6 l_1} - \frac{w_1 l_1^3}{24}, \quad . \tag{45}$$

$$S_0 = \frac{M_a - M_0}{l_1} + \frac{\Sigma W_1 d_1}{l_1} + \frac{w_1 l_1}{2}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \tag{46}$$

When the supports are on the same level, or when the spans are equal, these equations may be easily reduced to simpler forms, as before.

Case V. — *Load Uniformly Distributed over a Portion of Each Span. General Case.* (Fig. 164.)

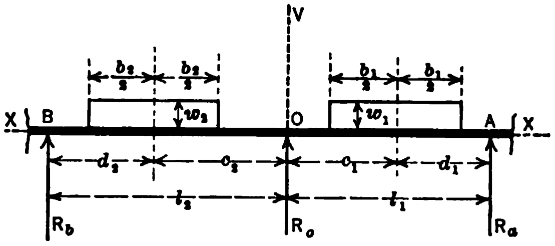


FIG. 164.

Let w_1 = the load intensity, b_1 = the distance over which the load extends, c_1 = the distance from O and d_1 = the distance from A to the center of the load in the span O–A, and let $W_1 = w_1 b_1$ equal the total load in this span.

Let w_2 = the load intensity, b_2 = the distance over which the load extends, c_2 = the distance from O and d_2 = the distance from B to the center of the load in the span O–B; and let $W_2 = w_2 b_2$ equal the total load in this span.

Using the same notation as in Case (I), we have, for values of x from O to $\left(c_1 - \frac{b_1}{2}\right)$ in the span O–A,

$$S = S_0, \qquad (47)$$

$$M = M_0 + S_0 x \text{ (Art. 73)}, \qquad (48)$$

$$EIi = M_0 x + \frac{S_0 x^2}{2} + c, \qquad (49)$$

where $c = EIi_0$, since $i = i_0$ when $x = 0$,

$$EIv = EIi_0 x + \frac{M_0 x^2}{2} + \frac{S_0 x^3}{6} + c', \qquad (50)$$

where $c' = 0$, since $v = 0$ when $x = 0$.

For values of x from $\left(c_1 - \frac{b_1}{2}\right)$ to $\left(c_1 + \frac{b_1}{2}\right)$,

$$S = S_0 - w_1\left(x - c_1 + \frac{b_1}{2}\right), \qquad (51)$$

$$M = M_0 + S_0x - \frac{w_1}{2}\left(x - c_1 + \frac{b_1}{2}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad (52)$$

$$EIi = M_0x + \frac{S_0x^2}{2} - \frac{w_1}{6}\left(x - c_1 + \frac{b_1}{2}\right)^3 + c'', \quad . \quad . \quad (53)$$

where $c'' = EIi_0$, since (53) and (49) give the same value for EIi when $x = \left(c_1 - \frac{b_1}{2}\right)$,

$$EIv = EIi_0x + \frac{M_0x^2}{2} + \frac{S_0x^3}{6} - \frac{w_1}{24}\left(x - c_1 + \frac{b_1}{2}\right)^4 + c''', \quad (54)$$

where $c''' = 0$, since (54) and (50) give the same value for EIv when $x = \left(c_1 - \frac{b_1}{2}\right)$.

For values of x from $\left(c_1 + \frac{b_1}{2}\right)$ to l_1,

$$S = S_0 - w_1b_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

$$M = M_0 + S_0x - w_1b_1(x - c_1), \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$EIi = M_0x + \frac{S_0x^2}{2} - \frac{w_1b_1}{2}(x - c_1)^2 + c^{\text{iv}}, \quad . \quad . \quad (57)$$

where $c^{\text{iv}} = EIi_0 - \frac{w_1b_1^3}{24}$, from the condition that (57) and (53) give the same value for EIi when $x = \left(c_1 + \frac{b_1}{2}\right)$,

$$EIv = EIi_0x + \frac{M_0x^2}{2} + \frac{S_0x^3}{6} - \frac{w_1b_1}{6}(x - c_1)^3 - \frac{w_1b_1^3x}{24} + c^{\text{v}}, \quad (58)$$

where $c^{\text{v}} = \frac{w_1b_1^3c_1}{24}$, from the condition that (58) and (54) give the same value for EIv when $x = \left(c_1 + \frac{b_1}{2}\right)$.

Substituting $x = l_1$ in (56), we obtain

$$M_a = M_0 + S_0l_1 - w_1b_1d_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and hence

$$S_0 = \frac{M_a - M_0}{l_1} + \frac{w_1b_1d_1}{l_1} = \frac{M_a - M_0}{l_1} + \frac{W_1d_1}{l_1}. \quad . \quad . \quad (60)$$

By substituting $x = l_1$ in (58),

$$EIv_a = EIi_0l_1 + \frac{M_0l_1^2}{2} + \frac{S_0l_1^3}{6} - \frac{w_1b_1d_1^3}{6} - \frac{w_1b_1^3l_1}{24} + \frac{w_1b_1^3c_1}{24}; \quad (61)$$

and eliminating S_0 between (61) and (59) and reducing,

$$EIv_a = EIi_0 l_1 + \frac{M_0 l_1^2}{3} + \frac{M_a l_1^2}{6} + \frac{w_1 b_1 d_1}{24}[4(l_1^2 - d_1^2) - b_1^2]; \qquad (62)$$

and solving for EIi_0, and substituting $W_1 = w_1 b_1$,

$$EIi_0 = \frac{EIv_a}{l_1} - \frac{M_0 l_1}{3} - \frac{M_a l_1}{6} - \frac{W_1 d_1}{24\, l_1}[4(l_1^2 - d_1^2) - b_1^2]. \quad . \quad (63)$$

For the span O–B we shall have, by analogy,

$$S_{-0} = \frac{M_b - M_0}{l_2} + \frac{W_2 d_2}{l_2} \quad . \quad . \quad . \quad . \quad . \quad (64)$$

and

$$EIi_{-0} = \frac{EIv_b}{l_2} - \frac{M_0 l_2}{3} - \frac{M_b l_2}{6} - \frac{W_2 d_2}{24\, l_2}[4(l_2^2 - d_2^2) - b_2^2]. \quad . \quad (65)$$

Adding (63) and (65) we have, for the three moment equation for this case,

$$0 = \frac{EIv_a}{l_1} + \frac{EIv_b}{l_2} - \frac{M_0}{3}(l_1 + l_2) - \frac{M_a l_1}{6} - \frac{M_b l_2}{6}$$
$$- \frac{W_1 d_1}{24\, l_1}[4(l_1^2 - d_1^2) - b_1^2] - \frac{W_2 d_2}{24\, l_2}[4(l_2^2 - d_2^2) - b_2^2]. \quad (66)$$

When the supports are on the same level (66) reduces to

$$M_a l_1 + 2\,M_0(l_1 + l_2) + M_b l_2 = -\frac{W_1 d_1}{4\, l_1}[4(l_1^2 - d_1^2) - b_1^2]$$
$$- \frac{W_2 d_2}{4 l_2}[4(l_2^2 - d_2^2) - b_2^2]; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

and when the spans are equal, if we let $l = l_1 = l_2$, equation (67) reduces to

$$(M_a + 4\,M_0 + M_b)\,l^2 = -\frac{W_1 d_1}{4}[4(l_1^2 - d_1^2) - b_1^2]$$
$$- \frac{W_2 d_2}{4}[4(l_2^2 - d_2^2) - b_2^2] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

When the loads are uniformly distributed over the entire length of each span, $b_1 = l_1$, $d_1 = \frac{l_1}{2}$, $b_2 = l_2$ and $d_2 = \frac{l_2}{2}$ and equations (60) and (63–68) inclusive reduce to the corresponding equations for Case (III).

When the loads are concentrated, $b_1 = 0$ and $b_2 = 0$ and equations (60) and (63–68) inclusive reduce to the corresponding equations for Case (I).

When there are two or more superimposed loads in a span, by the same method of reasoning as that employed in Case (II), the following expression for the three moment equation can be obtained,

$$0 = \frac{EIv_a}{l_1} + \frac{EIv_b}{l_2} - \frac{M_0}{3}(l_1 + l_2) - \frac{M_a l_1}{6} - \frac{M_b l_2}{6}$$
$$- \frac{\Sigma W_1 d_1 [4 (l_1^2 - d_1^2) - b_1^2]}{24\, l_1} - \frac{\Sigma W_2 d_2 [4 (l_2^2 - d_2^2) - b_2^2]}{24\, l_2}; \quad (69)$$

also, the following expressions for the shearing force and the slope at the origin,

$$S_0 = \frac{M_a - M_0}{l_1} + \frac{\Sigma W_1 d_1}{l_1}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (70)$$

$$EIi_0 = \frac{EIv_a}{l_1} - \frac{M_0 l_1}{3} - \frac{M_a l_1}{6} - \frac{\Sigma W_1 d_1 [4 (l_1^2 - d_1^2) - b_1^2]}{24\, l_1}. \quad (71)$$

The equations for this case may, therefore, be considered to be the general equations for a continuous beam, subjected to any system of concentrated and uniformly distributed loads, which can easily be reduced to the forms obtained for the four special cases previously considered.

119. Determination of the Bending Moments and Reactions at the Supports of any Continuous Beam. — The bending moments and reactions at the supports of any continuous beam of uniform cross section, subjected to concentrated and uniformly distributed loads, may be determined by the use of the three moment equation, in its various forms, and the equations for the shearing forces and slopes deduced in Art. (118). The method of procedure will be as follows: Let the sketch (Fig. 165) represent a continuous beam, having four spans, which is subjected to a system of concentrated and uniformly distributed loads as indicated.

By use of the three moment equation, with the origin taken successively at the supports 2, 3 and 4, three separate equations in terms of the unknown bending moments at the supports can be obtained. If the beam is free at the ends and does not overhang the end supports, the bending moments at the end supports will be equal to zero and, by solving the equations simultaneously, the bending moments at the intermediate supports can be found. If the beam overhangs the end supports, the bending moments at these supports can be calculated from the loads on the overhangs, leaving the bending moments at the intermediate supports to be determined as before. If the ends of the beam are fixed, the three moment equation can be applied with the origin taken at each intermediate support, as before; and the additional equations, required to determine the bending moments at the ends, can be obtained by substituting $i_0 = 0$ in the equation for the slope at a support taken as the origin.

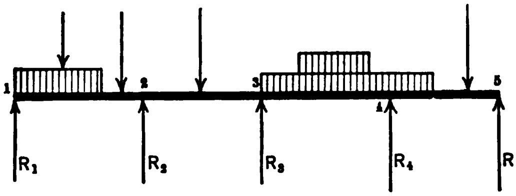


FIG. 165.

Having calculated the values of the bending moments at the supports, the shearing forces on both sides of each intermediate support and at the inside of each end support can be found by use of the formulas for the shearing force at the right, or left, of any support taken as an origin.

The supporting force for any intermediate support can then be found by adding the shearing forces on the two sides, keeping in mind the reversal of the signs of the shears on the two sides of the support. When the ends are free, the end supports will evidently be equal to the shearing forces at the ends, and, when the ends are fixed, the reactions at each end will be made up of a shearing force and a bending moment, the latter being found as previously described.

An inspection of the equations given in Art. (118) and of the process of deriving them will show:

(a) That any of the equations for the shearing force, or the slope, at the right, or left, of a support taken as an origin, are entirely independent of the loads or dimensions in any span, other than that to which the equations apply. Hence these can be applied with any support (intermediate or end support) taken as the origin. In fact these equations will be found to apply in the case of any simple beam, or a beam fixed at the ends, such as were discussed in Chapter V.

(b) That the different forms of the three moment equation will apply only when the origin is taken at one of the intermediate supports, since these equations embody the length of the span and the difference in level of the adjacent support, on each side of the origin.

(c) That, when the supports are on the same level, the bending moments, shearing forces and slopes at the different supports are independent of the size and material of the beam, provided the conditions are in accord with the limitations and assumptions of the beam theory.

(d) That, when the supports are not on the same level, the bending moments, shearing forces and slopes at the supports will depend on the material and the size and dimensions of the cross section of the beam. In many cases a small change in the level of one, or more, supports will be found to affect the bending moments and shearing forces to a very considerable extent.

For this reason continuous beams are not suitable to use in types of construction where a small change in level of supports, due to unequal settlement of foundations or other causes, will seriously affect the magnitudes of the stresses.

(e) That, when the cross sections of the beam vary, or the load is distributed non-uniformly, a form of the three moment equation can be derived, provided the moment of inertia and the load intensity can be expressed as integrable functions of x. Ordinarily in cases of this kind, it is necessary to make an approximate solution by dividing the beam up into sections and taking average values for the load intensity and the moment of inertia for each section, the approximation evidently depending on the number of the sections into which the beam is divided.

(f) If, for a continuous girder which is of non-uniform section or is constructed in such a manner that the central axis is not straight, the supports are made to conform to the shape of the girder, the shearing forces and bending moments throughout the length of the girder will vary in the same manner as when the central axis is a straight line and the supports are on the same level.

120. Determination of the Greatest Fiber Stress and the Greatest Deflection. — After having determined the bending moments and the shearing forces at the supports, the shearing force and bending moment at any section in any span of a continuous beam can be found by the method indicated in Art. (73). The sections in the different spans at which the shearing force is equal to zero will be sections at which the bending moment has maximum values. Ordinarily the bending moments at the intermediate supports, or at the fixed ends, will be negative and the bending moments at the sections of zero shear in the different spans will be positive. It is possible, however, that the loading and the conditions at the supports may be such that the signs in either case will be reversed.

Having calculated the maximum value of the bending moment in each span, as well as the values of the bending moments at the supports, the greatest bending moment in the entire beam can be determined by inspection; and the greatest outside fiber stress can be calculated by the usual formula.

To determine the greatest deflection in any span it is necessary to write the bending moment equations, in terms of the bending moment and shearing force at one end of the span (Art. 73), and by integration to determine the slope and deflection equations; the constant of integration for the slope equation being determined from one of the expressions for the slope at a support which have already been deduced. The point, or points, of greatest deflection, above or below the horizontal axis, can then be found by placing the slope equation equal to zero, after which the greatest deflection in the span can be calculated. By determining in this manner the greatest deflection in each span, the greatest deflection in the entire beam from any horizontal axis as a datum line is readily obtained.

It is evident from the foregoing analysis that it is unnecessary to calculate the supporting forces of a continuous beam, in order to determine the greatest fiber stress or the greatest deflection. If desired, however, the magnitude of any intermediate support may be obtained by adding the shearing forces on either side together, taking account of signs. The magnitude of an intermediate supporting force can be expressed in terms of the bending moments, at the support and the two adjacent supports, using the notation for the general case (Case V), as follows:

$$R_0 = S_0 + S_{-0} = \frac{M_a - M_0}{l_1} + \frac{M_b - M_0}{l_2} + \frac{\Sigma W_1 d_1}{l_1} + \frac{\Sigma W_2 d_2}{l_2}, \quad (1)$$

where $\frac{\Sigma W_1 d_1}{l_1}$ and $\frac{\Sigma W_2 d_2}{l_2}$ are evidently the components at the support, which would be obtained by treating the adjacent spans as two simple beams.

For an end support, when the beam is free,

$$R_0 = S_0 + S_{-0} = \frac{M_a - M_0}{l_1} + \frac{\Sigma W_1 d_1}{l_1} + S_{-0}, \quad . \quad . \quad (2)$$

where S_{-0} = the shear due to the load on the overhanging end, if any, the previous observation in regard to the term $\frac{\Sigma W_1 d_1}{l_1}$ applying as before.

After having obtained the supporting forces, the expressions for the shearing force, bending moment, slope and deflection at any section of a continuous beam can evidently be written in terms of all the forces acting on the portion of the beam between the section and either end, as in the case of any simple beam. This method involves, in general, the use of more complicated equations than the one previously outlined.

121. Continuous Beams with Equal Spans and Loads Uniformly Distributed. — The formulas for continuous beams of this type, having the supports on the same level, are more generally used than those for any other type.

For a beam, with any number of equal spans and a uniformly distributed load of the same intensity w throughout its entire length, having the supports on the same level, the three moment equation (Case III) will reduce to the form

$$M_a + 4\,M_0 + M_b = -\frac{wl^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the expression for the shearing force at the right of the origin will take the form

$$S_0 = \frac{M_a - M_0}{l} + \frac{wl}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In applying these equations in the following cases, the bending moments at the supports will be denoted by M_1, M_2, M_3, etc., the subscripts corresponding to the numbers at the supports, the maximum positive values of the bending moments between the supports will be denoted by M', M'', M''', etc., the primes corresponding to the number of the span. The shearing forces at the right of the supports will be denoted by positive subscripts and those at the left by negative subscripts.

(a) *Two equal spans* (Fig. 166).

Applying equation (1) with the origin at the support (2)

$$M_1 = 0, \qquad M_2 = -\frac{wl_2}{8}, \qquad M_3 = 0. \quad . \quad . \quad . \quad (3)$$

Applying equation (2) with the origin at the support (1),

$$S_1 = \tfrac{3}{8}\,wl; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and, following the usual convention of signs for shearing forces,

$$S_{-2} = \tfrac{3}{8}\,wl - wl = -\tfrac{5}{8}\,wl; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and, from symmetry,

$$S_2 = \tfrac{5}{8}\,wl, \qquad S_{-3} = -\tfrac{3}{8}\,wl. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

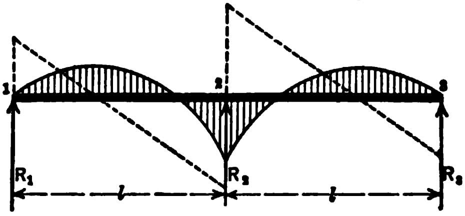


FIG. 166.

A maximum value of the bending moment in the span (1–2) will occur at a distance $\frac{3\,l}{8}$ from the support (1) and will be equal to

$$M' = \frac{3\,wl}{8} \times \frac{3\,l}{8} - \frac{3\,wl}{8} \times \frac{3\,l}{16} = \frac{9}{128}\,wl^2, \quad . \quad . \quad . \quad (7)$$

and, from symmetry, the maximum positive bending moment in the span (2–3),

$$M'' = \frac{9}{128}\,wl^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The supporting forces will be equal to

$$R_1 = \tfrac{3}{8}\,wl, \qquad R_2 = \tfrac{5}{4}\,wl, \qquad R_3 = \tfrac{3}{8}\,wl. \quad . \quad . \quad . \quad (9)$$

The greatest bending moment for the entire beam will evidently be that at the middle support,

$$M_2 = -\frac{wl^2}{8}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and the greatest shearing force will be that on either side of the middle support,

$$S_2 = -S_{-2} = \frac{5\,wl}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

(b) *Three equal spans* (Fig. 167).

Applying equation (1) with the origin at the support (2),

$$M_3 + 4\,M_2 = -\frac{wl^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and applying (1) again, with the origin at the support (3),

$$4\,M_3 + M_2 = -\frac{wl^2}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Solving (12) and (13) simultaneously

$$M_2 = -\frac{wl^2}{10}, \quad M_3 = -\frac{wl^2}{10}; \quad . \; . \; . \; . \quad (14)$$

also,

$$M_1 = 0, \qquad M_4 = 0. \quad . \; . \; . \; . \; . \; . \; . \quad (15)$$

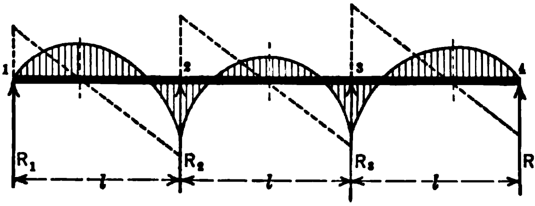


FIG. 167.

Applying equation (2) with the origin at the support (1),

$$S_1 = \tfrac{2}{5}\,wl \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

and, following the usual convention of signs,

$$S_{-2} = \tfrac{2}{5}\,wl - wl = -\tfrac{3}{5}\,wl. \quad . \; . \; . \; . \; . \quad (17)$$

From the symmetry of loading,

$$S_2 = \frac{wl}{2}, \qquad S_{-3} = -\frac{wl}{2}, \quad . \; . \; . \; . \; . \; . \quad (18)$$

and also,

$$S_3 = \tfrac{3}{5}\,wl, \quad S_{-4} = \tfrac{2}{5}\,wl. \quad . \; . \; . \; . \; . \; . \quad (19)$$

A maximum value of the bending moment in the span (1–2) will be obtained at the section, distant $\frac{2\,l}{5}$ from the support (1), and will be equal to

$$M' = \frac{2\,wl}{5} \times \frac{2\,l}{5} - \frac{2\,wl}{5} \times \frac{l}{5} = \frac{2\,wl^2}{25}. \quad . \; . \; . \quad (20)$$

From symmetry, the maximum value in the span (3–4) will in like manner be equal to

$$M''' = \frac{2\,wl^2}{25}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

A maximum value of the bending moment will also be obtained at the middle section of the span (2–3) and will be equal to

$$M'' = -\frac{wl^2}{10} + \frac{wl}{2} \times \frac{l}{2} - \frac{wl}{2} \times \frac{l}{4} = \frac{wl^2}{40}. \quad . \; . \; . \quad (22)$$

The supporting forces will be equal to

$$R_1 = \tfrac{2}{5}\,wl, \quad R_2 = \tfrac{11}{10}\,wl, \quad R_3 = \tfrac{11}{10}\,wl, \quad R_4 = \tfrac{2}{5}\,wl. \quad . \quad (23)$$

The greatest bending moment for the entire beam will be that at either intermediate support,

$$M_2 = M_3 = -\frac{wl^2}{10}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and the greatest shearing force will be in either end span, at the intermediate support,

$$S_3 = -S_{-2} = \frac{3\,wl}{5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

By applying equations (1) and (2) to the supports in succession, the values of the bending moments, shearing forces and supporting forces for similar beams, with any number of equal spans, can be easily determined. In every case the greatest bending moment will occur at one of the intermediate supports and the greatest shearing force will occur in the end span, at the first intermediate support. The bending moments at the supports will in each case be expressed in the form

$$k_m wl^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where k_m = a numerical coefficient, and the expressions for the shearing forces at the supports will take the form

$$k_s wl, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where k_s = a numerical coefficient.

The following diagram on page (278) shows, in tabulated form, the numerical values, disregarding signs, for k_m and k_s at the successive supports of continuous beams, having from two to seven equal spans, subjected to uniformly distributed loads. .

The central axis of the beam and the supports in each case are represented in the conventional manner, the span numbers being indicated by figures placed in the middle of each span. These numbers in each case may also be taken as the numbers at the supports of the beam indicated above. The values of k_m for each support are placed directly over the center of the support, while the values of k_s are indicated below the central axis. To save repetition, the numerators of the fractions, representing the values of k_s on each side of each support, are shown, while under the middle of the support the common denominator for both fractions is given. By adding the numerators on the two sides of any support the coefficient of wl, giving the magnitude of the supporting force at that point, is evidently obtained.

For example, in the case of the beam with four spans the magnitude of the bending moment over the second support will be equal to

$$M_2 = \tfrac{3}{28} wl^2;$$

and the magnitude of the shear to the right of the same support will be

$$S_2 = \tfrac{15}{28} wl,$$

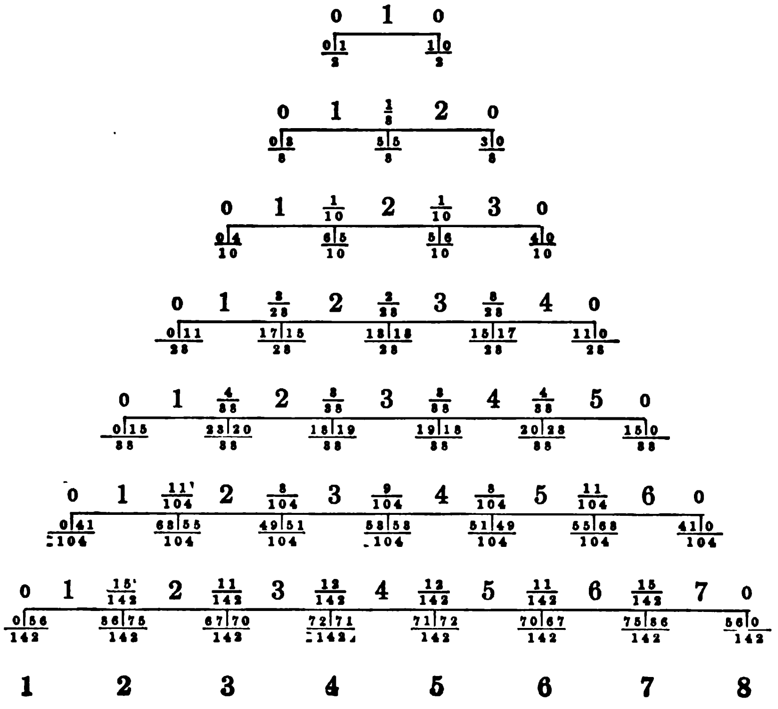
while that to the left will be

$$S_{-2} = \tfrac{17}{28} wl;$$

and the supporting force at this point will be equal to

$$R_2 = \tfrac{32}{28} wl.$$

It should be noted that, as the number of spans increases, the values of the shearing forces and bending moments in the central spans become nearly equal to the values of these quantities for a beam fixed at the ends, subjected to a uniformly distributed load (Art. 101).



0 1 0
0|1 over 2, 1|0 over 2
0 1 1/8 2 0
0|3 over 8, 5|5 over 8, 3|0 over 8
0 1 1/10 2 1/10 3 0
0|4 over 10, 6|5 over 10, 5|6 over 10, 4|0 over 10
0 1 3/28 2 2/28 3 3/28 4 0
0|11 over 28, 17|15 over 28, 13|13 over 28, 15|17 over 28, 11|0 over 28
0 1 4/38 2 3/38 3 3/38 4 4/38 5 0
0|15 over 38, 23|20 over 38, 18|19 over 38, 19|18 over 38, 20|23 over 38, 15|0 over 38
0 1 11/104 2 8/104 3 9/104 4 8/104 5 11/104 6 0
0|41 over 104, 63|55 over 104, 49|51 over 104, 53|53 over 104, 51|49 over 104, 55|63 over 104, 41|0 over 104
0 1 15/142 2 11/142 3 12/142 4 12/142 5 11/142 6 15/142 7 0
0|56 over 142, 86|75 over 142, 67|70 over 142, 72|71 over 142, 71|72 over 142, 70|67 over 142, 75|86 over 142, 56|0 over 142
1 2 3 4 5 6 7 8

122. Approximate Method of Determining Shearing Forces and Bending Moments in a Continuous Beam.—If the loading on a continuous beam is complex, or if the conditions regarding the level of the supports, or the constraining forces at the ends, when the ends are not free, are not definitely known, an approximate solution for the shearing forces and bending moments in any one span can be made by estimating the position of the points of inflexion and following the usual method employed in the case of simple and cantilever beams.

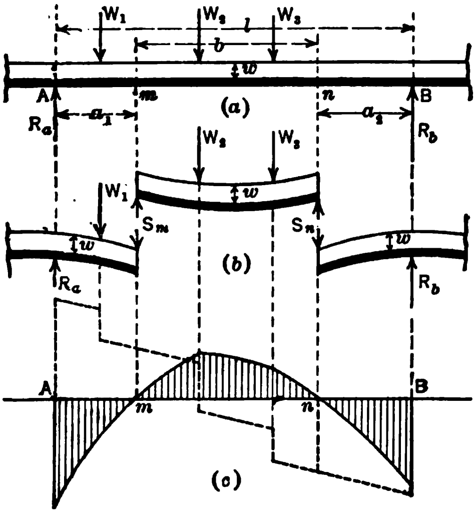


FIG. 168.

For example, let the sketch (Fig. 168a) indicate a span of a continuous beam, subjected to a system of concentrated and distributed loads. Assume points of inflexion at m and n at distances a_1 and a_2, respectively, from the supports. On this basis the portion of the beam between m and n can then be treated as a simple beam (Fig. 168b), which is supported on the ends of two cantilever beams A–m and B–n, extending inward from the ends of the span. The supporting forces for the portion m–n can be found in the same manner as in the case of any simple beam, and, having these, the greatest bending moment in m–n can easily be determined. The greatest bending moments and the greatest shearing forces in the portions A–m and B–n will be at the supports R_a

and R_b and can be determined in the same manner as for any cantilever beam.

The forms of the bending moment and shearing force diagrams, resulting from this analysis are indicated in the sketch (Fig. 168c). The accuracy of the results will depend on the accuracy in locating the points of inflexion m and n. The choice of the locations of m and n will be solely a matter of judgment, based on a knowledge of the conditions at the supports, the loads in the neighboring spans and the positions the points of inflexion in the more common types of continuous beams.

The method can evidently be used with equal facility in the case of a beam fixed at the ends, the beam having one or more spans.

123. Problems. — Continuous Beams. — In each of the following problems the beam is of uniform section with the central axis a straight line before loading. In all cases, for which sketches are shown, the supporting forces are indicated as acting upwards. It sometimes occurs, however, with certain systems of loading, that a downward force is required at one or more supports to maintain the level at the supports. In such cases negative values will be obtained for the downward supporting forces. Particular attention should be given to the solutions of the numerical problems, which illustrate the application of the theorem of three moments and the method of determining shearing forces, bending moments and deflexions in a few representative cases.

Problem 1.

Determine the greatest bending moment, the greatest shearing force, the supporting forces and the points of inflexion for the continuous beam, carried on three supports at the same level and loaded as indicated (Fig. 169). Neglect the weight of the beam.

Solution. — For this case the origin must be taken at the support (2) and the general form of the three moment equation (equation 44, Art. 118) will reduce to

$$0 = 2\,M_0\,(l_1 + l_2) + M_a l_1 + M_b l_2 + \frac{W_1 d_1}{l_1}\,(l_1^2 - d_1^2) + \frac{w_2 l_2^3}{4}$$

and, since the bending moments at the end supports are both equal to zero, by substituting in the above equation, we shall obtain

$$0 = 70\,M_2 + \frac{10{,}000 \times 10}{20}\,(400 - 100) + \frac{1000 \times 3375}{4}\,;$$

and solving for M_2,

$$M_2 = -\,33{,}480 \text{ ft. lbs.}$$

By substituting in equation (46) (Art. 118), taking the origin at the support (1), we obtain

$$S_1 = \frac{-33{,}480}{15} + \frac{1000 \times 15}{2} = 5268 \text{ lbs.}$$

and hence

$$S_{-2} = 5268 - 15{,}000 = -9732 \text{ lbs.}$$

By substituting again in equation (46), taking the origin at the support (2),

$$S_2 = \frac{33{,}480}{20} + \frac{10{,}000 \times 10}{20} = 6674 \text{ lbs.}$$

and hence

$$S_{-3} = 6674 - 10{,}000 = -3326 \text{ lbs.}$$

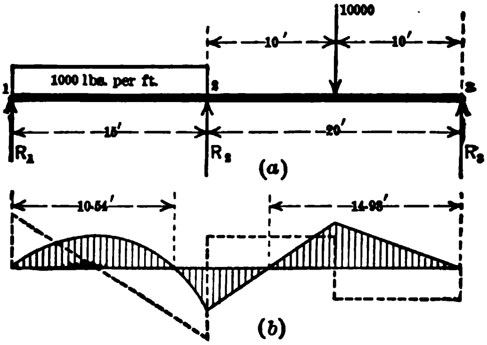


FIG. 169.

It should be observed that, if the general equation for S_{-0} had been used to obtain the values of S_{-2} and S_{-3}, the results would have been positive in each case instead of negative. The supporting forces will evidently be

$$R_1 = 5268 \text{ lbs.,} \quad R_2 = 9732 + 6674 = 16{,}406 \text{ lbs.,} \quad R_3 = 3326 \text{ lbs.}$$

The greatest shearing force in the beam is evidently the shearing force S_{-2}, at the left of the middle support.

To determine the maximum positive value of the bending moment in the span (1–2), we have, for the expression for the shearing force at any distance x from the support (1) as an origin,

$$S = 5268 - 1000\,x;$$

and placing $S = 0$ and solving for x,

$$x = 5.27 \text{ ft.}$$

Hence the maximum value of the bending moment will be equal to

$$M' = 5270 \times 5.27 - 500 \times \overline{5.27}^2 = 13{,}880 \text{ ft. lbs.}$$

To determine the maximum value of the positive bending moment in the span (2–3), observe that the shearing force is zero under the concentrated load. Hence the maximum bending moment will be equal to

$$M'' = 3326 \times 10 = 33{,}260 \text{ ft. lbs.}$$

The bending moment M_2 at the middle support is, therefore, the greatest bending moment in the beam.

There will be two points of inflexion which can be found as follows: Taking the origin at the support (1), the equation for the bending moment in the span (1–2) will be

$$M = 5268\,x - 500\,x^2.$$

Placing $M = 0$ and solving for x, we have

$$x = 10.54 \text{ ft.}$$

Taking the origin at the support (3) and x positive when measured to the left, the equation for the bending moment for values of x from 10 to 20 will be

$$M = 3326\,x - 10{,}000\,(x - 10).$$

Placing $M = 0$ and solving for x, we have

$$x = 14.98 \text{ ft.}$$

Sketches showing the forms of the bending moment and shearing force diagrams for this case are shown in Fig. (169b).

Problem 2.

Determine the greatest bending moment, the greatest shearing force, the supporting forces and the points of inflexion for a beam supported at three points on the same level, with two equal spans, of length l, when subjected to each of the following load systems:

(a) Load uniformly distributed over one span only; w = load intensity;

(b) Two concentrated loads W, one in the middle of each span;

(c) Single concentrated load W in the middle of one span;

(d) Single concentrated load W at any distance kl from one of the end supports.

Neglect the weight of the beam in each case. Sketch the shearing force and bending moment diagrams for each case.

Problem 3.

Determine the bending moments at the supports and the supporting forces for the continuous beam carried on four supports at the same level, having the two end spans equal and subjected to a uniformly distributed load (Fig. 170).

Solution. — Let W = the intensity of the load, l = the length of each end span and nl = the length of the center span.

For the bending moments at the supports we shall have $M_1 = 0$, $M_4 = 0$; and $M_2 = M_3$, from the symmetry of the loading. Applying the three moment equation (41) (Art. 118), with the origin at the support (2), we shall have

$$M_3 nl + 2\,M_2\,(l + nl) = -\frac{wn^3l^3}{4} - \frac{wl^3}{4}$$

and solving,

$$M_2 = M_3 = -\frac{wl^2}{4}\left(\frac{1 + n^3}{2 + 3\,n}\right).$$

Taking the origin at the support (1) and substituting in equation (35) (Art. 118), we obtain

$$S_1 = -\frac{wl}{4}\left(\frac{1 + n^3}{2 + 3\,n}\right) + \frac{wl}{2} = \frac{wl}{4}\left(\frac{3 + 6\,n - n^3}{2 + 3\,n}\right).$$

Taking the origin at the support (2) and substituting in equation (38) (Art. 118),

$$S_{-2} = \frac{wl}{4}\left(\frac{1+n^3}{2+3n}\right) + \frac{wl}{2} = \frac{wl}{4}\left(\frac{5+6n+n^3}{2+3n}\right).$$

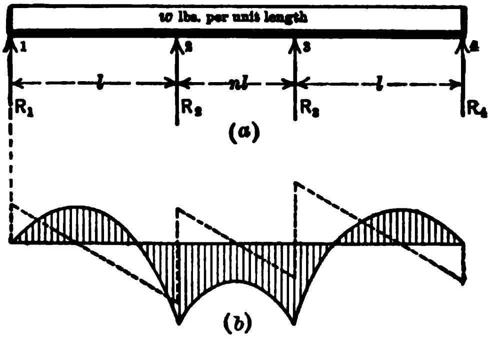


FIG. 170.

From the symmetry of loading it is evident that

$$S_2 = \frac{wnl}{2};$$

and that the shearing forces at the supports (3) and (4) are equal to the corresponding values at the supports (2) and (1).

For the supporting forces

$$R_1 = R_4 = \frac{wl}{4}\left(\frac{3+6n-n^3}{2+3n}\right),$$

$$R_2 = R_3 = \frac{wl}{4}\left(\frac{5+6n+n^3}{2+3n}\right) + \frac{wln}{2} = \frac{wl}{4}\left(\frac{5+10n+6n^2+n^3}{2+3n}\right).$$

A sketch showing the general forms of the bending moment and shearing force diagrams is given in Fig. (170b). It should be observed that when *n* is small the bending moment at the section in the middle span, at which the shearing force is zero, has a minimum instead of a maximum value, as indicated by the sketch.

Problem 4.

Deduce the expression for the bending moments at the supports and the supporting forces for the continuous beam given in Problem (3) for the following load systems:

(a) Load uniformly distributed over the two end spans only; *w* = load intensity;

(b) Load uniformly distributed over one end span only; *w* = load intensity;

(c) Load uniformly distributed over the middle span only; *w* = load intensity;

(d) Load uniformly distributed over the middle span and one end span only; *w* = the load intensity;

(e) Single concentrated load W, applied at any distance kl from one of the end supports;

(f) Single concentrated load W applied at the middle point of the center span (2–3).

Problem 5.

Given a continuous beam, fixed at one end and subjected to the system of concentrated and uniformly distributed loads as indicated (Fig. 171a), the supports being on the same level. The beam is a 12″ I-beam, 31.5 lbs. per ft., for which $I = 216$ (ins.)^4 and $\frac{I}{c} = 36$ (ins.)^3.

Determine:

(a) The bending moments at the supports;
(b) The shearing forces at the supports;
(c) The maximum positive bending moment in each span;
(d) The greatest outside fiber stress;
(e) The value of EIi at each support;
(f) The general equations for slope and deflection for each span and the overhanging end; and the greatest value of EIv for each span and the overhanging end;
(g) The greatest deflection in the entire beam, assuming $E = 28{,}000{,}000$ lbs. per sq. in.

Solution.—(a) The bending moment at the support (3) will be equal to

$$M_3 = -2000 \times 4 - 2400 \times 2 = -12{,}800 \text{ ft. lbs.} \quad . \quad . \quad (1)$$

Taking the origin at the support (2) and applying the three moment equation, in the form given in equation (44) (Art. 118), we obtain

$$0 = -\frac{M_2 32}{3} + \frac{12{,}800 \times 16}{6} - \frac{M_1\,16}{6} - \frac{5000 \times 10\,(256-100) + 3000 \times 4\,(256-16)}{6 \times 16}$$
$$-\frac{600 \times 4096}{24} - \frac{4000 \times 10\,(256-100)}{6 \times 16} - \frac{1200 \times 4096}{24},$$

which reduces to

$$0 = -4\,M_2 - M_1 - 168{,}494. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Taking the origin at the fixed end (1) and applying the equation for the slope at the support, in the form given in equation (45) (Art. 118), and noting that $i_0 = 0$, we obtain

$$0 = -\frac{M_1 16}{3} - \frac{M_2\,16}{6} - \frac{4000 \times 6\,(256-36)}{6 \times 16} - \frac{1200 \times 4096}{24},$$

which reduces to

$$0 = -2\,M_1 - M_2 - 97{,}425. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Solving equations (2) and (3) simultaneously,

$$M_1 = -31{,}601, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$
$$M_2 = -34{,}223. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(b) Taking the origin at (1) and applying the general equation for the shearing force at a support, as given in equation (46) (Art. 118),

$$S_1 = \frac{-34{,}223 + 31{,}601}{16} + \frac{4000 \times 6}{16} + \frac{1200 \times 16}{2} = 10{,}936 \text{ lbs.}, \quad (6)$$

and computing S_{-2} by the ordinary method,

$$S_{-2} = 10{,}936 - 4000 - 16 \times 1200 = -12{,}264 \text{ lbs.} \quad . \ . \ . \quad (7)$$

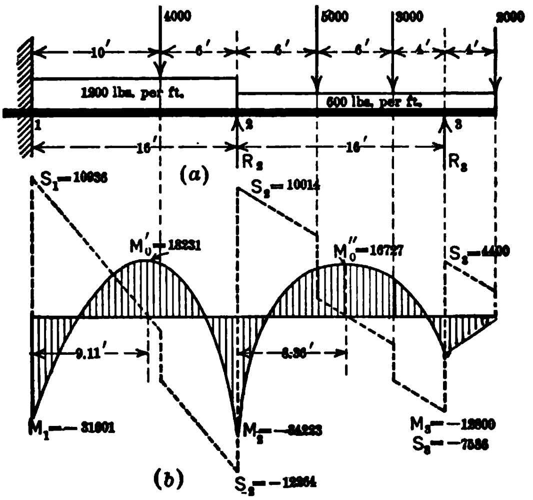


FIG. 171.

Taking the origin at (2) and applying the shearing force equation,

$$S_2 = \frac{-12{,}800+34{,}223}{16} + \frac{5000\times10+3000\times4}{16} + \frac{600\times16}{2} = 10{,}014 \text{ lbs.}, \quad (8)$$

and computing S_{-3},

$$S_{-3} = 10{,}014 - 5000 - 3000 - 600 \times 16 = -7586 \text{ lbs.} \quad . \ . \quad (9)$$

From the loads on the overhanging end,

$$S_4 = 2000 + 4 \times 600 = 4400 \text{ lbs.} \ . \ . \ . \ . \ . \ . \ . \quad (10)$$

Hence the greatest shearing force for the entire beam is at the left of the middle support and is equal in magnitude to 12,264 lbs.

The diagram showing the variation in shearing force, following the convention of signs for the ordinary beam theory, is shown by the dotted lines (Fig. 171b).

(c) Taking the origin at (1), the general equations for the shearing force and bending moment in the span (1–2) will be the following:

For values of x from 0 to 10,

$$S = 10{,}936 - 1200\,x, \ . \ . \ . \ . \quad . \ . \ . \ . \ . \quad (11)$$

$$M = -31{,}601 + 10{,}936\,x - 600\,x^2. \quad . \ . \ . \ . \ . \quad (12)$$

For values of x from 10 to 16,

$$S = 10{,}936 - 1200\,x - 4000, \qquad (13)$$

$$M = -31{,}601 + 10{,}936\,x - 600\,x^2 - 4000\,(x - 10). \qquad (14)$$

Placing (11) equal to zero, we obtain, for the section of zero shear,

$$x = 9.11 \text{ ft.} \qquad (15)$$

and, substituting in (12) and reducing, the maximum positive value of M for the span (1–2) is found to be

$$M' = 18{,}231 \text{ ft. lbs.} \qquad (16)$$

Taking the origin at (2), the general equations for the shearing force and bending moment in the span (2–3) will be the following:

For values of x from 0 to 6,

$$S = 10{,}014 - 600\,x, \qquad (17)$$

$$M = -34{,}223 + 10{,}014\,x - 300\,x^2. \qquad (18)$$

For values of x from 6 to 12,

$$S = 10{,}014 - 600\,x - 5000, \qquad (19)$$

$$M = -34{,}223 + 10{,}014\,x - 300\,x^2 - 5000\,(x - 6). \qquad (20)$$

For values of x from 12 to 16,

$$S = 10{,}014 - 600\,x - 5000 - 3000, \qquad (21)$$

$$M = -34{,}223 - 10{,}014\,x - 300\,x^2 - 5000\,(x - 6) - 3000\,(x - 12). \qquad (22)$$

Placing (19) equal to zero, the value of x for the section of zero shear is found to be

$$x = 8.36 \text{ ft.} \qquad (23)$$

and, substituting in (20) and reducing, the maximum positive value of the bending moment in the span (2–3) is found to be

$$M'' = 16{,}727 \text{ ft. lbs.} \qquad (24)$$

Taking the origin at (3), the equations for the overhanging end will be

$$S = 4400 - 600\,x, \qquad (25)$$

$$M = -12{,}800 + 4400\,x - 300\,x^2. \qquad (26)$$

A sketch of the bending moment diagram is shown in Fig. (171b).

(d) Comparing the values of M' and M'' with the values of the bending moments at the supports, the greatest bending moment for the entire beam is found to be at the middle support; and is equal in magnitude to 34,223 ft. lbs. Hence the greatest outside fiber stress is equal to

$$f = \frac{34{,}223 \times 12}{36} = 11{,}400 \text{ lbs. per sq. in.} \qquad (27)$$

(e) The slope at the fixed end being zero,

$$EIi_1 = 0. \qquad (28)$$

Applying equation (45) (Art. 118), with the support (2) as an origin, and applying the corresponding equation for the slope at the left of a support, with support (3) as an origin, the following values for EIi_2 and EIi_3 are obtained:

$$EIi_2 = \frac{34{,}223 \times 16}{3} + \frac{12{,}800 \times 16}{6} - \frac{5000 \times 10\,(256-100) + 3000 \times 4\,(256-16)}{6 \times 16}$$

$$- \frac{600 \times 4096}{24} = 3011, \quad . \ . \ . \ . \ . \quad (29)$$

$$EIi_{\llcorner 3} = \frac{12{,}800 \times 16}{3} + \frac{34{,}223 \times 16}{6} - \frac{5000 \times 6\,(256-36) + 3000 \times 12\,(256-144)}{6 \times 16}$$

$$- \frac{600 \times 4096}{24} = -\,53{,}619. \quad . \ . \ . \ . \quad (30)$$

Hence

$$\therefore EIi_3 = 53{,}619. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (31)$$

(f) The general equations for slope and deflexion can be obtained by integration from the bending moment equations (c), the constants being determined by the usual methods employed in dealing with ordinary beams. For the span (1–2), taking the origin at (1), the equations will be the following:

For values of x from 0 to 10,

$$EIi = -\,31{,}601\,x + 5468\,x^2 - 200\,x^3, \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (32)$$

$$EIv = -\,15800\,x^2 + 1823\,x^3 - 50\,x^4. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (33)$$

For values of x from 10 to 16,

$$EIi = -\,31{,}601\,x + 5468\,x^2 - 200\,x^3 - 2000\,(x-10)^2, \quad . \ . \quad (34)$$

$$EIv = -\,15800\,x^2 + 1823\,x^3 - 50\,x^4 - 667\,(x-10)^3. \quad . \ . \ . \quad (35)$$

By placing (32) equal to zero, the value of x, for the point of zero slope, is found to be

$$x = 8.28 \text{ ft.} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (36)$$

and, by substituting this value in (33), the maximum value of EIv for the span is found to be

$$EIv' = -\,283{,}000. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (37)$$

Taking the origin at (2), the equations for the span (2–3) will be the following:

For values of x from 0 to 6,

$$EIi = 3011 - 34{,}223\,x + 5007\,x^2 - 100\,x^3, \quad . \ . \ . \ . \quad (38)$$

$$EIv = 3011\,x - 17{,}112\,x^2 + 1669\,x^3 - 25\,x^4. \quad . \ . \ . \quad (39)$$

For values of x from 6 to 12,

$$EIi = 3011 - 34{,}223\,x + 5007\,x^2 - 100\,x^3 - 2500\,(x-6)^2, \quad . \quad (40)$$

$$EIv = 3011\,x - 17{,}112\,x^2 + 1669\,x^3 - 25\,x^4 - 833\,(x-6)^3. \quad . \quad (41)$$

For values of x from 12 to 16,

$$EIi = 3011 - 34{,}223\,x + 5007\,x^2 - 100\,x^3 - 2500\,(x-6)^2 - 1500\,(x-12)^2, \quad . \quad (42)$$

$$EIv = 3011\,x - 17{,}112\,x^2 + 1669\,x^3 - 25\,x^4 - 833\,(x-6)^3 - 500\,(x-12)^3. \quad . \quad (43)$$

By placing (40) equal to zero, the value of x, for the point of zero slope, is found to be equal to

$$x = 8.69 \text{ ft.} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (44)$$

and, by substituting in (41) the maximum value of EIv for the span is found to be

$$EIv'' = -330{,}000. \quad \dots \dots \dots \quad (45)$$

Taking the origin at (3), the equations for the overhanging end will be

$$EIi = 53{,}619 - 12{,}800\,x + 2200\,x^2 - 100\,x^3, \quad \dots \quad (46)$$

$$EIv = 53{,}619\,x - 6400\,x^2 + 733\,x^3 - 25\,x^4. \quad \dots \quad (47)$$

An inspection of (46) will show that the greatest deflection in this section will come at the end of the beam; and, substituting $x = 4$ in (42),

$$EIv''' = 152{,}000. \quad \dots \dots \dots \quad (48)$$

(g) Comparing the maximum values of EIv, the greatest deflection will evidently occur in the span (2–3) and will be equal to

$$v'' = -\frac{330{,}000}{EI}, \quad \dots \dots \dots \quad (49)$$

where the linear units are all expressed in ft. Hence to obtain the value of v'' in inches we have

$$v'' = \frac{330{,}000 \times \overline{12}^4}{28{,}000{,}000 \times 144 \times 216} \times 12 = -0.094''. \quad \dots \quad (50)$$

For the sake of completeness, the equations for S, M, EIi and EIv have been written for every section of the span, although some of the equations were not required in making a solution of the problem.

Problem 6.

Find the greatest deflexion of the beam given in Problem (1), assuming $E = 28{,}000{,}000$ lbs. per sq. in. and $I = 200$ (ins.)^4.

Problem 7.

Find the greatest bending moment and the greatest deflexion in the beam given in Problem (1), assuming $E = 28{,}000{,}000$ lbs. per sq. in., $I = 200$ (ins.)^4:

(a) When the middle support is ¼" below the level of the end supports.

(b) When the middle support is ¼" above the level of the end supports.

Problem 8.

Solve Problem (5), assuming that the end of the beam is free, instead of fixed, at the support (1), all other conditions remaining the same.

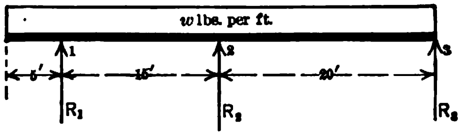


FIG. 172.

Problem 9.

A standard riveted plate girder, made up of a 24" × ⅜" web plate and 4 – 5" × 3½" × ½" angles, is supported at three points at the same level and loaded uniformly over its entire length, as shown (Fig. 172).

(a) Find the maximum load per ft. of length (w), including the weight of the girder, provided the greatest allowable fiber stress is 12,000 lbs. per sq. in.; given $\frac{I}{c} = 204$ (ins.)^3.

(b) Find the maximum deflexion, given $E = 28{,}000{,}000$ lbs. per sq. in. and the total depth of the girder $= 24\frac{1}{2}''$.

Problem 10.

Solve Problem (9), assuming that the support (1) is $\frac{1}{2}''$ below the level of the supports (2) and (3).

Problem 11.

Find the bending moments at the supports and the supporting forces for a four-span continuous beam, loaded as shown (Fig. 173), assuming that the supports are maintained at the same level.

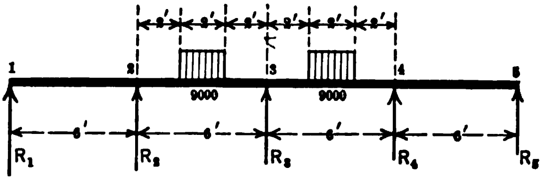


FIG. 173.

Problem 12.

Solve Problem (11), assuming that the beam is subjected to a total load of 16,000 lbs., uniformly distributed over a length of 8 ft., the center of the load being over the support (3).

Problem 13.

(a) Find a suitable I-beam to use for the continuous beam (Problem 11), provided the greatest allowable fiber stress = 12,000 lbs. per sq. in.

(b) Find a suitable I-beam to use for the continuous beam (Problem 12) assuming the greatest allowable fiber stress = 12,000 lbs. per sq. in.

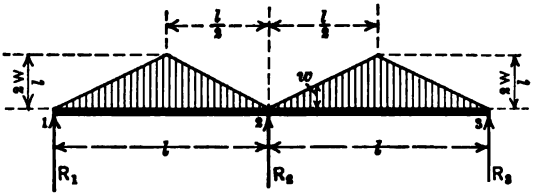


FIG. 174.

Problem 14.

Deduce the equation for the bending moment at the middle support and determine the supporting forces and the maximum values of the bending moments in each span of a continuous beam, having two equal spans subjected to equal loads W, distributed over each span as shown (Fig. 174).

Solution. — Let w = the load intensity at any point in the beam. Taking the origin at (2) and following the usual notation we shall have for values of x from 0 to $\frac{l}{2}$:

$$w = ax, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the constant $a = \frac{4W}{l^2}$,

$$S = S_2 - \frac{ax^2}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$M = M_2 + S_2x - \frac{ax^3}{6} \quad \text{(Art. 73).} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting in the equation of the elastic curve and integrating,

$$EIi = M_2x + \frac{S_2x^2}{2} - \frac{ax^4}{24}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the constant of integration being equal to zero, since the slope at the middle support is equal to zero. Integrating again

$$EIv = \frac{M_2x^2}{2} + \frac{S_2x^3}{6} - \frac{ax^5}{120}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the constant being equal to zero.

For values of x from $\frac{l}{2}$ to l,

$$w = ax - 2a\left(x - \frac{l}{2}\right) = a(l - x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$S = S_2 - \frac{W}{2} - \int_{\frac{l}{2}}^{x} w\,dx = S_2 - \frac{al^2}{8} + \frac{a}{2}(l - x)^2 \Big|_{\frac{l}{2}}^{x}$$

$$= S_2 - \frac{al^2}{4} + \frac{a(l - x)^2}{2} \quad \text{(Art. 73).} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$M = \int s\,dx = S_2x - \frac{a(l - x)^3}{6} - \frac{al^2x}{4} + c', \quad . \quad . \quad . \quad (8)$$

and, putting (8) equal to (3) when $x = \frac{l}{2}$, we obtain $c' = M_2 + \frac{al^3}{8}$.

$$EIi = \int M\,dx = M_2x + \frac{S_2x^2}{2} + \frac{a(l - x)^4}{24} - \frac{al^2x^2}{8} + \frac{al^3x}{8} + c'', \quad . \quad (9)$$

and, putting (9) equal to (4) when $x = \frac{l}{2}$, we obtain $c'' = -\frac{7al^4}{192}$.

$$EIv = EI\int i\,dx = \frac{M_2x^2}{2} + \frac{S_2x^3}{6} - \frac{a(l - x)^5}{120} - \frac{al^2x^3}{24} + \frac{al^3x^2}{16} - \frac{7al^4x}{192} + c''' \quad (10)$$

and, putting (10) equal to (5) when $x = \frac{l}{2}$, we obtain $c''' = \frac{al^5}{128}$.

When $x = l$, (8) reduces to

$$M_3 = 0 = M_2 + S_2l - \frac{al^3}{8};1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

and hence

$$S_2 = -\frac{M_2}{l} + \frac{al^2}{8} = -\frac{M_2}{l} + \frac{W}{2}; \quad \ldots \ldots \quad (12)$$

and, when $x = l$, (10) reduces to

$$EIv_2 = 0 = \frac{M_2 l^2}{2} + \frac{S_2 l^3}{6} - \frac{al^5}{128}. \quad \ldots \ldots \quad (13)$$

Eliminating S_2 between (12) and (13), we have

$$0 = \frac{M_2 l^2}{3} + \frac{5\,al^5}{384}; \quad \ldots \ldots \quad (14)$$

and hence

$$M_2 = -\frac{5\,al^3}{128} = -\frac{5\,Wl}{32}. \quad \ldots \ldots \quad (15)$$

Substituting in (12),

$$S_2 = \frac{5\,W}{32} + \frac{W}{2} = \frac{21}{32}W; \quad \ldots \ldots \quad (16)$$

and hence

$$S_1 = \frac{11}{32}W. \quad \ldots \ldots \quad (17)$$

The supporting forces will be equal to

$$R_1 = \frac{11}{32}W, \quad R_2 = \frac{21}{16}W, \quad R_3 = \frac{11}{32}W. \quad \ldots \ldots \quad (18)$$

Taking the origin at the support (1), the shearing force and bending moment equations, for values of x from 0 to $\frac{l}{2}$, will be

$$S = \frac{11}{32}W - \frac{ax^2}{2} = \frac{11}{32}W - \frac{2\,Wx^2}{l^2}, \quad \ldots \ldots \quad (19)$$

$$M = \frac{11}{32}Wx - \frac{ax^3}{6} = \frac{11}{32}Wx - \frac{2\,Wx^3}{3\,l^2}. \quad \ldots \ldots \quad (20)$$

Putting (19) equal to zero, we obtain for the section of zero shear

$$x = \frac{\sqrt{11}}{8}l = 0.415\,l; \quad \ldots \ldots \quad (21)$$

and substituting in (20) we have, for the maximum value of the positive bending moment in the span (1–2) and also for the span (2–3),

$$M' = \frac{11\sqrt{11}}{384}Wl = 0.095\,Wl. \quad \ldots \ldots \quad (22)$$

Problem 15.

Using the notation in Art. (118), deduce the following form of the three moment equation for a continuous beam, loaded as shown (Fig. 174), when the spans are unequal, the supports being on the same level.

$$M_a l_1 + 2\,M_0\,(l_1 + l_2) + M_b l_2 = -\frac{5\,a_1 l_1^4}{64} - \frac{5\,a_2 l_2^4}{64} = -\frac{5}{16}(W_1 l_1^2 + W_2 l_2^2).$$

Problem 16.

Determine the supporting forces and the greatest bending moment for the continuous beam given in Problem (14) if the load on one span is omitted, the distribution of the load on the other span being as shown in Fig. (174).

CHAPTER VII.

COMBINED STRESSES.

124. Combined Stresses. — In the discussion of the applications of the theory of bending, it has been shown (Arts. 75 and 100) that the values of the shearing force, bending moment, slope and deflection, at any cross section of a beam subjected to bending by the action of a system of transverse loads or couples, can be obtained by calculating the values of these quantities due to each load, or couple, acting separately and adding the results, algebraically. It is plainly evident that the value of the normal or the shearing stress intensity at any point in the cross section could be obtained in the same manner, by adding together the components of the stress intensity at the point due to each load acting separately. The discussion thus far, however, has been confined to load systems, comprised of transverse forces and couples, in which all the forces act in a single plane.

The method of analysis may be extended to cases in which the external forces are not restricted to transverse forces and couples, nor confined necessarily to a single plane. Since the resultant of the stress on any cross section of a stationary member must be in equilibrium with the external forces acting on the portion of the member on one side of the section, it is evident that, when the system of external forces can be divided into parts such that the component of the stress on the section due to each part acting alone can be determined, the resultant or combined stress on the section can be found by adding together in the same manner as in the case of bending under transverse loads.

In the preceding chapters, an analysis has been made of the stresses due to three types of load systems.

(a) *When the resultant of the system of external forces, acting on the part of a member on one side of a given cross section, is a single force, whose line of action passes through the center of gravity of the* *section.* In this case, the stress on the section is a uniformly distributed normal stress and the intensity at any point is equal to

$$\frac{P}{A} \text{ (Arts. 50–52).}$$

(b) *When the resultant of the external forces acting on one side of the cross section is a couple, in a plane intersecting the cross section along an axis of symmetry.* In this case, the stress on the section is a uniformly varying normal stress, the resultant of which is also a couple, and the intensity at any point is equal to

$$\frac{My}{I} \text{ (Art. 66).}$$

(c) *When the external forces acting on the part of the member on one side of a cross section are perpendicular to the central axis of the member and lie in a single plane which intersects the cross section in an axis of symmetry.* In this case, the resultant of the system consists of a couple and a shearing force and the resultant of the stress on the section is made up of a uniformly varying normal stress, as in the preceding case, and a shearing stress (Art. 69).

We shall now consider a number of cases in which the stress on a cross section of a member can be determined by dividing the external forces, acting on the portion on one side of the section, into two or more of the above-mentioned systems and finding the resultant stress intensities at different points in the section by combining the direct and bending stress intensities due to the component systems. The greatest combined stress intensity in every case is assumed to be within the elastic limit of the material.

125. Axial and Transverse Loads. — Let the sketch (Fig. 175) represent a member which is subjected to the action of a system of transverse forces, in a plane intersecting any cross section AB in an axis of symmetry YY, in combination with axial forces, causing compression in the member.

Divide the external forces acting on the part to the left of the section AB into two groups, viz., the transverse forces and the axial load P. If we assume that the deflection of the central axis due to the transverse forces is so small that it can be neglected, the stress due to the load P will be a uniform compression stress and its intensity at any point in the section will be equal to

$$f_1 = \frac{P}{A}, \quad \ldots \ldots \ldots \quad (1)$$

where A = the area of the cross section.

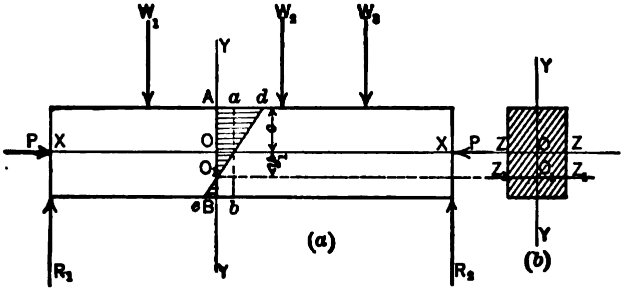


FIG. 175.

The normal stress due to the transverse loads will be uniformly varying and its intensity at any point, at a vertical distance y from the center of gravity of the section, will be equal to

$$f_2 = \pm \frac{My}{I}, \quad \ldots \ldots \ldots \quad (2)$$

where I = the moment of inertia of the cross section about the horizontal axis through the center of gravity. *The plus sign indicates compression and the minus sign tension, the symbols M and y being used in this chapter to represent magnitudes only.*

Adding (1) and (2) the resultant normal stress intensity at any point in the cross section will be equal to

$$f = f_1 + f_2 = \frac{P}{A} \pm \frac{My}{I} \cdot \ldots \ldots \ldots \quad (3)$$

The resultant normal stress on the section will evidently be uniformly varying and the neutral axis will pass through some point O_1, at a distance OO_1 from the center of gravity.

The variation in the resultant stress intensity is clearly shown in Fig. (175a), where the ordinates between AB and ab represent the uniform stress components f_1 and the ordinates between ab and de the uniformly varying stress components f_2. The neutral axis of the resultant stress will be the line $Z_1O_1Z_1$ (Fig. 175b), parallel to the principal axis ZOZ through the center of gravity. If we let y_1 = the distance OO_1, it is evident that

$$\frac{P}{A} = \frac{My_1}{I},$$

and hence

$$y_1 = \frac{PI}{MA} = \frac{P}{M}\rho^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where ρ = the radius of gyration of the cross section about the axis ZOZ.

The compressive stress intensity will evidently be greatest at the side of the section which is farthest from O_1 and will be equal in magnitude to

$$f = \frac{P}{A} + \frac{Mc}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where c = the distance from the axis through the center of gravity to the most strained fiber in compression.

If the cross section is symmetrical with respect to the axis ZOZ, the minimum intensity of stress, or the greatest intensity of the tensile stress, on the cross section will be equal to

$$f' = \frac{P}{A} - \frac{Mc}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

When the axial loads are applied at the ends of the member as indicated (Fig. 175) the cross section at which the fiber stress is greatest will evidently be the section of greatest bending moment.

The shearing stress on the section may be assumed to be distributed as in the case of a beam subjected to ordinary bending, the intensity at any point being represented by the usual formula (Art. 89).

When the deflection of the central axis, due to the transverse loads, is so great that a serious error results from assuming that the force P passes through the center of gravity of the cross section, allowance can be made for the displacement of the cross section, as explained in the next article.

The foregoing analysis will evidently apply equally well in the case in which the axial load P produces tension instead of compression, the plus sign being used to indicate a tensile instead of a compressive stress.

126. Loads Parallel to the Axis. — Eccentric Loading. — When the loads are parallel to the central axis of a member, the resultant of the forces acting on the part on one side of any given cross section can be found by the usual method of dealing with any system of parallel forces.

We have already considered:

(a) The case in which the resultant is a single force, whose line of action coincides with the axis of the member. (Arts. 51–52.)

(b) The case in which the resultant is a couple, located in a plane, intersecting the cross section at an axis of symmetry (Art. 66).

Two additional cases will now be considered:

(c) *When the resultant is a single force, parallel to the central axis, its line of action intersecting the cross section at an axis of symmetry.*

(d) *When the resultant is a single force, parallel to the central axis, not intersecting the cross section in an axis of symmetry, and the cross section has two axes of symmetry at right angles.*

In both of these cases, the load on the member is said to be *eccentric* and the distance between the line of action of the resultant force and the central axis is called the *eccentricity* of the load.

(c) In this case the force can be resolved into an equal parallel force, coinciding with the central axis, and a couple, acting in the plane of symmetry. Hence, if we neglect the deflection due to the bending produced by the couple, the normal stress on the cross section will be the resultant of a uniform stress, due to the component force acting along the axis, and a uniformly varying stress, due to the couple.

For an illustration, let the force P represent the resultant of a system of forces, acting on the portion on one side of the cross section AB and parallel to the central axis ZZ of the member shown (Fig. 176), the line of action of P intersecting the cross section at a point on the axis of symmetry YY, at a distance a from the center of gravity O. If the force P is resolved into a parallel force through O and a couple, the moment of the couple will be equal to $M = Pa$ and, if we use the notation of Art. (125), the normal stress intensity at any point in the cross section, at a distance y from the center of gravity, will be equal to

$$f = f_1 + f_2 = \frac{P}{A} \pm \frac{My}{I} = \frac{P}{A} \pm \frac{Pay}{I}, \quad . \quad . \quad . \quad . \quad (1)$$

where the plus sign indicates compression, The expression for the distance y_1, between the center of gravity and the neutral axis X_1X_1, will be

$$y_1 = \frac{PI}{MA} = \frac{P}{M}\rho^2 = \frac{\rho^2}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

and the magnitude of the greatest fiber stress

$$f = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A}\left(1 + \frac{ac}{\rho^2}\right), \quad . \quad . \quad . \quad . \qquad (3)$$

where ρ = the radius of gyration of the cross section about the principal axis XX.

If the curvature of the central axis due to bending is to be taken into account, the arm a of the couple will be increased by an amount v, equal to the displacement of the center of gravity due to the deflection of the member; and the couple M will be increased by an amount equal to Pv. Hence equation (1) may be written

$$f = \frac{P}{A} \pm \frac{P\,(a + v)\,y}{I}, \quad . \quad . \quad . \qquad (4)$$

where v will evidently be always positive. In using equation (4), a value of v, which is slightly approximate, can be found by substituting $M = Pa$ in the deflection equation (Art. 97).

In this case it is obvious that there is no shearing stress on any cross section.

If the forces are applied at the ends of the member and the stress component due to deflection is neglected, the state of stress on every cross section will be the same.

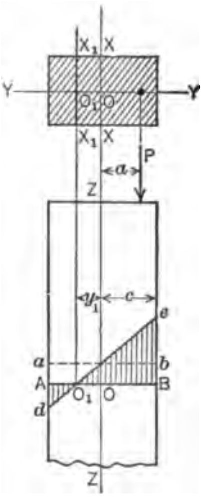


Fig. 176.

The foregoing analysis will apply equally well when the axial component produces tension instead of compression.

It will be evident that if the deflection of the beam, in the case considered in Art. (125), is to be taken into account, it is simply necessary to add to the bending couple, the couple due to deflection, estimated as above; the formula for the normal stress intensity at any point being written,

$$f = \frac{P}{A} \pm \frac{My}{I} \pm \frac{Pvy}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

which is another form of equation (4).

(d) In this case, neglecting the deflection due to bending, let the line of action of the resultant force P intersect any cross section $ABCD$ at a point whose coördinates, with respect to two axes of symmetry OX and OY, are (a, b), as indicated (Fig. 177). Let (x, y) be the coördinates of any point in the cross section.

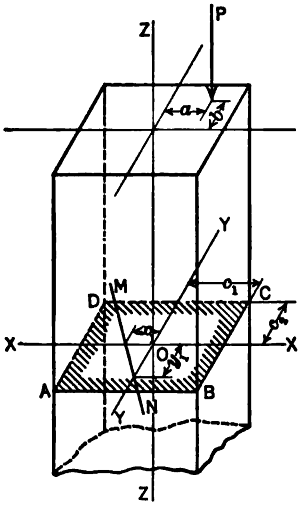


FIG. 177.

The force P may be resolved into an equal parallel force, acting along the central axis OZ of the member, and two couples,

$$M_y = Pa \text{ and } M_x = Pb,$$

in planes perpendicular to the axes OY and OX, respectively.

Proceeding by the same method as before, letting I_x and I_y equal the moments of inertia of the cross section about the axes OX and OY, respectively, the stress intensity at the point (x, y), due to the axial component P, will be equal to

$$f_1 = \frac{P}{A};$$

that due to the component couple M_x will be equal to

$$f_2 = \pm\frac{M_x y}{I_x} = \pm\frac{Pby}{I_x};$$

and that due to the component couple M_y will be equal to

$$f_3 = \pm\frac{M_y x}{I_y} = \pm\frac{Pax}{I_y}.$$

Hence the resultant stress intensity at the point (x, y) will be equal to

$$f = f_1 + f_2 + f_3 = \frac{P}{A} \pm \frac{M_x y}{I_x} \pm \frac{M_y x}{I_y}, \quad . \ . \ . \quad (6)$$

the plus sign indicating compression and the minus sign tension.

The greatest intensity of the resultant stress on the cross section shown will evidently occur at the point C, whose coördinates (c_1, c_2) are the greatest values of x and y, respectively. When the shape of the section is such that the point at which x has the greatest value is not the same as that at which the value of y is greatest, the point in the cross section at which the resultant stress intensity is a maximum can be determined by inspection.

The resultant stress on the section will be uniformly varying and the neutral axis will be a straight line, indicated by the line MN in the sketch, cutting the coördinate axes obliquely and not, in general, perpendicular to the plane of loading. Let $(x_1, 0)$ and $(0, y_1)$ be the coördinates of the points of intersection of MN and the axes OX and OY, respectively. For all points on the axis OX the component stress $f_2 = 0$; and hence, at the intersection of OX with the neutral axis,

$$\frac{P}{A} = \frac{M_y x_1}{I_y}$$

and

$$x_1 = \frac{PI_y}{M_y A} = \frac{P}{M_y} \rho_y^2 = \frac{\rho_y^2}{a}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $\rho_y =$ the radius of gyration about the axis OY.

Similarly, for all points on OY, $f_3 = 0$; and hence, at the intersection of OY with the neutral axis,

$$\frac{P}{A} = \frac{M_x y_1}{I_x}$$

and

$$y_1 = \frac{PI_x}{M_x A} = \frac{P}{M_x} \rho_x^2 = \frac{\rho_x^2}{b}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $\rho_x =$ the radius of gyration about the axis OX.

The foregoing analysis will evidently apply equally well when the axial stress is tension instead of compression. In either case there will be no shearing stress and, if the deflection is neglected, the state of stress on every cross section will be the same.

127. Maximum Eccentricity without Reversing Stress. — When a member is subjected to eccentric loading, as explained in Art. (126), the distance between the center of gravity of any cross section and the neutral axis of the stress on the section has been shown to vary inversely as the eccentricity of the load. When the eccentricity is large, the neutral axis will lie within the limits of the cross section, the stress over part of the section being compression and over the remainder tension, as indicated by the sketch (Fig. 178a). When the eccentricity is small, the neutral axis will lie outside of the limits of the section and the stress over the whole section will be of the same sign, as indicated (Fig. 178b).

It is sometimes essential to determine the eccentricity for which the neutral axis will coincide with one side of the cross section, the stress on the section being of the same sign throughout and varying in intensity from zero to a maximum, as indicated (Fig. 178c).

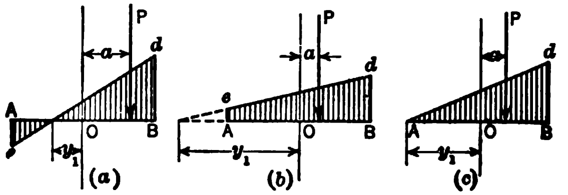


Fig. 178.

When the resultant load acts in a plane intersecting the cross section at an axis of symmetry we have, by transforming equation (2) (Art. 126),

$$a = \frac{\rho^2}{y_1} = \frac{I}{Ay_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

and, by substituting for y_1 the distance from the center of gravity to the edge of the cross section, the value of the eccentricity required to satisfy the last of the above-mentioned conditions can be obtained.

If the cross section is rectangular, of breadth b and depth h, when $y_1 = \frac{h}{2}$

$$a = \frac{bh^3}{12\,bh} \times \frac{2}{h} = \frac{h}{6}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

If the cross section is *circular*, of diameter d, when $y_1 = \frac{d}{2}$

$$a = \frac{4\,d^4}{64\,d^2} \times \frac{2}{d} = \frac{d}{8}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

For other sections, the eccentricity required to fulfill the same condition may easily be obtained by the solution of equation (1).

128. Oblique Loads. — When the external forces acting on a member are inclined to the central axis at angles less than 90° and the lines of action lie in a plane cutting any cross section in an axis of symmetry, the forces can be resolved into components perpendicular and parallel to, or coinciding with, the central axis. By the methods of analysis, stated in Arts. (125–126), the resultant stress on the cross section can then be determined.

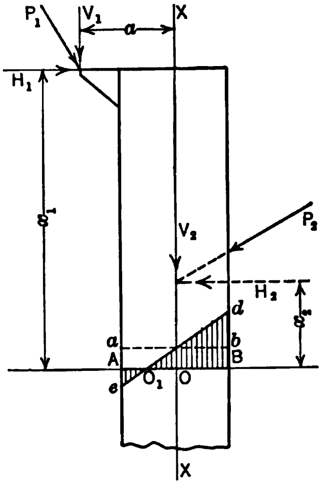


FIG. 179.

For example, take the upright member fixed at the lower end and subjected to the oblique loads P_1 and P_2 (Fig. 179), acting in a plane of symmetry. To determine the stress on a cross section A–B resolve P_1 and P_2 into H and V components as indicated. The component V_1 can in turn be resolved into an equal component, acting along the central axis OX, and a couple V_1a. Hence the resultant of the external forces acting above the section AB will be made up of a single force

$$P = V_1 + V_2, \quad . \quad . \quad . \quad (1)$$

acting along the axis OX, a couple

$$M = H_1x_1 - V_1a - H_2x_2, \quad . \quad (2)$$

which is the bending moment at the section, and a shearing force at the section,

$$S = H_1 - H_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The normal stress intensity at any point in the cross section will, therefore, be equal to

$$f = \frac{P}{A} \pm \frac{My}{I}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and the greatest intensity of stress and the position of the neutral axis can be determined as in the preceding cases.

If required, the shearing stress intensity can be calculated by the usual formula (Art. 89).

If a_1 and a_2 represent the moment arms of the forces P_1 and P_2 with respect to the center of gravity O of the cross section, the value of M will evidently be equal to

$$M = P_1a_1 - P_2a_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Occasionally the last expression gives a simpler solution for M than that given by equation (2). The cross section at which the maximum combined stress intensity occurs can be determined by inspection. It should be observed that this section is not necessarily the section of greatest bending moment.

129. Transverse Loads not in the Same Plane. — When the cross section of a member has two axes of symmetry at right angles and the external forces act in different planes, passing through the central axis, each force can be resolved into components in the two planes at right angles, which intersect any cross section in the axes of symmetry. In this manner, the entire system of external forces can be resolved into two systems acting in the two planes of symmetry and the resultant stress on any cross section can be found by combining the stresses due to each of the component force systems. For example, assume the member (Fig. 180) to be in equilibrium under the action of a system of forces P_1, P_2, etc., perpendicular to the central axis ZZ, any cross section $ABCD$ having two axes of symmetry, XX and YY.

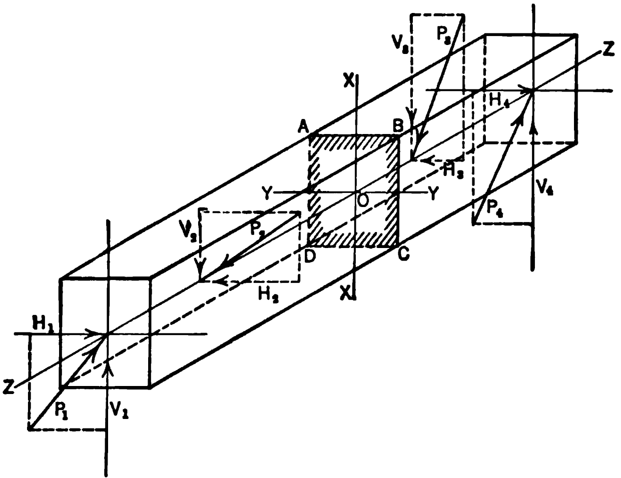


FIG. 180.

By resolving each force into V and H components in the planes containing XX and YY, respectively, the system of external forces can be resolved into two balanced systems, one of which will cause bending in the plane ZOX and the other in the plane ZOY. If we let M_x = the bending moment at the section $ABCD$, due to the H components, M_y = the bending moment due to the V components, I_x = the moment of inertia of the section with respect to the axis XX and I_y = the moment of inertia with respect to YY, the resultant intensity of the normal stress at any point in the cross section, whose coördinates are (x, y), will be equal to

$$f = \pm \frac{M_x y}{I_x} \pm \frac{M_y x}{I_y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The resultant normal stress on the cross section will be uniformly varying, with the neutral axis passing through the center of gravity, and the greatest intensity of the stress will evidently occur at the opposite corners of the section.

If we let M = the magnitude of the resultant bending moment at the cross section $ABCD$ it is evident that

$$M = \sqrt{M_x^2 + M_y^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, if θ = the angle between the moment axes of M and M_x,

$$\cos\theta = \frac{M_x}{M}, \text{ and } \sin\theta = \frac{M_y}{M} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When written in terms of the resultant bending moment M, equation (1) becomes

$$f = \pm \frac{My\cos\theta}{I_x} \pm \frac{Mx\sin\theta}{I_y} = M\left(\pm \frac{y\cos\theta}{I_x} \pm \frac{x\sin\theta}{I_y}\right). \quad (4)$$

If (x', y') are the coördinates of any point on the neutral axis,

$$\frac{M_x y'}{I_x} - \frac{M_y x'}{I_y} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which may be taken, with proper assumptions in regard to signs, as the equation of the neutral axis. The neutral axis will not, in general, be perpendicular to the plane of the resultant couple M.

When the form of the cross section is such that the point at which the coördinate x is greatest is not the same as that at which the coördinate y is greatest, the point at which the resultant stress intensity is a maximum can be determined by inspection.

When the greatest values of M_x and M_y occur at the same cross section, that section will evidently be the one on which the combined stress intensity has its greatest value. When the greatest values of M_x and M_y do not occur at the same cross section, the section on which the combined stress is a maximum must be determined by inspection.

The components of the shearing stress intensity at any point in a cross section, due to the H and V load components may be calculated by the usual formula (Art. 89) and combined to give approximately the resultant intensity of the shearing stress.

Bending by Couples. — A special case under the preceding occurs when the force system is composed of two couples, not in a plane of symmetry, acting at the ends of the bar. In this case the couples may be resolved into components M_x and M_y in planes perpendicular to the axes of X and Y, respectively, and the normal stress intensity at any point in a cross section found by the use of equation (1). In this case no shear will accompany the bending.

Oblique Loads. — If the loads are oblique, instead of perpendicular to the central axis, each load can be resolved into a component acting along the axis and one perpendicular to it, after the method in Art. (128). The transverse components can then be resolved as indicated above and the formula for the combined stress intensity will take the form

$$f = \frac{P}{A} \pm \frac{M_x y}{I_x} \pm \frac{M_y x}{I_y}, \quad \ldots \ldots \quad (6)$$

the effect of the deflection, caused by the transverse components, being neglected.

130. Resilience Due to Combined Stresses. — When the material is perfectly elastic, the resilience of a member subjected to combined stress will evidently be equal to the algebraic sum of the works done by the different components into which the system of external forces may be divided, provided each of the components of the force system is gradually applied.

For example: when a member is subjected to eccentric loading, as in Case (c) (Art. 126), if the external forces are applied at the ends of the member, and we let A = the area of the cross section and l = the length, the resilience due to the axial force P will be equal to

$$\frac{P^2 l}{2\,AE} \text{ (Art. 15)}; \quad \ldots \ldots \quad (1)$$

and the resilience due to the couple M will be equal to

$$\frac{M^2 l}{2\,EI} \text{ (Art. 109)}. \quad \ldots \ldots \quad (2)$$

Hence the total resilience will be equal to

$$R = \frac{P^2 l}{2\,AE} + \frac{M^2 l}{2\,EI} = \frac{P^2 l}{2\,E}\left(\frac{1}{A} + \frac{a^2}{I}\right) = \frac{P^2 l}{2\,AE}\left(1 + \frac{a^2}{\rho^2}\right), \quad . \quad (3)$$

where a = the eccentricity of the loading and ρ = the radius of gyration of the cross section about the axis through the center of gravity.

In a similar manner, the following expression for the resilience of a member loaded eccentrically as in Case (d) (Art. 126) may be obtained, the notation being the same as in the article referred to.

$$R = \frac{P^2l}{2\,AE} + \frac{M_x{}^2l}{2\,EI_x} + \frac{M_y{}^2l}{2\,EI_y} = \frac{P^2l}{2\,AE}\left(1 + \frac{b^2}{\rho_x{}^2} + \frac{a^2}{\rho_y{}^2}\right)\,. . \quad (4)$$

Similar expressions for the resilience in the other cases of combined stress, discussed in this chapter, can be obtained by adding the expressions for the resilience due to the axial loads and those due to bending under the load systems in the different planes.

131. Truss Members Subjected to Combined Stresses. — In the simple trusses, which were discussed under the subject of

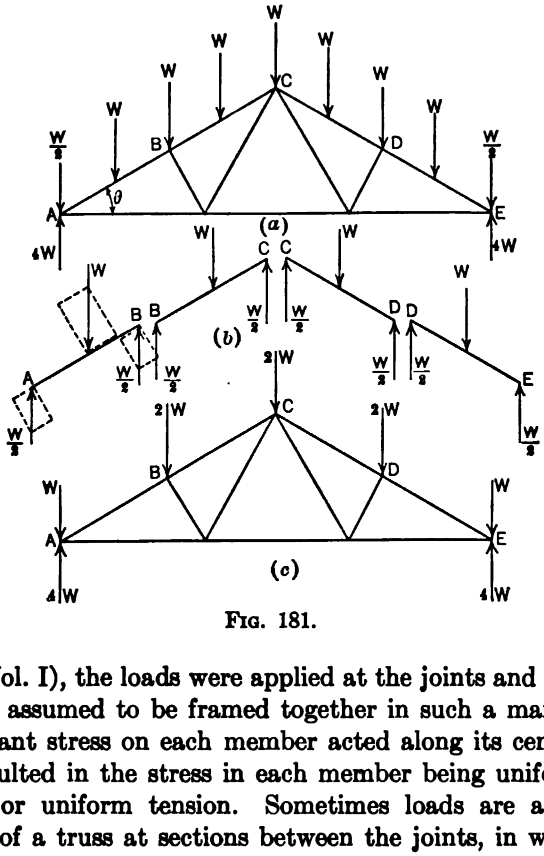


FIG. 181.

Statics (Vol. I), the loads were applied at the joints and the members were assumed to be framed together in such a manner that the resultant stress on each member acted along its central axis, which resulted in the stress in each member being uniform compression, or uniform tension. Sometimes loads are applied to members of a truss at sections between the joints, in which case bending will take place, combined with compression or tension.

An illustration of a truss loaded in this manner is given in Fig. (181), where the load on the truss is divided into equal parts W, concentrated at points on the upper chords, at equal distances apart, with the exception of the loads at the ends A and E, which are each equal to $\frac{W}{2}$. The members of the upper chords are subjected to combined bending and axial stresses, but all the other members are joined together in such a manner that each one is subjected to axial tension, or axial compression, only.

The analysis for the stresses, neglecting deflections due to bending, can be made as follows: First separate from the load system, the forces causing bending in the members of the upper chords. For the member AB these forces will be the load W, acting at its middle point, and the reactions $\frac{W}{2}$ at A and B, required to balance the load W, as indicated (Fig. 181b). The stress in the member AB due to these forces can be found by the method of analysis for bending under oblique loads (Art. 128). On resolving the forces into components, perpendicular to and coinciding with the central axis AB, it will be evident that the greatest intensity of the compressive stress in the member will occur at the top of a cross section just to the left of the intersection of the line of action of the load W and the central axis; and, if we let l = the length of AB, the stress intensity at this point will be equal to

$$f_1 = \frac{P}{A} + \frac{Mc}{I} = \frac{W \sin\theta}{2A} + \frac{Wl\cos\theta c}{4I}. \quad . \quad . \quad . \quad (1)$$

The same expression will evidently represent the resultant intensity of the tensile stress at the bottom of a section just to the right of W, provided the cross section is symmetrical with respect to the neutral axis and the plus sign is taken to indicate tension instead of compression.

If the members of the upper chords are all of the same cross section, the maximum compressive stress intensity f_1, due to the loads W at the middle points, will be the same for all four members.

Having determined the component stresses in the members, due to the loads causing bending, the axial stresses in the different members of the truss as a whole can be determined by adding to the loads at the joints the reactions, due to the intermediate loads between the joints, and calculating the stresses in the members from the resultant joint loads, as indicated in Fig. (181c). This is evidently equivalent to resolving each intermediate load into two parallel components, acting at the adjacent joints.

Finally, in order to determine the resultant stress on any cross section of an upper chord member, the uniform stress component, due to the axial load, must be combined with the component stress, due to bending. In the member *AB*, for example, the axial stress will evidently be equal to

$$P_1 = \frac{3\,W}{\sin\theta}$$

and the stress intensity on any cross section will be equal to

$$f_2 = \frac{P_1}{A} = \frac{3\,W}{A\sin\theta}. \quad \dots\dots \quad (2)$$

Therefore, to obtain the greatest intensity of stress for the entire member, the value of f_2 must be added to the value of f_1, which will give

$$f = f_1 + f_2 = \frac{W\sin\theta}{2\,A} + \frac{Wl\cos\theta c}{4\,I} + \frac{3\,W}{A\sin\theta}. \quad \dots \quad (3)$$

The stress in *BC* can be found in a similar manner.

In this analysis, each upper chord has been assumed to be made up of two parts hinged at the joints. When the chord is continuous, as is practically always the case, the foregoing analysis for the stress due to bending is evidently approximate. It is hardly worth while, however, to undertake the more complex solution required if the member is to be treated as a continuous beam, owing to the uncertainty in regard to the conditions at the joints.

132. Trussed Beams. — Beams are sometimes trussed for the purpose of increasing the strength and stiffness. Such beams may be considered to be simple forms of trusses in which a part of the members are subjected to bending. Hence the method of determining the stresses is similar to that given in Art. (131).

The following case may be taken as an illustration. The beam *AB* (Fig. 182) is braced with a strut *CD* at the middle point, the strut being supported by the tie rods *AD* and *BD*, and is subjected to a uniformly distributed load of intensity w. Let l = the length of each span of the beam and $W = 2\,wl$ equal the total load. The supporting forces will each be equal to $\frac{W}{2}$.

AB is the member of the truss which is subjected to bending and will be treated as a beam which is continuous over the support *C*. We will assume that, when the truss is subjected to the total load *W*, the points *A*, *C* and *B* are on the same level. Then, if the system of forces acting on the truss as a whole is resolved into two component systems, one component including the forces producing bending in the beam *AB* and the other including the forces producing axial stresses only in the members of the truss, the forces producing bending in *AB* will be those shown in Fig. (182b) (see Case b, Art. 102) and those producing the axial stresses in the members will be those shown in Fig. (182c).

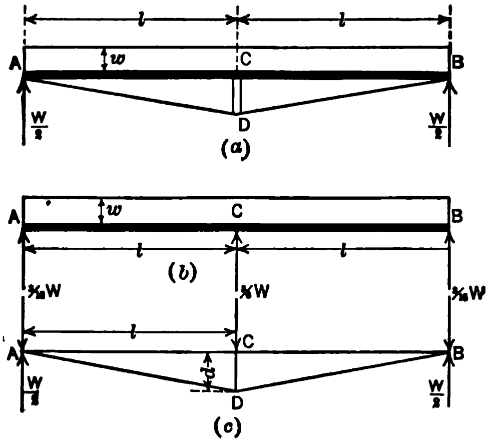


FIG. 182.

The greatest bending moment in the beam *AB* will occur at the point *C* and will be equal to

$$M = -\frac{wl^2}{8} = -\frac{Wl}{16} \text{ (Art. 102).} \quad . \quad . \quad . \quad . \quad (1)$$

If we let d = the length of the strut *CD*, the axial stress in the beam *AB* will be equal to

$$P = \frac{5\,Wl}{16\,d}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence the greatest intensity of the combined stress will be equal to

$$f = \frac{P}{A} + \frac{Mc}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which is the intensity of the compression at the lower side of the cross section at C.

The stress in the tie rod AD will be equal to

$$P_1 = \frac{5\,Wl_1}{16\,d}, \quad \ldots \ldots \ldots \quad (4)$$

where $$l_1 = \sqrt{l^2 + d^2},$$

and the stress in the strut CD will be equal to

$$P_2 = \frac{5}{8}W. \quad \ldots \ldots \ldots \quad (5)$$

The foregoing analysis could easily be made, if the final deflection at C instead of being zero were assumed to have any small finite value, by determining the corresponding values of the reactions at A, C and B, as indicated in Art. (102). In any case, the accuracy of the result will evidently depend on the correctness of the estimate of the final deflection at the point C, as the bending stress in AB will be affected to a considerable degree by a small change in level at this point.

Solution by Method of Least Work. — A solution of the foregoing problem can also be made by applying the method of least work as follows:

Let A = the area and I = the moment of inertia of the cross section and E = the modulus of elasticity of the member AB; A_1 = area of the cross section and E_1 = the modulus of elasticity of the tie rod ADB; A_2 = the area of the cross section and E_2 = the modulus of elasticity of the strut CD. Let P, P_1 and P_2 equal the axial stresses in the members ACB, ADB and CD, respectively, and let $l_1 = \sqrt{l^2 + d^2}$ equal the length of the two sections AD and BD of the tie rod.

Expressing the axial stresses in terms of the stress in the strut, we have

$$P = \frac{P_2 l}{2\,d} \quad \ldots \ldots \ldots \quad (6)$$

and

$$P_1 = \frac{P_2 l_1}{2\,d}. \quad \ldots \ldots \ldots \quad (7)$$

The expression for the resilience of the strut may then be written

$$R_2 = \frac{P_2{}^2\,d}{2\,A_2E_2} \text{ (Art. 15)}; \quad \ldots \ldots \quad (8)$$

that for the total resilience of the two sections of the tie rod will be

$$R_1 = 2\frac{P_1^2 l_1}{2\,A_1E_1} = \frac{P_2^2 l_1^3}{4\,A_1\,d^2E_1}; \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and for the component of the resilience of the beam AB, due to the axial stress P,

$$R' = 2\frac{P^2 l}{2\,AE} = \frac{P_2^2 l^3}{4\,A\,d^2E}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The component of the resilience of AB, due to bending, will be

$$R'' = 2\int_0^l \frac{M^2}{2\,EI}dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

where M = the bending moment at any section at a distance x from the end support. The expression for M may be written

$$M = \left(wl - \frac{P_2}{2}\right)x - \frac{wx^2}{2}$$

and then

$$M^2 = \left(wl - \frac{P_2}{2}\right)^2 x^2 - w\left(wl - \frac{P_2}{2}\right)x^3 + \frac{w^2x^4}{4}.$$

Substituting the value of M^2 in (11) and integrating,

$$\begin{aligned} R'' &= \frac{1}{EI}\left[\left(wl - \frac{P_2}{2}\right)^2 \frac{l^3}{3} - w\left(wl - \frac{P_2}{2}\right)\frac{l^4}{4} + \frac{w^2l^5}{20}\right] \\ &= \frac{l^3}{3\,EI}\left[\frac{2\,w^2l^2}{5} - \frac{5\,P_2\,wl}{8} + \frac{P_2^2}{4}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12) \end{aligned}$$

Adding (8), (9), (10) and (12), the expression for the total resilience of the trussed beam becomes

$$\begin{aligned} R = R_2 + R_1 + R' + R'' &= \frac{P_2^2\,d}{2\,A_2E_2} + \frac{P_2^2 l_1^3}{4\,A_1\,d^2E_1} \\ &+ \frac{P_2^2 l^3}{4\,A\,d^2E} + \frac{l^3}{3\,EI}\left[\frac{2}{5}w^2l^2 - \frac{5}{8}P_2wl + \frac{P_2^2}{4}\right]. \qquad (13) \end{aligned}$$

Differentiating,

$$\frac{dR}{dP_2} = P_2\left[\frac{d}{A_2E_2} + \frac{l_1^3}{2\,A_1\,d^2E_1} + \frac{l^3}{2\,A\,d^2E}\right] + \frac{l^3}{3\,EI}\left[-\frac{5}{8}wl + \frac{P_2}{2}\right], \quad (14)$$

and, placing the derivative equal to zero and solving for P_2, we obtain

$$P_2 = \frac{\frac{5}{24}wl^4}{\frac{d}{A_2E_2} + \frac{l_1^3}{2\,A_1\,d^2E} + \frac{l^3}{2\,E}\left(\frac{1}{Ad^2} + \frac{1}{3\,I}\right)}, \quad . \quad . \qquad (15)$$

which gives the stress in the strut CD. Having this stress, the axial stresses in the tie rods and the beam can be obtained from equations (7) and (6). The reactions at the ends of the beam can then be found and the greatest bending moment and greatest fiber stress can be computed in the usual manner.

133. Stresses in Hooks. — An *approximate* solution for the stress on a cross section of a hook, or an open link can be made by the same method of analysis as that employed in dealing with eccentric loading (Art. 126). For example, if the open link (Fig. 183) is subjected to a pull P, along a line at a distance a from the center of gravity O of the cross section AB, the force P acting on the part of the link on one side of AB can be resolved into an equal parallel force P, whose line of action passes through O, and a couple $M = Pa$. The stress on the cross section due to the component P will be uniform and its intensity will be equal to

$$f_1 = \frac{P}{A}. \quad . \quad . \quad (1)$$

If the stress intensity due to the couple is calculated by use of the formula

$$f_2 = \frac{My}{I}, \quad . \quad . \quad (2)$$

the equation for the combined stress intensity at any point in the cross section will be the usual expression

$$f = \frac{P}{A} \pm \frac{My}{I}, \quad . \quad . \quad (3)$$

the plus sign indicating tension.

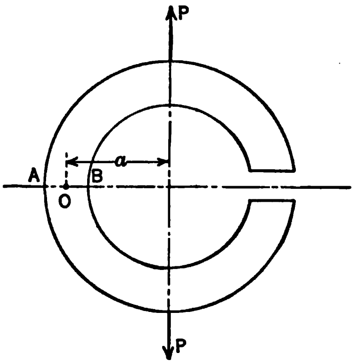


FIG. 183.

The approximation, when this equation is used in computing the stress in a hook, or link, or any curved bar, is due to the fact that, when the assumptions of the simple beam theory are applied in the case of the curved bar, the stress on the cross section will be found to be non-uniformly varying and hence the formula $f = \frac{My}{I}$ will not give the correct value for the intensity.

The error of approximation in cases of this kind will depend on the curvature of the central axis of the member, being comparatively slight when the radius of curvature is large and increasing as the radius diminishes.

If the formula is used, therefore, it must be regarded as empirical, the value of f not being a true value of the stress intensity, but a factor which must be chosen to suit the conditions of the problem.

A more exact analysis of the stresses in curved members will be found in Chapter XI.

134. Problems — Combined Stresses. — The following problems will serve to illustrate the application of the methods of determining combined stresses, which have been discussed in this chapter. When not otherwise stated, the loads are to be assumed to act in a plane intersecting the different cross sections of the member in their axes of symmetry. The eccentricity of the loading due to the deflection of the member is to be neglected unless otherwise specified.

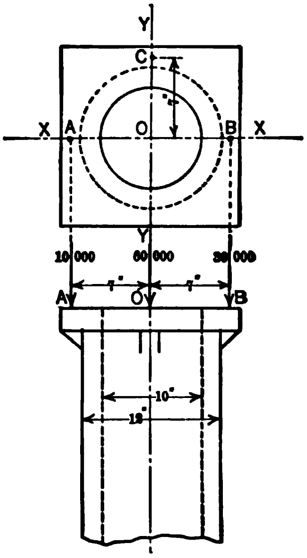


FIG. 184.

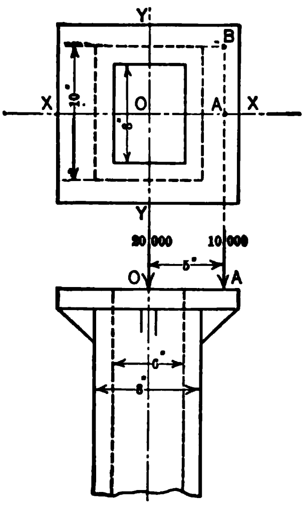


FIG. 185.

Problem 1.

Find the greatest intensity of stress in a cross section of the hollow circular column due to the vertical loads of 10,000 lbs., 60,000 lbs. and 30,000 lbs. acting at the points A, O, B, respectively (Fig. 184). Locate the neutral axis of the stress.

Problem 2.

Solve Problem (1), assuming that an additional load of 20,000 lbs. is applied at the point C.

Problem 3.

Find the greatest intensity of stress on the cross section of the hollow rectangular column (Fig. 185) due to the vertical loads of 20,000 lbs. and 10,000 lbs., acting at the points O and A respectively. Locate the neutral axis of the stress.

Problem 4.

Solve Problem (3), assuming that an additional load of 10,000 lbs. is applied at the point B. Determine the angle which the neutral axis makes with the line of intersection of the plane containing O and the resultant of the loads and the cross section.

Problem 5.

A column 10″ square is subjected to vertical loading as shown (Fig. 186). The maximum allowable fiber stress is 1000 lbs. per sq. inch. Find the greatest allowable value of P.

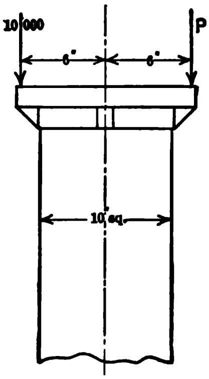


FIG. 186.

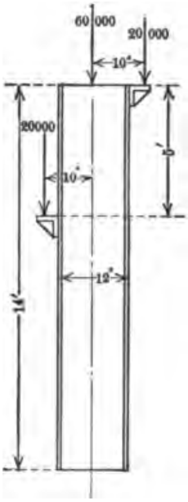


FIG. 187.

Problem 6.

Solve Problem (5), assuming that the load of 10,000 lbs. (Fig. 186) is replaced by a load of 40,000 lbs.

Problem 7.

A 12″ Bethlehem H column supports a central vertical load of 60,000 lbs. and two vertical loads of 20,000 lbs. each on brackets as indicated (Fig. 187). Find the greatest intensity of stress in the column due to the loading. $I_0 = 500$ (ins.)^4. $A = 19$ sq. ins. Locate the neutral axis in the cross section of greatest stress.

Problem 8.

Solve Problem (7), assuming that an additional vertical load of 30,000 lbs. is applied at the top of the column at a distance of 3" from the central axis in a plane at right angles to the plane of the loads shown (Fig. 187); the moment of inertia of the cross section with respect to the other axis of symmetry being equal to $I_0' = 168$ (ins.)^4 and the width of the flange of the H being equal to 12".

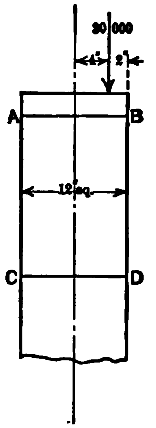


FIG. 188.

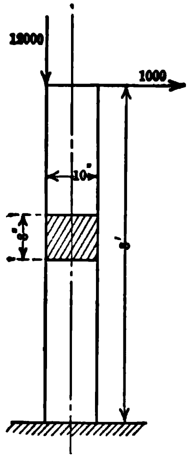


FIG. 189.

Problem 9.

A thick steel plate *rests* on the top of a wood column, 12" × 12" cross section (Fig. 188). If a concentrated vertical load of 30,000 lbs. is applied to the plate, determine: (a) the maximum intensity of the compressive stress between the plate and the column at the section *AB*; (b) the maximum intensity of the compressive stress on the cross section *CD*. Assume the stress on *AB* to be uniformly varying.

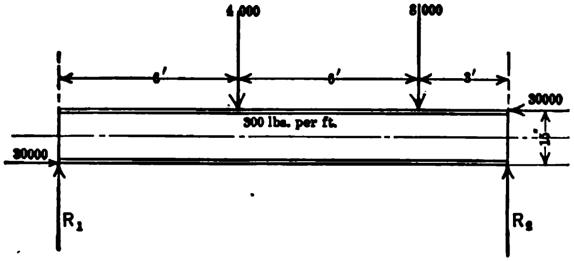


FIG. 190.

Problem 10.

A wooden strut, 8″ × 10″ cross section and 8 ft. long, is rigidly held at the base and subjected to system of loads shown (Fig. 189). Find the greatest intensity of stress.

Problem 11.

Solve Problem (10), assuming that the load of 1000 lbs. is replaced by a load of 2000 lbs.

Problem 12.

Find the maximum fiber stress in a standard 15″ I-beam, subjected to the system of forces shown (Fig. 190). $I_0 = 442$ (ins.)^4. $A = 12.5$ sq. ins.

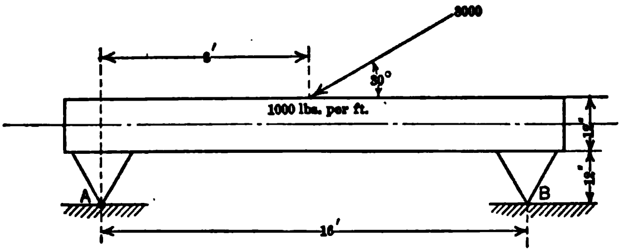


FIG. 191.

Problem 13.

A 12″ I-beam is supported and loaded as shown (Fig. 191). Find the maximum intensity of fiber stress, assuming the reaction at the point B to be vertical. $I = 216$ (ins.)^4. $A = 9.26$ sq. ins.

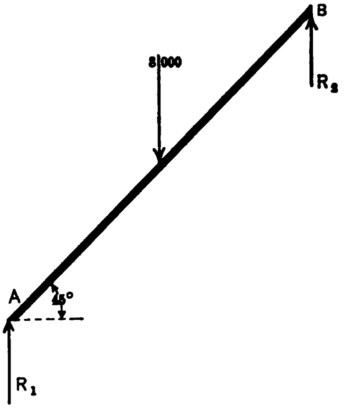


FIG. 192.

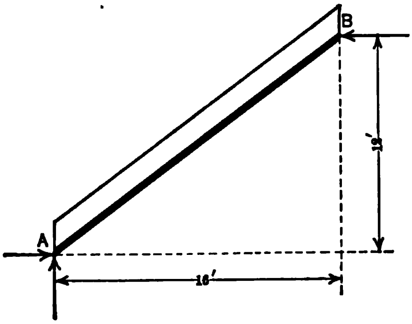


FIG. 193.

Problem 14.

Find the maximum fiber stress in the inclined wooden beam *AB*, 8″ × 10″ cross section and 10 ft. long, subjected to a single concentrated vertical load of 8000 lbs. at the center. (Fig. 192.) Assume the reactions parallel to the resultant load.

Problem 15.

Solve Problem (14), replacing the load of 8000 lbs. with a load of 12,000 lbs., uniformly distributed along the central axis of the beam.

Problem 16.

The beam *AB* (Fig. 193), 6" × 10" cross section, is subjected to a uniformly distributed vertical load of 5000 lbs. Find the maximum intensity of stress on the middle cross section, assuming the supporting force at *B* to be horizontal.

Problem 17.

Find the section at which the greatest intensity of stress occurs in the beam given in Problem (16). Find the greatest intensity of stress on this section.

Problem 18.

A 12" I-beam, 31.5 lbs. per ft., is placed in a vertical position and subjected to the system of loads shown (Fig. 194). Assuming that the supporting force at *A* is horizontal and the beam is hinged at *B*, find the greatest fiber stress. $I = 216$ (ins.)^4. $A = 9.26$ sq. ins.

Problem 19.

A beam 6" × 10" cross section is subjected to an oblique load of 4000 lbs. and supported at the ends as shown (Fig. 195). Find the maximum fiber stress and locate the neutral axis.

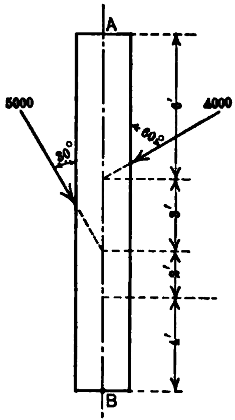


FIG. 194.

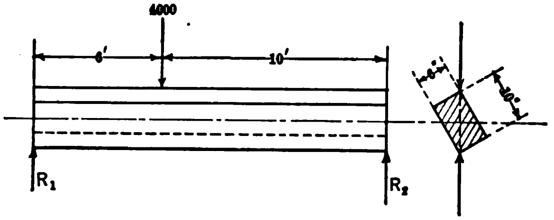


FIG. 195.

Problem 20.

A beam, 8" × 10" cross section (Fig. 196), is supported at the ends and subjected to a horizontal load of 2000 lbs. and a vertical load of 3000 lbs., acting as shown. Find the greatest fiber stress and locate the neutral axis in the section at which the greatest stress occurs.

Problem 21.

Determine the greatest fiber stress in the beam given in Problem (20) if, in addition to the loads given, two equal and opposite forces of 20,000 lbs., acting along the axis, are applied at the ends, the forces producing compression in the member. Locate the neutral axis in the section at which the fiber stress is a maximum.

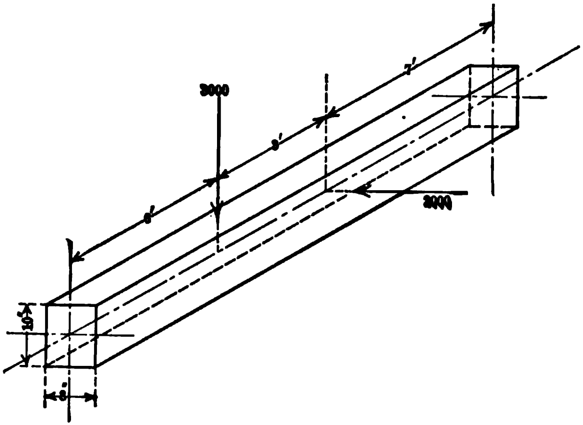


Fig. 196.

Problem 22.

Solve Problem (20), assuming that an additional load of 2400 lbs. is applied at a cross section 12 ft. from the end A, acting in the direction of the diagonal from the upper left to the lower right-hand corner of the section.

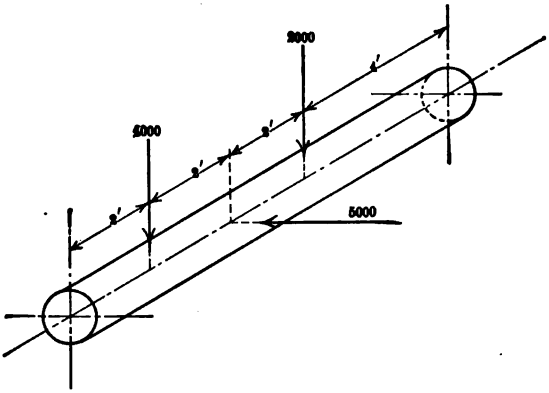


Fig. 197.

Problem 23.

A round bar, 4" diameter (Fig. 197), is supported at the ends and subjected to vertical loads of 4000 lbs. and 2000 lbs. and a horizontal load of 5000 lbs. Find the greatest fiber stress.

Problem 24.

Determine the greatest fiber stress in the bar given in Problem (23) if, in addition to the transverse loads, two equal and opposite forces of 30,000 lbs. acting along the axis are applied at the ends. the forces producing compression in the member.

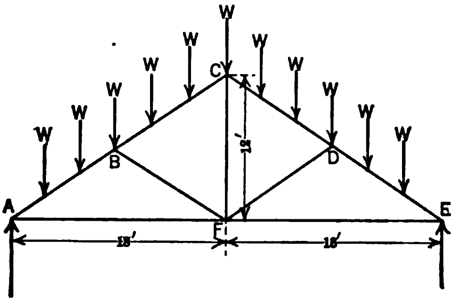


FIG. 198.

Problem 25.

Find the greatest intensity of stress in the members of the truss subjected to the system of vertical loads W spaced equal distances apart as shown (Fig. 198), assuming $W = 1000$ lbs. The members AB, BC, CD, BF and DF are 6″ × 10″ cross section; and the members AF, EF and CF are steel rods, 1″ diameter.

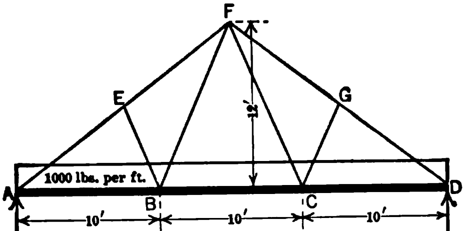


FIG. 199.

Problem 26.

The lower chord of the truss (Fig. 199) is subjected to a uniformly distributed load of 1000 lbs. per ft. Find the greatest fiber stress in the members AB, BC and CD. The cross section of these members is 8″ × 12″.

Problem 27.

Solve Problem (26), assuming that $ABCD$ is a continuous beam and that the points A, B, C and D are on the same level.

Problem 28.

Solve Problem (26) if, in addition to the load on the lower chord, loads of 1000 lbs. each are applied at the joints *E*, *F*, and *G* and also at the middle point of each member of the upper chords. Also determine the greatest fiber stress in the members *AE* and *EF* if the cross section of these members is 8″ × 10″.

Problem 29.

The trussed beam *AB* (Fig. 200) supports a uniformly distributed load of 1000 lbs. per ft. of length. The section of the beam is 8″ × 12″. The tie rod *ADB* is 2″ diameter. The strut *CD* has a cross section of 6 sq. ins. Find the greatest fiber stress in *AB*, assuming no deflection at the center (Art. 102); also the tensile stress intensity in the tie rod *ADB*.

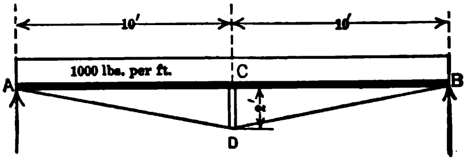


Fig. 200.

Problem 30.

Solve Problem (29) if the deflection of the beam at its center due to the applied load is 0.25″. $E = 1,200,000$ lbs. per sq. in.

Problem 31.

Solve Problem (29) by the method of least work (Art. 132), assuming for the tie rod $E = 30,000,000$ lbs. per sq. in. and for the strut $E = 15,000,000$ lbs. per sq. in.

Problem 32.

Solve Problem (29), replacing the uniformly distributed load with three concentrated loads of 5000 lbs. each, applied at the middle of each of the spans *AC* and *CB* and at the point *C*.

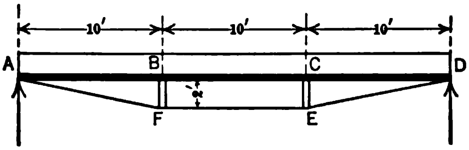


Fig. 201.

Problem 33.

A trussed beam (Fig. 201) is constructed of two timber beams, 8″ × 12″ cross section, placed side by side, and two parallel tie rods each 2″ diameter and two struts *BF* and *CE*, each having a cross section of 8 sq. ins. If the beam is subjected to a uniformly distributed load of 1600 lbs. per ft. determine the greatest fiber stress in the beam and the intensity of the tensile stress in the tie rods, assuming no deflection at the points *B* and *C*.

Problem 34.

Solve Problem (33) by the method of least work, assuming $E = 1{,}200{,}000$ lbs. per sq. in. for the beam, $E = 30{,}000{,}000$ lbs. per sq. in. for the tie rods and $E = 12{,}000{,}000$ lbs. per sq. in. for the struts.

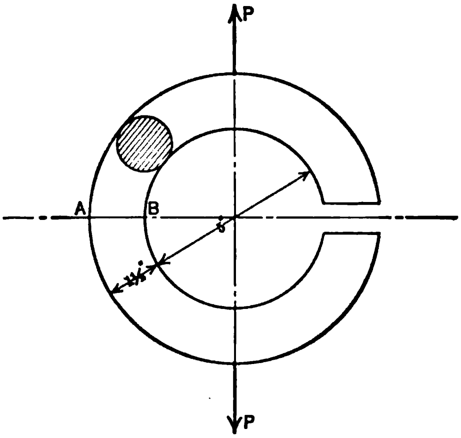


FIG. 202.

Problem 35.

Given an open circular link loaded as shown (Fig. 202). If the greatest allowable fiber stress = 12,000 lbs. per sq. in., find the allowable value of P, by the approximate solution (Art. 133).

CHAPTER VIII.

GENERAL THEORY OF FLEXURE.

135. Unsymmetrical Bending. — Thus far the discussion of the theory of bending has been restricted, at first to cases in which the loads have been in a single plane, intersecting every cross section in an axis of symmetry, and then to cases in which all the external forces could be resolved into components in two planes at right angles intersecting every cross section along two axes of symmetry.

In practice, nearly all members which are intended to resist bending moments of any considerable magnitude are designed with cross sections having one axis of symmetry at least; and usually with cross sections symmetrical with respect to two axes at right angles.

Minor members which are sometimes subjected to bending moments of small magnitude are quite often designed with unsymmetrical cross sections. In such cases an analysis of the stresses due to bending under different load systems will be of value.

136. Principal Axes and Moments of Inertia. — Passing through any point in a plane area, there are always two axes at right angles, with respect to one of which the moment of inertia of the area is a maximum and to the other a minimum, the product of inertia with respect to the two axes being equal to zero. The moments of inertia about these axes are called principal moments of inertia; and the axes are called principal axes.

Axes of symmetry are always principal axes passing through the center of gravity of the area and, for any unsymmetrical area, the principal axes and moments of inertia, with the center of gravity as the origin, can always be found when the moments and product of inertia with respect to any pair of rectangular axes are known. Throughout the discussion in this chapter, the center of gravity of a cross section will be taken as an origin and the axes XX and YY as principal axes, I_x and I_y will be taken to represent the principal moments of inertia and I_1, I_2 and K, the moments and product of inertia, respectively, about any pair of rectangular axes, 1–1 and 2–2, inclined at an angle α with the principal axes.

Then from Art. (124) (Vol. I),

$$I_x = I_1 \cos^2\alpha + I_2 \sin^2\alpha - 2K \sin\alpha \cos\alpha, \quad . \quad . \quad . \quad (1)$$

$$I_y = I_1 \sin^2\alpha + I_2 \cos^2\alpha + 2K \sin\alpha \cos\alpha \quad . \quad . \quad . \quad (2)$$

and

$$\tan 2\alpha = \frac{2K}{I_2 - I_1} \text{ (Art. 125, Vol. I).} \quad . \quad . \quad . \quad (3)$$

Conversely, the values of the moments and product of inertia, with respect to the axes 1–1 and 2–2, in terms of the principal moments of inertia will be represented by the expressions:

$$I_1 = I_x \cos^2\alpha + I_y \sin^2\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$I_2 = I_y \cos^2\alpha + I_x \sin^2\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$K = (I_x - I_y) \sin\alpha \cos\alpha. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In the following discussion, whenever a principal axis is referred to, it is to be understood that the axis passes through the center of gravity.

137. Bar of Unsymmetrical Cross Section Subjected to Simple Bending. — If the same assumptions are made as in the case of a bar of symmetrical section, bent under the action of couples at the ends (Art. 64), *the stress on any cross section of a straight bar of unsymmetrical section, which is bent under the action of terminal couples, will be uniformly varying and the neutral axis will pass through the center of gravity* (Art. 60). *Moreover, the resultant couple formed by the stress on the cross section must be in a plane coinciding with, or parallel to, the plane of the terminal couples.* There will be no shearing stress on any cross section.

If the principal axes of the section are unknown, they can be located by the method referred to in Art. (136), provided the moments and product of inertia of the cross section with respect to any two axes at right angles can be found.

In determining the distribution of the normal stress, two cases will be considered:

(a) *When the plane of the terminal couple intersects the cross section at, or in a line parallel to, a principal axis.* — In this case, since the couple, formed by the stress on a cross section, is in a plane containing, or parallel to, the principal axis, it follows, conversely, from Art. (60) that the neutral axis is the principal axis of the section which is perpendicular to the plane of the couple.

Therefore, the normal stress intensity at any point in the cross section, at a distance y from the neutral axis, will be given by the usual formula

$$f = \frac{My}{I}, \quad \ldots \quad \ldots \quad (1)$$

where I = the moment of inertia of the cross section about the neutral axis, and the greatest stress intensity will be equal to

$$f = \frac{Mc}{I}, \quad \ldots \quad \ldots \quad (2)$$

where c = the distance to the fiber which is farthest from the neutral axis.

(b) *When the plane of the terminal couple does not intersect a cross section at, or in a line parallel to, a principal axis.* — In this case let M represent the terminal couple, acting in a plane perpendicular to the axis OA (Fig. 203), and let θ = the angle between the moment axis of M and the principal axis OX. Resolving M into components in planes perpendicular to the principal axes, we have

$$M_x = M \cos\theta \quad . \quad . \quad (3)$$

and

$$M_y = M \sin\theta . \quad . \quad . \quad (4)$$

Let (x, y) be the coördinates of any point q in the cross section with respect to the principal axes.

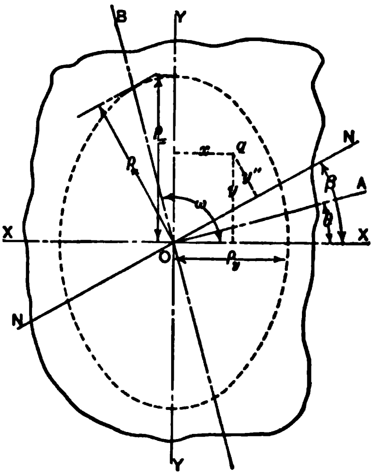


FIG. 203.

If we call a *compressive stress plus* and a *tensile stress minus* and follow the usual system of signs, in designating values of x, y and the functions of θ, the component stress intensity at the point q due to the component couple M_x will, according to Case (a), be equal to

$$f_1 = \frac{M_x y}{I_x} = \frac{My \cos\theta}{I_x} \quad \ldots \ldots \quad (5)$$

and that due to the couple M_y will be equal to

$$f_2 = \frac{M_y x}{I_y} = -\frac{Mx \sin\theta}{I_y} . \quad \ldots \ldots \quad (6)$$

The resultant stress intensity at the point will, therefore, be equal to

$$f = f_1 + f_2 = \frac{M_x y}{I_x} - \frac{M_y x}{I_y} = \frac{My\cos\theta}{I_x} - \frac{Mx\sin\theta}{I_y}$$
$$= M\left(\frac{y\cos\theta}{I_x} - \frac{x\sin\theta}{I_y}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

If we designate the coördinates of any point on the neutral axis by the symbols (x', y'), the equation of the neutral axis may evidently be written

$$M\left(\frac{y'\cos\theta}{I_x} - \frac{x'\sin\theta}{I_y}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which reduces to

$$y' = x'\frac{I_x}{I_y}\tan\theta = x'\left(\frac{\rho_x}{\rho_y}\right)^2\tan\theta, \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where ρ_x and ρ_y represent the radii of gyration of the section about the principal axes OX and OY, respectively.

If we let β = the angle between the neutral axis ON and the principal axis OX (Fig. 203),

$$\tan\beta = \frac{y'}{x'},$$

and hence equation (9) may be written,

$$\tan\beta = \frac{I_x}{I_y}\tan\theta = \left(\frac{\rho_x}{\rho_y}\right)^2\tan\theta. \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The angle ω, between the axis OX and the line of intersection OB, of the plane of the resultant couple and the cross section, will be equal to $(90° + \theta)$. Hence $\tan\theta = -\cot\omega$ and, substituting in (10) and transposing,

$$\frac{\tan\beta}{\cot\omega} = \tan\beta\tan\omega = -\left(\frac{\rho_x}{\rho_y}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If the inertia ellipse (Art. 127, Vol. I) is constructed for the principal axes OX and OY, as indicated (Fig. 203), it is evident from (11) that ON and OB will be conjugate axes of the ellipse. The formula for the fiber stress (equation 7) may be expressed in terms of the couple, formed by the stress on the cross section, and the moment of inertia of the section, about the neutral axis, as follows:

The component in a plane perpendicular to ON, of the resultant couple M, is equal to

$$M\cos(\beta - \theta);$$

and the moment of inertia of the section, about the neutral axis *ON*, is equal to

$$I_n = I_x \cos^2 \beta + I_y \sin^2 \beta \text{ (Art. 136).}$$

Hence $$M \cos (\beta - \theta) = a'' I_n \text{ (Art. 60)}, \quad . \quad . \quad . \quad . \quad (12)$$

where a'' = the intensity of stress at a distance unity from the neutral axis, and, if we let y'' = the distance of any point q from the neutral axis, the stress intensity at that point will be equal to

$$f = \frac{M \cos (\beta - \theta) y''}{I_n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

When the axis *OA* coincides with the principal axis *OX*, $\theta = 0$ and, from (10), $\beta = 0$ and equation (13) reduces to the ordinary form

$$f = \frac{My}{I_x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The case of the bar with a symmetrical cross section, bent by couples not in a plane of symmetry (Art. 129), may evidently be considered to be a special case under the foregoing one.

The greatest fiber stress at any cross section will occur at the point which is farthest from the neutral axis, and, if that point can be determined by inspection, the stress intensity can be calculated directly from equation (7), without determining the position of the neutral axis. When the cross section is a rectangle, for example, opposite corners will be points of greatest stress whatever the position of the bending couple.

When the point of maximum stress intensity cannot be located by inspection, the neutral axis may be plotted by use of equation (10). The coördinates of the point farthest from the neutral axis may then be measured and the greatest fiber stress calculated by the use of equation (7) or (13). If the ellipse of inertia is constructed, the neutral axis may be found by drawing the diameter *NON* which is conjugate to *OB* (Fig. 203) and the value of $I_n = A\rho_n^2$ can be computed directly from the value of ρ_n, measured from the ellipse.

138. Bar of Unsymmetrical Cross Section Subjected to Ordinary Bending. — When a straight bar of unsymmetrical section is bent under the action of transverse forces, passing through the axis of the bar, the forces can be resolved into two systems in planes at right angles and the resultant bending moment at any cross section can be obtained in the same manner as for the symmetrical bar (Art. 129).

In like manner the greatest resultant bending moment in the bar can be obtained. The stress on any cross section, due to bending, may then be found by the method used in the case of simple bending, the stress intensity at any point being given by the ordinary formula for fiber stress, provided the plane of the resultant bending moment intersects the cross section at a principal axis (Case a), and otherwise by the formulas derived for (Case b) (Art. 137).

In general, the normal stress on a cross section will be accompanied by a shearing stress. For an unsymmetrical section, however, the ordinary analysis for determining the intensity of the shearing stress fails to give satisfactory results; and no solution for shearing stress will be attempted.

The approximation in the results obtained by the foregoing analysis of the bending of an unsymmetrical bar is probably somewhat greater than in the case of a symmetrical beam subjected to ordinary bending (Art. 115).

The cases discussed in Art. (129) may evidently be considered as special cases under this and the preceding article, in which the cross sections are symmetrical, instead of unsymmetrical.

139. Combined Direct and Bending Stresses in Unsymmetrical Bars. — When the load system, acting on an unsymmetrical bar, includes forces which are parallel to, or oblique to, the axis of the bar, the intensity of the combined direct and bending stress at any point in a cross section can be found by a method similar to that employed in the cases discussed in Arts. (125–128).

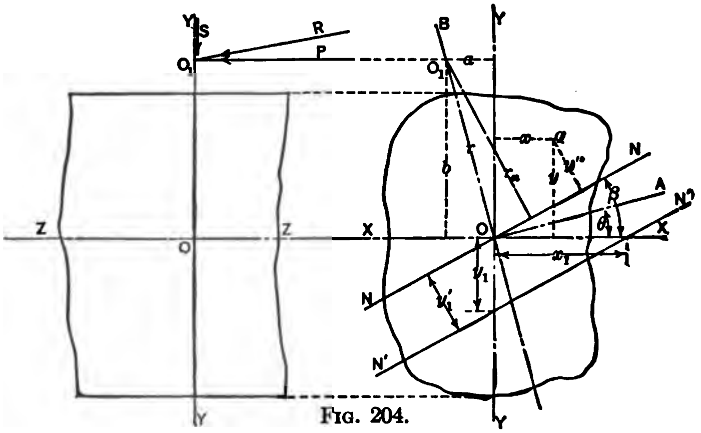


FIG. 204.

A general case of this kind will be one in which all the forces, acting on the part of the bar on one side of any cross section AB, can be combined into a single resultant force R, acting in a plane containing the central axis ZZ of the bar, as indicated (Fig. 204); where O_1 is the trace of the line of action of R in the plane of the cross section and OO_1 is the line of intersection of the plane, containing R and the central axis, and the cross section.

The resultant R may be resolved into a force P, normal to the section AB, and a shearing force S. The normal force P can then be resolved into an equal and parallel force, acting along the central axis ZZ, and a couple

$$M = Pr, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $r = OO_1$.

Let OX and OY represent the principal axes of the cross section and OA the moment axis of the couple M. Let $\theta =$ the angle AOX and let (a, b) be the coördinates of O_1 and (x, y) the coördinates of any other point q in the cross section, with respect to the principal axes. Let $A =$ the area, I_x and I_y equal the principal moments of inertia and ρ_x and ρ_y equal the principal radii of gyration of the cross section. Then, if a compressive stress is called positive and a tensile stress negative, the component stress intensity at q, due to the axial component P, will be equal to

$$f_1 = \frac{P}{A}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the component stress intensity at q, due to the bending moment M, will be equal to

$$f_2 = \frac{M_x y}{I_x} - \frac{M_y x}{I_y} = M\left(\frac{y \cos\theta}{I_x} - \frac{x \sin\theta}{I_y}\right) \text{ (Art. 137)}, \quad . \quad (3)$$

where $M_x = M \cos\theta = Pb$

and $M_y = M \sin\theta = -Pa.$

The resultant stress intensity at the point q will, therefore, be equal to

$$f = f_1 + f_2 = \frac{P}{A} + \frac{M_x y}{I_x} - \frac{M_y x}{I_y} = \frac{P}{A} + M\left(\frac{y \cos\theta}{I_x} - \frac{x \sin\theta}{I_y}\right)$$

$$= P\left[\frac{1}{A} + r\left(\frac{y \cos\theta}{I_x} - \frac{x \sin\theta}{I_y}\right)\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we let *NON* represent the neutral axis of the component stress due to bending only and β = the angle *NOX*,

$$\tan \beta = \frac{I_x}{I_y} \tan \theta = \left(\frac{\rho_x}{\rho_y}\right)^2 \tan \theta \text{ (Art. 137).} \quad . \quad . \quad . \quad (5)$$

The neutral axis $N'N'$ of the resultant stress will be a line parallel to *NON* and, if we let $(x_1, 0)$ and $(0, y_1)$ be the coördinates of the points of intersection of $N'N'$ and the principal axes *OX* and *OY*, respectively, we shall obtain from equation (4)

$$0 = \frac{P}{A} - \frac{M x_1 \sin \theta}{I_y}$$

and

$$0 = \frac{P}{A} + \frac{M y_1 \cos \theta}{I_x};$$

whence

$$x_1 = \frac{P I_y}{A M \sin \theta} = \frac{P \rho_y^2}{M \sin \theta} = -\frac{\rho_y^2}{a} \quad . \quad . \quad . \quad (6)$$

and

$$y_1 = -\frac{P I_x}{A M \cos \theta} = -\frac{P \rho_x^2}{M \cos \theta} = -\frac{\rho_x^2}{b}. \quad . \quad (7)$$

Expressed in terms of the moment of inertia of the cross section with respect to the neutral axis, the resultant stress intensity at any point q will be equal to

$$f = \frac{P}{A} + \frac{M \cos (\beta - \theta)\, y''}{I_n} = \frac{P}{A} + \frac{P r_n y''}{I_n} = \frac{P}{A}\left(1 + \frac{r_n y''}{\rho_n^2}\right), \quad (8)$$

where $r_n = r \cos (\beta - \theta)$ is the perpendicular distance between O_1 and the axis *NON*, ρ_n = the radius of gyration of the cross section about *NON*; and y'' is positive for points on the same side of *NON* as O_1 and negative for points on the opposite side. Equation (8) should be compared with equation (3) (Art. 126).

To obtain the value of y_1'', the distance between the center of gravity and the neutral axis $N'N'$, we have, from equation (8), for points on $N'N'$

$$\frac{P}{A} + \frac{M \cos (\beta - \theta)\, y_1''}{I_n} = 0;$$

whence

$$y_1'' = -\frac{P I_n}{A M \cos (\beta - \theta)} = -\frac{P \rho_n^2}{M \cos (\beta - \theta)} = -\frac{\rho_n^2}{r_n}. \quad . \quad (9)$$

Case (d) (Art. 126) might evidently be considered as a special case under the foregoing. In order to calculate the greatest stress intensity on the section, it is necessary to determine the coördinates of the point at which the intensity is a maximum, either by inspection, or by locating the neutral axis NON of the stress due to bending, as indicated in Art. (137). The solution can then be made by the use of either of the equations (4) or (8).

140. Section Moduli Lines. — By solving equation (7) (Art. 137), we obtain

$$\frac{M}{f} = \frac{I_x I_y}{I_y y \cos\theta - I_x x \sin\theta} = S_q, \quad . \quad . \quad . \quad . \quad (1)$$

where S_q may be considered to be a section modulus (Art. 77) for any point q in a cross section, the coördinates of which with respect to the principal axes are (x, y) (Fig. 203).

If the plane of the bending couple is rotated in such a manner that the angle θ, between the moment axis OA and the principal axis OX, varies from 0 to 90°, the value of S_q will evidently vary from

$$S_q' = \frac{I_x}{y}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

when $\theta = 0$, to

$$S_q'' = -\frac{I_y}{x}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

when $\theta = 90°$, the negative sign indicating that the stress intensity at q, when $\theta = 90°$, is of the opposite sign to the stress intensity, when $\theta = 0$. For points in the second quadrant for which the values of the coördinate x are negative, the signs of S_q' and S_q'' would evidently be the same.

If vectors, representing the values of S_q for different values of θ, are laid off along the corresponding positions of the trace OB, of the plane of the bending couple in the cross section, they will all terminate in the same straight line; for, on writing equation (1) in the form

$$S_q[(I_y y)\cos\theta - (I_x x)\sin\theta] - (I_x I_y) = 0, \quad . \quad . \quad . \quad (4)$$

it is evident, since S_q and θ are the only variables, that it is the polar equation of a straight line; for which S_q is the length and θ the angle of inclination of the radius vector to any point. Such a line may be called a *section-modulus line* for the point q or, for the sake of brevity, an S_q line.

As an illustration; let OB (Fig. 205) represent the intersection of the plane of the bending couple M in a cross section, whose principal axes are OX and OY, and let θ = the angle between the axis OX and the moment axis OA, of the couple M.

Let q be any point in the section whose coördinates are (x, y). The value of S_q, for any angle θ, can be computed from (1) and the values S_q' and S_q'', for the principal axes, from (2) and (3). For the point in question, S_q' is represented by the vector Oc, S_q'' by the vector Od, and the value S_q, for any value of the angle θ, by the vector Oe; and the straight line dce is the section modulus line for the point q.

It should be observed *that positive values of S_q are laid off along OB to the right and negative values to the left, as the directions appear when looking along the moment axis from A towards O.*

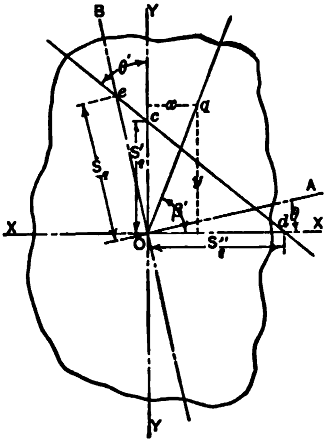


FIG. 205.

The section modulus line will extend to infinity in both directions, since S_q becomes infinite when the value of θ is such that the neutral axis passes through q. Therefore, if we let θ' = the value of θ when $S_q = \infty$, by putting the denominator of the fraction in equation (1) equal to zero, we obtain

$$\tan\theta' = \frac{I_y y}{I_x x} = \frac{I_y}{I_x}\tan\beta', \quad . \quad . \quad (5)$$

where β' = the angle between OX and the radius vector through q. When $S_q = \infty$, OB is evidently parallel to cd and hence θ' = angle between cd and the axis OY.

On comparing (5) with equation (10) (Art. 137), it will be evident that cd is parallel to the diameter, of the ellipse of inertia for the cross section, which is conjugate to the diameter through Oq. In other words the section modulus line for the point q is parallel to the neutral axis of the stress due to a bending couple whose plane intersects the cross section in the line Oq.

When the section modulus line for any point q has been drawn, the stress intensity at the point, due to any bending moment M, may evidently be obtained from the formula

$$f = \frac{M}{S_q}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where S_q is equal to the distance measured, along the trace of the plane of the bending couple, from O to the S_q line. The sign of the stress can be easily determined by inspection.

It is evident from the foregoing analysis that, if the magnitude of a bending moment remains constant while its plane is rotated about the central axis of the bar, the greatest stress intensity at any given point q will occur when the plane is at right angles to the section modulus line for the point; and the stress intensity will be zero when the plane is parallel to that line.

141. Section Moduli Polygon. — When a cross section is bounded by straight lines a polygon, circumscribing the whole section, may be constructed by drawing straight lines between the extreme corners. A polygon $ABCDE$ (Fig. 206) would represent such a polygon for the angle cross section shown.

If section moduli lines are constructed for each of the vertices, as explained (Art. 140), they will intersect, forming a polygon, known as the *section moduli polygon*, or simply, the S polygon.

In such a polygon, the section moduli lines for the points A, B, C, etc., may be designated as the S_a line, the S_b line, etc., and the vertices may be lettered (ab), (bc), etc., to indicate the intersection of the S_a and S_b lines, the S_b and S_c lines, etc.

The plane of any bending moment M will intersect the S polygon at two points; and the radii vectors to these two points will give the two values, S_1 and S_2, of the section modulus required to obtain the greatest intensity of the compressive and tensile stresses due to the bending; viz.,

$$f_c = \frac{M}{S_1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

and

$$f_t = \frac{M}{S_2}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Moreover, the section moduli lines which are intersected and the direction of rotation of the bending couple will indicate the points in the cross section at which the maximum intensities f_c and f_t occur. *When the plane of the bending couple passes through a vertex of the S polygon, the stress intensities at the two corresponding vertices of the polygon circumscribing the cross section will evidently be the same; and the neutral axis of the stress will be parallel to the straight line connecting the vertices.* Hence the stress intensities at all points on this straight line will be the same and therefore the section moduli lines for all points on the line will pass through the same vertex of the S polygon.

For example; if the intersection OB_1 (Fig. 206), of the plane of the bending couple whose moment axis is OA_1 and the cross section, passes through the vertex (ab) of the S polygon, the greatest intensity of the compressive stress will occur at points on the side AB of the cross section: also, if a section modulus line were drawn for any point on the line AB, it would pass through the point (ab). The neutral axis in this case is parallel to AB, and hence the greatest intensity of the tensile stress will occur at points on the side CD of the section, which is, evidently, farthest from the neutral axis. Moreover, since CD is parallel to AB, the vertex (cd) of the S polygon must lie in the line OB_1 extended on the opposite side of the origin.

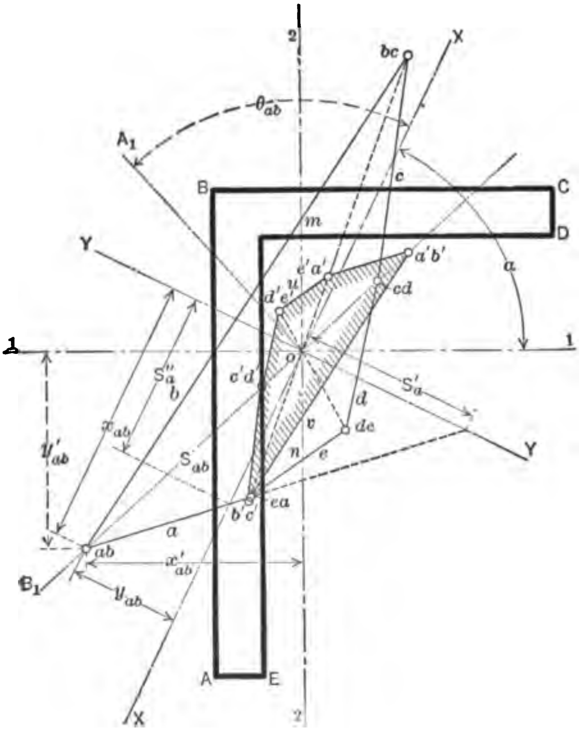


FIG. 206.

Similarly, the vertices of the S polygon, corresponding to any two parallel sides of the polygon circumscribing the cross section, such as (bc) and (ea), will lie on straight lines passing through the origin.

Construction of the S Polygon. — The S polygon for any given cross section may be constructed by laying off the values of S_a' and S_a'', S_b' and S_b'', etc., computed from equations (2) and (3) (Art. 140), along the principal axes of the section, as indicated (Fig. 206), and drawing the sides of the S polygon through the ends of these vectors. Greater accuracy can usually be obtained by locating the vertices (ab), (bc), etc., of the S polygon, as follows:

Let the coördinates with respect to the principal axes, OX and OY, of the points A and B (Fig. 206) be (x_a, y_a) and (x_b, y_b). Let S_{ab} represent the value of the section modulus and θ_{ab} the value of θ at the point of intersection of the S_a and S_b lines, designated a and b in the diagram, and let (x_{ab}, y_{ab}) be the coördinations of this point of intersection, with respect to the principal axes. Then from equation (1) (Art. 140) we shall have

$$S_{ab} = \frac{I_x I_y}{I_y y_a \cos\theta_{ab} - I_x x_a \sin\theta_{ab}} = \frac{I_x I_y}{I_y y_b \cos\theta_{ab} - I_x x_b \sin\theta_{ab}}. \quad (3)$$

But $\quad x_{ab} = -S_{ab}\sin\theta_{ab},\ y_{ab} = S_{ab}\cos\theta_{ab}$

and $\quad \frac{x_{ab}}{y_{ab}} = -\tan\theta_{ab}$; and hence, from equation (3),

$$x_{ab} = \frac{-I_x I_y \sin\theta_{ab}}{I_y y_a \cos\theta_{ab} - I_x x_a \sin\theta_{ab}} = \frac{I_x I_y}{I_y y_a \frac{y_{ab}}{x_{ab}} + I_x x_a}$$

and

$$I_y y_a y_{ab} + I_x x_a x_{ab} = I_x I_y. \quad \ldots\ldots \quad (4)$$

Similarly, from (3),

$$I_y y_b y_{ab} + I_x x_b x_{ab} = I_x I_y; \quad \ldots\ldots \quad (5)$$

and, solving (4) and (5) simultaneously,

$$x_{ab} = -\frac{(y_a - y_b)\, I_y}{x_a y_b - x_b y_a} \quad \ldots\ldots \quad (6)$$

and

$$y_{ab} = \frac{(x_a - x_b)\, I_x}{x_a y_b - x_b y_a}. \quad \ldots\ldots \quad (7)$$

When more convenient, the coördinates of the point of intersection of the section modulus lines with respect to a pair of rectangular axes, making any angle α with the principal axes, may be obtained by transforming the coördinates and substituting the values of the moments and product of inertia, with respect to these axes, in equations (6) and (7) and solving, as follows:

Let the axes 1–1 and 2–2 (Fig. 206) be any pair of rectangular axes, making the angle α with the principal axes OX and OY. Let I_1 = the moment of inertia about 1–1, I_2 = the moment of inertia about 2–2 and K = the product of inertia with respect to these axes. Let (x_{ab}', y_{ab}'), (x_a', y_a') and (x_b', y_b') be the coördinates of the points (ab), A and B, with respect to the axes 1–1 and 2–2, respectively. Then, by substituting the values of x_{ab}, y_a, y_b, etc., given by the usual equations of transformation, $x_{ab} = x_{ab}' \cos\alpha + y_{ab}' \sin\alpha$, $y_{ab} = y_{ab}' \cos\alpha - x_{ab}' \sin\alpha$, etc., in equation (6) and reducing, we obtain

$$x_{ab}'\cos\alpha + y_{ab}'\sin\alpha = \frac{-[(y_a'-y_b')\cos\alpha - (x_a'-x_b')\sin\alpha]\,I_y}{x_a'y_b' - x_b'y_a'}; \qquad (8)$$

and similarly from (7),

$$y_{ab}'\cos\alpha - x_{ab}'\sin\alpha = \frac{[(x_a'-x_b')\cos\alpha + (y_a'-y_b')\sin\alpha]\,I_x}{x_a'y_b' - x_b'y_a'}. \qquad (9)$$

Solving (8) and (9), simultaneously,

$$x_{ab}' = \frac{(x_a'-x_b')(I_y-I_x)\sin\alpha\cos\alpha - (y_a'-y_b')(I_x\sin^2\alpha + I_y\cos^2\alpha)}{x_a'y_b' - x_b'y_a'} \qquad (10)$$

But $I_x \sin^2\alpha + I_y \cos^2\alpha = I_2$ and $(I_y - I_x)\sin\alpha\cos\alpha = K$ (Art. 136). Substituting these values in (10) and observing, that when α is positive in the transformation equations for the coördinates, x_{ab}, y_{ab}, etc., it will be negative in the above formula for K, and vice versa, we shall have

$$x_{ab}' = \frac{(x_a' - x_b')\,K - (y_a' - y_b')\,I_2}{x_a'y_b' - x_b'y_a'}. \quad . \quad . \quad . \qquad (11)$$

In a similar manner we may obtain

$$y_{ab}' = \frac{(x_a' - x_b')\,I_1 - (y_a' - y_b')\,K}{x_a'y_b' - x_b'y_a'}. \quad . \quad . \quad . \qquad (12)$$

When the cross section is bounded by a curve, the S polygon can be constructed by plotting the section moduli lines for a series of points on the curve and drawing a curve tangent to these lines.

It should be observed that when the linear unit is the inch, all dimensions of the S polygon will be expressed in $(\text{ins.})^3$ (equation 1, Art. 140), and hence the polygon may be constructed to any convenient scale, which is large enough to give values of S_1 and S_2 with sufficient accuracy, when measured from the diagram.

142. S Polygon for Combined Stresses. — When the stress on a cross section of a member is a combined direct and bending stress (Art. 139), the section moduli polygon for the bending stress may be constructed, in the same manner as when the stress in the member is due to bending only (Art. 141).

If the axial stress is compression, the formulas for the maximum and minimum (algebraic) intensities of stress will evidently take the forms

$$f_c = \frac{P}{A} + \frac{M}{S_1} = P\left(\frac{1}{A} + \frac{r}{S_1}\right) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$f_t = \frac{P}{A} - \frac{M}{S_2} = P\left(\frac{1}{A} - \frac{r}{S_2}\right), \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where S_1 and S_2 are the radii vectors of the S polygon for compression and tension, respectively, measured along the line of intersection of the plane of loading and the cross section.

For any plane of loading, the limit of the eccentricity of the resultant longitudinal force P (Fig. 204) at which the stress will cease to be of the same sign throughout the cross section can be obtained by placing equation (2) equal to zero and solving as follows:

$$\frac{P}{A} - \frac{M}{S_2} = \frac{P}{A} - \frac{Pr'}{S_2} = 0,$$

whence

$$r' = \frac{S_2}{A}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where r' = the distance from the center of gravity of the section to the trace of the line of action of P, when the stress intensity at the vertex of the cross section corresponding to the section modulus line to which the radius vector S_2 is measured, is equal to zero.

It is evident from the form of equation (3) that, if values of r' are plotted for all possible values of θ, a polygon will be described in which each one of the sides is parallel to a side of the S polygon and is located on the opposite side of the origin. The vertices of the polygon will evidently be located on straight lines through the origin and the vertices of the S polygon, the corresponding vertices of the two polygons being in opposite directions from the origin.

Core of the Section. — A polygon constructed in this manner is called the *core* or the *kernel* of the section. If the resultant normal force intersects the cross section within the boundary of this polygon, the normal stress will be of the same sign throughout the limits of the section. If the resultant normal force acts through a vertex of the core, the neutral axis of the stress on the section will evidently coincide with one of the sides of the polygon, circumscribing the cross section.

In constructing the core, it is evidently necessary to locate the vertices only, by applying equation (3) to the radii vectors to each of the vertices of the S polygon. If desired, the coördinates of the vertices may be obtained by dividing the corresponding coördinates of the vertices of the S polygon (equations 6 and 7) or (equations 11 and 12) by the area A; and plotting the same on opposite sides of the coördinate axes.

It should be observed that when the linear unit is the inch, all dimensions of the polygon forming the core of the section will be expressed in (ins.); and hence the scale of the core will be the same as the scale of the section.

The *core* of the section (Fig. 206) is represented by the polygon whose vertices are marked $(a'b')$ $(b'c')$, etc., the radii vectors to which are, respectively,

$$r_{ab}' = \frac{S_{ab}}{A}, \quad r_{bc}' = \frac{S_{bc}}{A}, \text{ etc.}$$

143. Problems — General Flexure. — The problems may be divided into three classes, similar to those mentioned when the bending is in a plane of symmetry (Art. 83); the angle between the plane of loading and an axis through the center of gravity of the cross section being known in each case.

First: When the dimensions of the cross section and the resultant of the external forces acting on the portion of the member on one side of the section are known, and the greatest fiber stress is to be determined.

Second: When the dimensions of the cross section of a member and the greatest allowable fiber stress are known, and the moment of resistance of the section is to be determined.

Third: When the greatest allowable fiber stress and the resultant of the external forces acting on the portion of the member on one side of the section are known and the required section modulus of a cross section of a given type is to be determined.

Problem 1.

A steel angle, having a section 5″ × 3½″ × ½″, similar to that shown in Fig. (206), is subjected to bending by transverse loads acting in a plane intersecting the cross section in the axis 2-2, passing through the center of gravity and perpendicular to the face BC. Find the greatest intensity of the compressive and tensile stresses, if the greatest bending moment is equal to 30,000 in. lbs., the loads being applied in such a manner that the top of the angle, as shown (Fig. 206), is in compression.

First Solution. — Computing the values of the moments and product of inertia with respect to the axes 1-1 and 2-2, we have $I_1 = 10.00$ (ins.)^4, $I_2 = 4.05$ (ins.)^4, $K = 3.69$ (ins.)^4. Hence

$$\tan 2\,\alpha = \frac{2 \times 3.69}{4.05 - 10} = -1.240 \text{ (Art. 136);}$$

and

$\alpha = 64.5°$ (nearly), $\cos \alpha = 0.431$ and $\sin \alpha = 0.903$.

$$\therefore I_x = 10 \times \overline{0.431}^2 + 4.05 \times \overline{0.903}^2 - 2 \times 3.69 \times 0.431 \times 0.903 = 2.29 \text{ (ins.)}^4$$

and

$$I_y = 10 \times 0.903^2 + 4.05 \times \overline{0.431}^2 + 2 \times 3.69 \times 0.431 \times 0.903 = 11.78 \text{ (ins.)}^4.$$

In this case the angle between the moment axis OA_1, of the bending couple, and the axis OX is equal to $\theta = -64.5°$; and $\cos\theta = 0.431$, $\sin\theta = -0.903$ and $\tan\theta = -2.097$.

Substituting in equation (10) (Art. 137) we obtain

$$\tan\beta = -\frac{2.29}{11.78} \times 2.097 = -0.408 \text{ and } \beta = -22.2° \text{ (nearly).}$$

If the neutral axis is laid off on Fig. (206) it will be evident that the greatest intensity of the compressive stress will occur at the point B, and the greatest intensity of the tensile stress at the point E. The coördinates of these points with respect to the principal axes are

$$x_b = 1.11, \quad y_b = 1.54 \text{ and } x_e = -3.19, \; y_e = -1.06.$$

Substituting in equation (7) (Art. 137) we obtain for the intensity of the compressive stress at the point B,

$$f_c = 30{,}000\left(\frac{1.54 \times 0.431}{2.29} + \frac{1.11 \times 0.903}{11.78}\right) = 11{,}200 \text{ lbs. per sq. in.,}$$

and for the intensity of the tensile stress at the point E,

$$f_t = 30{,}000\left(-\frac{1.06 \times 0.431}{2.29} - \frac{3.19 \times 0.903}{11.78}\right) = -13{,}300 \text{ lbs. per sq. in.}$$

Second Solution. — Plot the section modulus lines for the points B and E by laying off the values,

$$S_b{}' = \frac{I_x}{y_b} = \frac{2.29}{1.54} = 1.49, \quad S_b{}'' = \frac{I_y}{x_b} = \frac{11.78}{1.11} = 10.61 \text{ (Art. 140)}$$

and

$$S_e{}' = \frac{I_x}{y_e} = -\frac{2.29}{1.06} = -2.16, \quad S_e{}'' = \frac{I_y}{x_e} = -\frac{11.78}{3.19} = -3.69,$$

along the principal axes; and scale off the lengths of the radii vectors to these lines, measured along the line of intersection 2-2 of the plane of loading and the cross section. The length of the radius vector to the S_b line will be $Om = S_1 = 2.67$ and that of the radius vector to the S_e line will be $On = S_2 = -2.25$. Hence

$$f_c = \frac{M}{S_1} = \frac{30{,}000}{2.67} = 11{,}200 \text{ lbs. per sq. in.}$$

and

$$f_t = \frac{M}{S_2} = -\frac{30{,}000}{2.25} = -13{,}300 \text{ lbs. per sq. in.}$$

Third Solution. — Plot the S polygon for the section, by computing the coordinates of the vertices from equations (11) and (12) (Art 141), as follows:

$$x_{ab}' = \frac{(3.34 + 1.66)\,4.05}{-0.91 \times 1.66 - 0.91 \times 3.34} = -4.43,$$

$$y_{ab}' = \frac{(3.34 + 1.66)\,3.69}{-0.91 \times 1.66 - 0.91 \times 3.34} = -4.05,$$

$$x_{bc}' = \frac{-(0.91 + 2.59)\,3.69}{-0.91 \times 1.66 - 2.59 \times 1.66} = 2.22,$$

$$y_{bc}' = \frac{-(0.91 + 2.59)\,10}{-0.91 \times 1.66 - 2.59 \times 1.66} = 6.02,$$

$$x_{cd}' = \frac{-(1.66 - 1.16)\,4.05}{2.59 \times 1.16 - 2.59 \times 1.66} = 1.56,$$

$$y_{cd}' = \frac{-(1.66 - 1.16)\,3.69}{2.59 \times 1.16 - 2.59 \times 1.66} = 1.43,$$

$$x_{de}' = \frac{(2.59 + 0.41)\,3.69 - (1.16 + 3.34)\,4.05}{-2.59 \times 3.34 + 0.41 \times 1.16} = 0.88,$$

$$y_{de}' = \frac{(2.59 + 0.41)\,10 - (1.16 + 3.34)\,3.69}{-2.59 \times 3.34 + 0.41 \times 1.16} = -1.64,$$

$$x_{ea}' = \frac{(-0.41 + 0.91)\,3.69}{0.41 \times 3.34 - 0.91 \times 3.34} = -1.10,$$

$$y_{ea}' = \frac{(-0.41 + 0.91)\,10}{0.41 \times 3.34 - 0.91 \times 3.34} = -2.99.$$

The values of S_1 and S_2 can then be measured as before.

Problem 2.

If the working fiber stress in tension is equal to the working fiber stress in compression, find the plane of loading for which the moment of resistance of the angle section in Problem (1) will be greatest.

Solution. — In this case the plane of loading must intersect the S polygon (Fig. 206) in such a manner that $S_1 = S_2$ and, at the same time, S_1 and S_2 must have the greatest value possible under this condition. An inspection of the S polygon will show that when the angle between the moment axis OA_1 and the principal axis OX is 85.5° (nearly), or, the angle between OA_1 and the axis 1-1 is 150° (nearly), the above condition is fulfilled and

$$S_1 = S_2 = 3.7 \text{ (ins.)}^3 \text{ (nearly).}$$

The maximum value of M under this condition will, therefore, be equal to

$$M_0 = 3.7f.$$

The plane of loading is inclined at an angle of 60° with the axis 1-1 and the greatest fiber stress occurs at the points A and C.

Problem 3.

Find the plane of loading for which the moment of resistance of the angle section in Problem (1), for a given working fiber stress, is the smallest.

Solution. — In this case the plane of loading will be perpendicular to the side of the S polygon (Fig. 206) which is nearest the center of gravity O; which is evidently the side d. The angle between the moment axis OA_1 and the principal axis OX will be 12.5° (nearly), or, the angle between OA_1 and the axis 1-1 will be 77° (nearly), and the value of $S_1 = 1.2$ (ins.)^3 (nearly). The greatest fiber stress f will occur at the point D and the value of M, in terms of f, will be

$$M = 1.2f.$$

Problem 4.

If the steel angle (Problem 1) is subjected to a compressive force $P = 12{,}000$ lbs., parallel to its central axis and intersecting the cross section at a point on the axis 2-2 at a distance $r = 1.5''$ from the center of gravity, find the greatest intensity of the fiber stress.

First Solution. — The area of the cross section $A = 4$ sq. ins. and the values of the moments of inertia, the coördinates of the vertices of the cross section, etc., are given in Problem (1). The bending moment is equal to $M = Pr = 12{,}000 \times 1.5 = 18{,}000$ in. lbs. It is evident from the solution of Problem (1) that the greatest compression stress intensity due to bending will occur at the point B; and, by substituting the values of the coördinates of this point in equation (4) (Art. 139), we obtain for the greatest intensity of the combined stress:

$$\begin{aligned} f_c &= \frac{12{,}000}{4} + 18{,}000\left(\frac{1.54 \times 0.431}{2.29} + \frac{1.11 \times 0.903}{11.78}\right) \\ &= 3000 + 6700 = 9700 \text{ lbs. per sq. in.} \end{aligned}$$

The greatest intensity of the tensile stress will occur at the point E and will be equal to

$$\begin{aligned} f_t &= \frac{12{,}000}{4} - 18{,}000\left(\frac{1.06 \times 0.431}{2.29} + \frac{3.19 \times 0.903}{11.78}\right) \\ &= 3000 - 8000 = -5000 \text{ lbs. per sq. in.} \end{aligned}$$

The coördinates of the point of intersection of the resultant force P and the cross section, with respect to the principal axes, will be $a = 1.35$, $b = 0.65$. Substituting in equations (6) and (7) (Art. 139) we obtain, for the points of intersection of the neutral axis with the principal axes,

$$x_1 = -\frac{\rho_y^2}{a} = -\frac{11.78}{4 \times 1.35} = -2.18'', \quad y_1 = -\frac{\rho_x^2}{b} = -\frac{2.29}{4 \times 0.65} = -0.88''.$$

The angle $\beta = -5.5°$, between the neutral axis and the principal axis OX, has been computed in the solution of Problem (1).

Second Solution. — The S polygon may be constructed, as in the solution of Problem (1), from which the radius vector for the greatest compressive stress, $S_1 = 2.67$, can be measured. The greatest intensity of the combined stress will then be equal to

$$f_c = \frac{P}{A} + \frac{M}{S_1} = \frac{12{,}000}{4} + \frac{18{,}000}{2.67} = 3000 + 6700 = 9700 \text{ lbs. per sq. in.}$$

For the greatest intensity of the tensile stress the radius vector measured from the S polygon will be equal to $S_2 = 2.25$ and hence

$$f_t = \frac{P}{A} - \frac{M}{S_2} = \frac{12{,}000}{4} - \frac{18{,}000}{2.25} = 3000 - 8000 = -5000 \text{ lbs. per sq. in.}$$

Problem 5.

If the steel angle in Problem (1) is subjected to an eccentric load, parallel to its central axis in the plane containing the axis 2–2, find the greatest possible eccentricity of the load under the condition that the stress shall be of the same sign throughout the limits of the cross section.

First Solution. — It is evident from the solution of Problem (1) that, so long as the plane of loading contains the axis 2–2 of the cross section, the greatest intensity of the combined stress will occur at either the point B or the point E and, therefore, the greatest allowable eccentricity of the load under the conditions of the problem will occur when the neutral axis of the stress passes through one of these points.

Let r_2' = the eccentricity of the load when the neutral axis passes through E. Then, from equation (4) (Art. 139),

$$0 = \frac{1}{A} + r_2'\left(\frac{y_e \cos\theta}{I_x} - \frac{x_e \sin\theta}{I_y}\right) = \frac{1}{4} - r_2'\left(\frac{1.06 \times 0.431}{2.29} + \frac{3.19 \times 0.903}{11.78}\right).$$

$$\therefore r_2' = 0.56''.$$

Similarly, if we let r_1' = the eccentricity of the load when the neutral axis passes through B we shall have

$$0 = \frac{1}{4} + r_1'\left(\frac{1.54 \times 0.431}{2.29} + \frac{1.11 \times 0.903}{11.78}\right).$$

$$\therefore r_1' = -0.67'',$$

the negative sign indicating that r_1' is measured in the negative direction along the axis 2–2. If we lay off $Ov = r_1$ and $Ou = r_2$ (Fig. 206), it is evident that, if the load intersects the cross section at any point in the axis 2–2 between the points u and v, the stress will be of the same sign throughout the cross section.

Second Solution. — The values of r_1' and r_2' might also be obtained from the values, $S_1 = Om$ and $S_2 = On$, measured from the S polygon (Fig. 206), by use of equation (2) (Art. 142), as follows:

When the neutral axis passes through E, we have

$$0 = \frac{1}{A} - \frac{r_2'}{S_2} = \frac{1}{4} - \frac{r_2'}{2.25} \quad \text{and} \quad r_2' = 0.56'';$$

and, when the neutral axis passes through B, we have

$$0 = \frac{1}{A} - \frac{r_1'}{S_1} = \frac{1}{4} - \frac{r_1'}{2.67} \quad \text{and} \quad r_1' = 0.67'';$$

the values of r_1' and r_2' to be laid off on the opposite sides of the origin from S_1 and S_2, as before.

Third Solution. — The core of the cross section may be plotted as explained in Art. (142). For the cross section under discussion:

$$r_{ab}' = \frac{6.00}{4} = 1.50'', \quad r_{bc}' = \frac{6.45}{4} = 1.61'', \quad r_{cd}' = \frac{2.15}{4} = 0.54'',$$

$$r_{da}' = \frac{1.84}{4} = 0.46'', \quad r_{ea}' = \frac{3.20}{4} = 0.80'' \text{ (Fig. 206).}$$

The limits within which the resultant normal force must act will then be given directly from the intersections of the plane of loading with the core of the section.

NOTE: — The greatest possible eccentricity of the load under any conditions, when the stress is of the same sign throughout the cross section, will occur when the resultant acts through the point $(b'c')$, where

$$r_{bc}' = 1.61''.$$

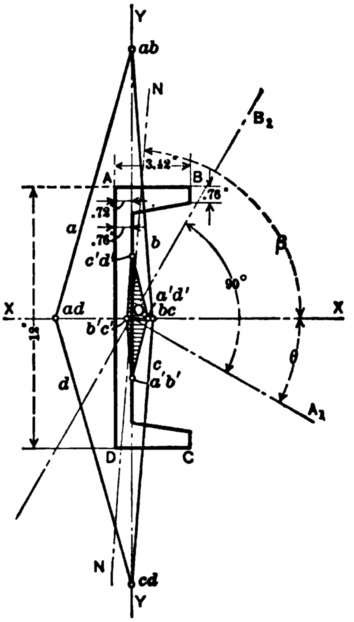


FIG. 207.

Problem 6.

A standard 12″ channel, weighing 40 lbs. per ft., having the cross section shown (Fig. 207), is subjected to a system of transverse loads in the plane intersecting the cross section in the axis YY. Calculate the moment of resistance, if the greatest allowable fiber stress is 15,000 lbs. per sq. in.

Solution. — In this case the axes XX and YY are principal axes and the values of the principal moments of inertia are $I_x = 196.9$ (ins.)^4 and $I_y = 6.6$ (ins.)^4. The area of the cross section is $A = 11.76$ sq. in.

The neutral axis of the stress is XX (Art. 138) and hence

$$M = \frac{fI_x}{c} = \frac{15{,}000 \times 196.9}{6} = 492{,}000 \text{ in. lbs.}$$

Problem 7.

Determine the moment of resistance of the channel given in Problem (6) when the plane of the loads intersects the cross section in the line OB_1, making an angle of 30° with OY (Fig. 207).

First Solution. — Let OA_1 represent the moment axis of the bending couple. Then $\theta = -30°$, $\cos\theta = 0.866$, $\sin\theta = -0.5$ and $\tan\theta = -0.577$. For the neutral axis of the stress,

$$\tan\beta = -\frac{196.9}{6.6} \times 0.577 = -17.21$$

and

$$\beta = -86.7°.$$

If the neutral axis NON is drawn, it is evident that the intensity of the stress will be greatest at the point B; and hence, from equation (7) (Art. 137),

$$f_c = 15{,}000 = M\left(\frac{6 \times 0.866}{196.9} + \frac{2.70 \times 0.5}{6.6}\right) = 0.231\,M.$$

$$\therefore\ M = 64{,}900 \text{ in. lbs.}$$

Second Solution. — Construct the S polygon for the section. In this case, since XX and YY are principal axes, the intercepts of the section moduli lines on these axes may be obtained directly from equations (2) and (3) (Art. 140), as follows:

$$S_a' = S_b' = -S_c' = -S_d' = \frac{196.9}{6} = 32.8 \text{ (ins.)}^3,$$

$$S_b'' = S_c'' = -\frac{6.6}{2.7} = -2.44 \text{ (ins.)}^3,$$

$$S_a'' = S_d'' = \frac{6.6}{0.72} = 9.17 \text{ (ins.)}^3$$

Since $S_a' = S_b'$, etc., the vertices of the S polygon are located on the principal axes. In laying off the intercepts particular attention must be paid to the rule for signs, stated in Art. (140).

On inspection of the S polygon it is evident that, for the given plane of loading, the stress intensity is greatest at the point B; and the value of S_1 for this point, measured from the diagram, will be found to be equal to

$$S_1 = 4.33.$$

Hence

$$15{,}000 = \frac{M}{4.33}$$

and

$$M = 64{,}900 \text{ in. lbs.}$$

Problem 8.

Solve Problem (7), when the plane of loading is such that $\theta = +30°$: (a) Analytically; (b) By use of the S polygon.

Problem 9.

Calculate the values of I_1, I_2 and K for the Z bar section shown (Fig. 208); and determine the values of α and the principal moments of inertia, I_x and I_y. Compute the coördinates of the vertices of the S polygon with respect to the coördinate axes 1-1 and 2-2 and construct the core of the section.

Problem 10.

Calculate the moment of resistance, in terms of the greatest fiber stress, of the Z bar section in Problem (9): (a) When the plane of loading intersects the section in the axis 2-2; (b) When the plane of loading intersects the section in the axis 1-1.

Problem 11.

Determine the position of the plane of loading when the moment of resistance of the section in Problem (9), for a given maximum fiber stress f, is a maximum. Calculate the value of the greatest moment of resistance in terms of f.

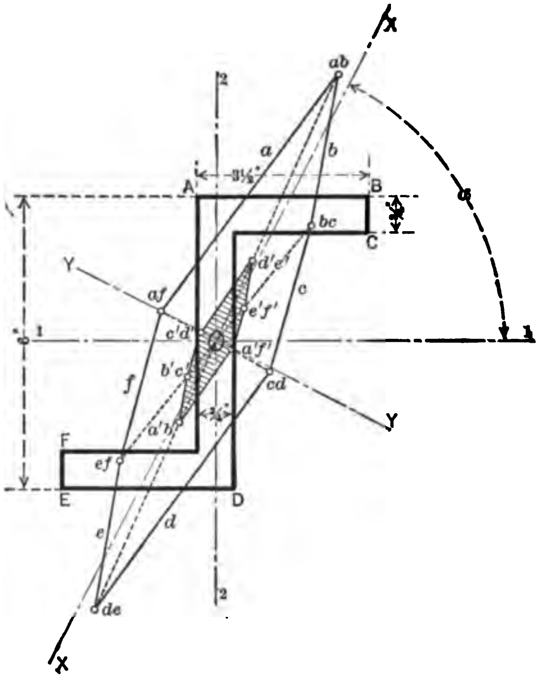


FIG. 208.

Problem 12.

Calculate the coördinates of the vertices of the S polygon and construct the core of the rectangular section, shown in Fig. (209).

Problem 13.

Calculate the coördinates of the vertices of the S polygon and construct the core of the standard I section, shown in Fig. (210). $I_x = 269.0$ (ins.)^4. $I_y = 13.8$ (ins.)^4. $A = 11.84$ sq. in.

Problem 14.

Calculate the moment of resistance of the I section given in Problem (13), when the plane of loading intersects the cross section at an angle of 30° with the principal axis YY, assuming the greatest fiber stress to be 16,000 lbs. per sq. in. and using the S polygon. Determine the position of the neutral axis.

Problem 15.

A beam having a standard I section is to be subjected to loading in a plane making an angle of 60° with the principal axis perpendicular to the web. If the greatest bending moment in this plane is 8000 ft. lbs., find the size of the section required, by plotting the section moduli lines for the point of maximum stress intensity only, for a series of standard sections, and determining the section which will give the required value of S_1 by interpolating the plot. Assume the working fiber stress f = 15,000 lbs. per sq. in.

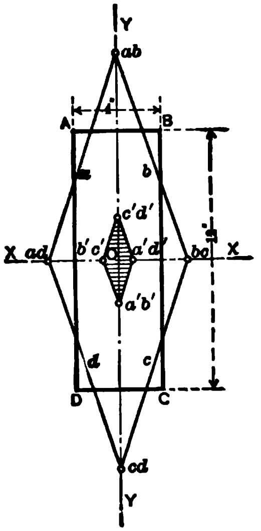


Fig. 209.

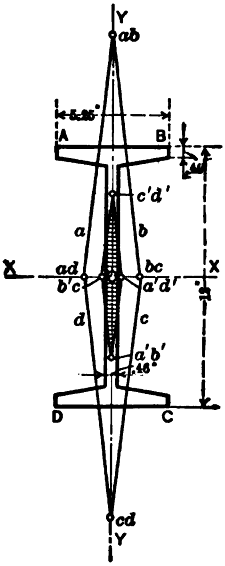


Fig. 210.

Problem 16.

Solve Problem (15), using standard channel sections and assuming that the plane of loading makes an angle of 75° with the principal axis perpendicular to the web, the plane being inclined in the direction indicated in Fig. (207).

Problem 17.

A load P, acting parallel to the central axis, is applied to a standard $5'' \times 3\frac{1}{2}'' \times \frac{1}{2}''$ angle through a riveted connection, shown in cross section (Fig. 211). The angle section is the same as that given in Problem (1). If the greatest allowable fiber stress f_c = 10,000 lbs. per sq. in., determine the greatest allowable value of P by use of the S polygon: (a) Assuming the resultant load to act through O_1, the intersection of the axis of the rivet and the middle layer of the plate; (b) Assuming the load to act through O_1', the intersection of the axis of the rivet and the surface of the angle; (c) Assuming the load to act through O_1'', the intersection of the axis of the rivet and the middle layer of the angle.

Problem 18.

Determine the greatest intensity of the tensile stress in the angle under each of the assumptions in Problem (17).

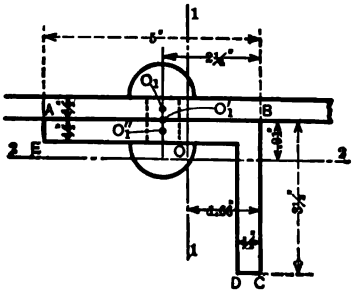


FIG. 211.

CHAPTER IX.

COLUMNS.

144. Columns and Struts. — Reference has been made (Art. 11) to the fact that when a piece of material is subjected to its *ultimate* or *breaking load* in compression the manner of failure and the breaking strength will depend upon the length of the piece. Under its ultimate load a short piece of a brittle material will crumble, breaking into small pieces; while a short piece of ductile or malleable material will gradually flatten out as the load is increased, without any definite breaking load being attained.

A long piece will bend laterally, or buckle, at the ultimate load, until it collapses and the breaking load in compression will be found to be much less than that of the short piece of the same cross section and material. Moreover, the breaking loads for pieces of a given cross section and material will be found to diminish as the lengths of the pieces are increased. If the material in the long piece is brittle rupture will occur at one or more sections after a certain amount of lateral deflection has taken place, but if the material is ductile the piece may continue to bend, after the ultimate load has been reached, until the ends are brought together.

A *column* may be defined as a vertical member which is subjected to compression by forces acting in the direction of its axis, whose length is large enough, compared with the dimensions of its cross section, for failure to take place by lateral bending. When a member which is not vertical is subjected to stress under similar conditions it may be called a *strut*, or brace.

In short columns and struts when the ultimate or breaking load is applied, the buckling is largely due to the unequal yielding of the fibers, which are subjected to stress beyond the elastic limit or even above the yield point of the material. In long columns failure at the ultimate load may be due to bending which takes place at stress intensities below the elastic limit.

Different theories, based upon assumptions similar to those made in the common theory of beams, have been proposed for determining the law of variation of the breaking strength with the length for different sizes and types of columns. The assumptions in these theories, however, are at greater variance with the actual conditions met with in practice than those made in the beam theory and hence the formulas which are deduced have to be regarded as empirical to a considerable degree. Under the usual conditions of practice purely empirical formulas, based upon the experimental determination of the ultimate strengths of columns under working conditions, as nearly as possible, can be made to represent the law of variation of the breaking strength fully as well as the formulas based on the column theories.

A discussion of the more common column theories and of different types of empirical formulas is given in the following Articles.

145. Long Thin Rods. — The behavior of a long thin rod, in equilibrium under equal and opposite forces applied at its ends, is of interest when considered in connection with the theory for determining the strength of long columns. Such a rod can be held in equilibrium in this manner in a number of different shapes without the stress intensity on any cross section exceeding the elastic limit. Some of these shapes will be in stable equilibrium, while others will be in unstable equilibrium and will change to different forms upon any disturbance in the forces acting. Under certain conditions the rod may remain perfectly straight and in other cases it may be bent into one of a number of curved forms, such as those shown (Fig. 212). For a rod of any given dimensions, subjected to forces of a given magnitude, one form only would

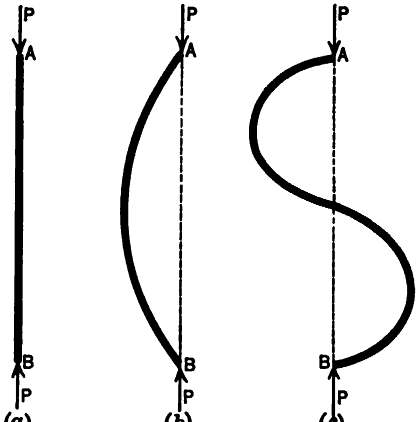


FIG. 212.

be in stable equilibrium, however, that being the form embodying a minimum of strain energy.

An analysis of rods of different dimensions, held in equilibrium under forces applied at the ends, would show that in some the straight form (Fig. 212a) would be the stable one and in others the curved form, shown in (Fig. 212b).

The derivation of the general equations of the elastic curves and the determination of the stable forms for thin rods of this kind is beyond the scope of this work.*

146. Euler's Theory — Long Columns. — The object of this theory is the determination of the magnitude of the centrally applied load required to produce a small lateral deflection in a long column when the greatest stress intensity on any cross section is less than the elastic limit of the material.

The result is affected to a large extent by the conditions at the ends of the column; the load required to bend the column being greater when the ends are "fixed" than when the ends are held so that the axis of the column may be free to incline to any angle with the vertical. This leads to the consideration of four separate cases, differing in the manner in which the ends of the columns are supported. In all of the cases the following conditions are imposed:

(a) The material is homogeneous.

(b) The cross section of the column is uniform.

(c) The line of action of the resultant of the load is vertical.

(d) The maximum stress intensity is below the elastic limit and the material follows the law of proportionality between stress intensity and strain.

Case I. — Column fixed at one end and free at the other. — The theory is first developed for an ideal case in which the column is fixed at the base, with the axis vertical, and is free to deflect at the top in any direction. The resultant of the load acts through the center of gravity of the top section and remains vertical, whatever the deflection of the column (Fig. 213). Under these conditions, if the column bends, the bending will evidently take place in the direction of the plane which is perpendicular to the principal axis of the cross section, about which the moment of inertia is a minimum. Assume that P = the magnitude of the load which, under the foregoing conditions, is required to maintain equilibrium when the column is deflected a small amount a at the top. Let l = the length of the column and let A = the area, I = the minimum moment of inertia and ρ = the minimum radius of gyration of a cross section. Let (x, y) be the coördinates of any point on the elastic curve OA and let r = the radius of curvature at this point.

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* A discussion of the forms of the elastic curve for thin rods held in equilibrium under the action of forces at the ends will be found in a number of treatises, among them the "Mathematical Theory of Elasticity," by A. E. H. Love.

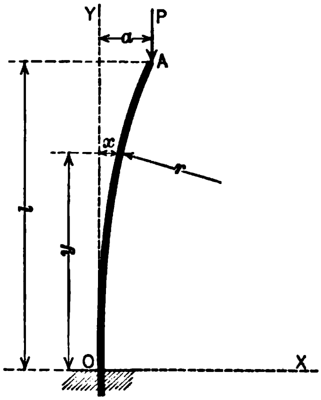

Fig. 213.

The bending moment at a cross section through this point will be equal to

$$M = P(a - x)$$

and, if the same assumptions are made as in the common theory of bending (Art. 95),

$$\frac{1}{r} = \frac{M}{EI} = \frac{P(a - x)}{EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

From the differential calculus, so long as a is small,

$$\frac{1}{r} = \frac{d^2x}{dy^2} \text{ (very nearly)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and equating (1) and (2),

$$\frac{d^2x}{dy^2} = \frac{P(a - x)}{EI}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which may be written

$$\frac{dx}{dy}\frac{d^2x}{dy^2}dy = \frac{P(a - x)}{EI}\frac{dx}{dy}dy.$$

Integrating,

$$\frac{1}{2}\left(\frac{dx}{dy}\right)^2 = -\frac{P}{2\,EI}(a - x)^2 + c.$$

When $x = 0$, $\frac{dx}{dy} = 0$ and hence $c = \frac{Pa^2}{2\,EI}$.

$$\therefore \quad \frac{dx}{dy} = \pm\sqrt{\frac{P}{EI}}\sqrt{2\,ax - x^2}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which may be written,

$$\frac{dx}{\sqrt{2\,ax - x^2}} = \pm\sqrt{\frac{P}{EI}}\,dy.$$

Integrating again,

$$\operatorname{vers}^{-1}\frac{x}{a} = \pm\, y\sqrt{\frac{P}{EI}} + c'.$$

When $y = 0$, $x = 0$, and hence $c' = 0$.

$$\therefore \quad x = a \operatorname{vers} y\sqrt{\frac{P}{EI}} = a\left(1 - \cos y\sqrt{\frac{P}{EI}}\right), \quad . \quad . \quad (5)$$

which gives the equation of the elastic curve in terms of the maximum deflection a.

When $y = l$, equation (5) becomes

$$a = a - a\cos l\sqrt{\frac{P}{EI}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

from which it is evident that $a = 0$, or, $\cos l\sqrt{\frac{P}{EI}} = 0$. Hence, if a has a finite value,

$$\cos l\sqrt{\frac{P}{EI}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and $$l\sqrt{\frac{P}{EI}} = \cos^{-1} 0 = \frac{\pi}{2} = \frac{3\,\pi}{2}, \text{ etc.}$$

Taking the smallest value of the angle and solving for P,

$$P = \left(\frac{\pi}{2\,l}\right)^2 EI = \left(\frac{\pi\rho}{2\,l}\right)^2 EA. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The average stress intensity on any cross section will evidently be equal to

$$\frac{P}{A} = \frac{\pi^2 E}{4}\left(\frac{\rho}{l}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The maximum stress intensity will occur at a section through the fixed end and will be greater than the above value by an amount depending on the value of a.

The load P may be called the *critical load;* for, a small increment added to P will produce a proportionately large increase in a and a corresponding increase in the maximum stress intensity on the section through the fixed end, resulting in the collapse of the column.

Hence the value of $\frac{P}{A}$, *given by equation* (9), *may be taken as the ultimate strength of the column, so long as it does not exceed the elastic limit of the material.*

When the average stress intensity, calculated from (9), is greater than the elastic limit, the theory assumes that the condition $a = 0$ (equation 6) must hold until the load on the column is sufficient to produce a stress intensity above the elastic limit and when this occurs the column will fail by buckling, due to the unequal yielding of its parts; and that under these conditions the ultimate strength will be represented by the simple expression

$$\frac{P}{A} = f_e, \quad \ldots \ldots \ldots \ldots \quad (10)$$

where f_e = the elastic limit of the material.

Therefore, the ultimate strength for a column of any dimensions will be the less of the values given by equations (9) *and* (10). For any given values of f_e and E there will be one value of $\frac{l}{\rho}$ for which the results given by these equations will be identical. Equating (9) and (10) and solving for $\frac{l}{\rho}$, this value evidently is

$$\frac{l}{\rho} = \frac{\pi}{2}\sqrt{\frac{E}{f_e}}. \quad \ldots \ldots \ldots \ldots \quad (11)$$

This may be called the critical value of $\frac{l}{\rho}$; for, when this ratio is less than the value given by (11), the ultimate strength is given by equation (10) and, when greater, the ultimate strength is given by equation (9).

Case II. Column free to turn at both end bearings. — In this case the column may be assumed to be supported on rounded ends, or frictionless hinges, at A and B (Fig. 214a) which offer no resistance to the turning of the ends of the axis to any angle of inclination with the vertical AB.

If the column bends the elastic curve will be symmetrical with respect to the axis OX through the middle point O.

If P = the magnitude of the equal and opposite, centrally applied, forces required to maintain equilibrium when the column is deflected laterally a small amount a, each half of the column must fulfill all the conditions imposed in Case I.

Hence, if l = the total length of the column and the notation adopted for Case I is otherwise followed, by substituting $\frac{l}{2}$ for l in equation (8) we obtain

$$P = \left(\frac{\pi}{l}\right)^2 EI = \left(\frac{\pi\rho}{l}\right)^2 EA, \quad \ldots\ldots \quad (12)$$

which gives the ultimate load, provided the average stress intensity,

$$\frac{P}{A} = \pi^2 E \left(\frac{\rho}{l}\right)^2, \quad \ldots\ldots\ldots \quad (13)$$

is less than the elastic limit. When the value given by (13) is above the elastic limit the ultimate strength is given by equation (10) as before.

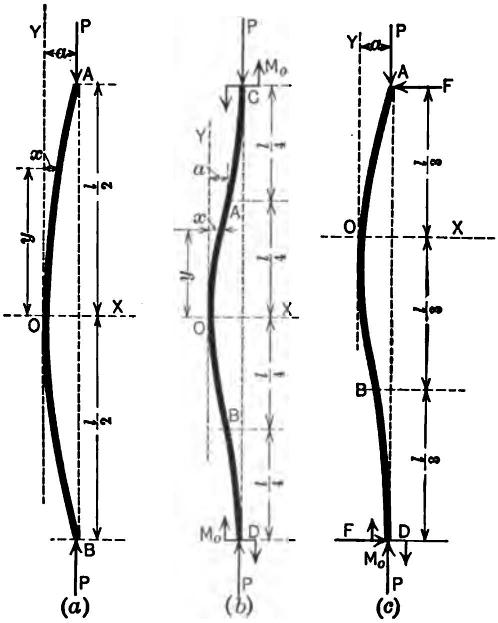

FIG. 214.

The critical value of $\frac{l}{\rho}$ for this case will evidently be equal to

$$\frac{l}{\rho} = \pi\sqrt{\frac{E}{f_c}}. \quad \ldots\ldots\ldots\ldots \quad (14)$$

Case III. Column fixed at both ends. — In this case the column is assumed to be held at the ends in such a manner that the tangents to the ends of the elastic curve remain vertical when the column bends (Fig. 214b). In order to maintain equilibrium under these conditions, when the column begins to bend, a couple will evidently be brought into action on each of the end sections, at C and D, in addition to the force P acting through the center of gravity. The stress on the center section through O will be the resultant of a couple and a central force P, respectively equal to the couple and force acting at either of the end sections; and points of inflexion will be located at A and B, halfway between O and the ends of the column.

The elastic curve will be symmetrical with respect to the axis OX, through the middle point O, and each quarter of the column will be in equilibrium under the same conditions as those imposed in Case I. Hence if l = the length of the column and the notation adopted in Case I is followed, by substituting $\frac{l}{4}$ for l in equation (8) we obtain

$$P = \left(\frac{2\pi}{l}\right)^2 EI = \left(\frac{2\pi\rho}{l}\right)^2 EA, \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

for the ultimate load on the column. The average stress intensity,

$$\frac{P}{A} = 4\pi^2 E \left(\frac{\rho}{l}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

must be, as before, less than the elastic limit. When the average intensity given by (16) is above the elastic limit the ultimate strength is given by equation (10).

The critical value of $\frac{l}{\rho}$ for this case is evidently equal to

$$\frac{l}{\rho} = 2\pi\sqrt{\frac{E}{f_c}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Case IV. Column free to turn at one end and fixed at the other. — This may be considered as an intermediate case between Cases II and III, the elastic curve taking the form indicated in Fig. (214c).

It should be noted that it differs from Case I in that the hinged end A is prevented from deflecting laterally, being held in line with the vertical through the fixed end D.

Since the greatest deflection a is small compared with the length of the column each third of the length may be assumed to fulfill very nearly all the conditions imposed in Case I and hence, by substituting $\frac{l}{3}$ for l in equation (8), the expression for the ultimate load becomes

$$P = \left(\frac{3\pi}{2l}\right)^2 EI = \left(\frac{3\pi\rho}{2l}\right)^2 EA \text{ (very nearly)}, \quad . \; . \quad (18)$$

and that for the average stress intensity on any cross section,

$$\frac{P}{A} = \frac{9\pi^2 E}{4}\left(\frac{\rho}{l}\right)^2. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (19)$$

As before when the value of the intensity given by (19) is above the elastic limit the ultimate strength is given by equation (10).

The critical value of $\frac{l}{\rho}$ for this case becomes

$$\frac{l}{\rho} = \frac{3\pi}{2}\sqrt{\frac{E}{f_c}}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

Summary. — It should be observed that the formula for the ultimate strength in each of the foregoing cases is in the form

$$\frac{P}{A} = kE\left(\frac{\rho}{l}\right)^2, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

where k has the values, $\frac{\pi^2}{4}$, π^2, $4\pi^2$, $\frac{9\pi^2}{4}$, in the four respective cases.

The expression for the critical value of $\frac{l}{\rho}$ in each case is in the form

$$\frac{l}{\rho} = \sqrt{k\frac{E}{f_c}}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

Since the ideal conditions imposed are rarely met, the values of k may be modified so that formulas in the form of equation (21) can be made to represent the ultimate strengths of a series of long columns, loaded under ordinary conditions, more nearly than the purely theoretical formulas.

147. Gordon's Formulas for Columns. — Another set of formulas for determining the ultimate, or breaking, loads for centrally loaded columns are known as Gordon's formulas.

These formulas have been developed for each of the last three cases considered in the preceding article. Although a rational derivation is attempted in each case, an assumption which is made is so inexact that the formulas must be regarded as empirical. The deduction of the formulas follows, the column in each case being assumed to be homogeneous and of uniform cross section.

Case I. Column free to turn at the ends. — In this case the column may be assumed to be supported on frictionless hinges, or on rounded ends, so that, when the column bends, the axis will take the form indicated (Fig. 215a). Let P = the ultimate load and v = the lateral deflection under this load, of the middle point of the axis of the column. Since the load is centrally applied the stress intensity will be a maximum on the middle cross section, and will evidently be equal to

$$f_c = \frac{P}{A} + \frac{(Pv)c}{I} \text{ (Art. 126)}, \quad \ldots\ldots \quad (1)$$

where $\frac{I}{c}$ = the minimum section modulus of the cross section. To determine the value of v the assumption is made that

$$v = k\frac{l^2}{c}, \quad \ldots\ldots\ldots\ldots \quad (2)$$

or, that the greatest lateral deflection of the column under the ultimate load varies as the square of the length; in the same manner as the greatest deflection of a transversely loaded beam, when subjected to a maximum fiber stress below the elastic limit (Art. 104).

Substituting in (1) the above value of v and for I its value

$$I = A\rho^2,$$

where ρ evidently equals the minimum radius of gyration of the cross section, we obtain

$$f_c = \frac{P}{A} + \frac{Pkl^2}{A\rho^2} = \frac{P}{A}\left[1 + k\left(\frac{l}{\rho}\right)^2\right]. \quad \ldots\ldots \quad (3)$$

Solving (3) the expression for the ultimate load reduces to

$$P = \frac{f_c A}{1 + k\left(\frac{l}{\rho}\right)^2}. \quad \ldots\ldots\ldots \quad (4)$$

Since the assumption (equation 2) is evidently erroneous when f_c is above the elastic limit and has not been verified for values of f_c below the elastic limit, (4) must be regarded as an empirical equation, for which the constants f_c and k can be determined from the results of experiments only.

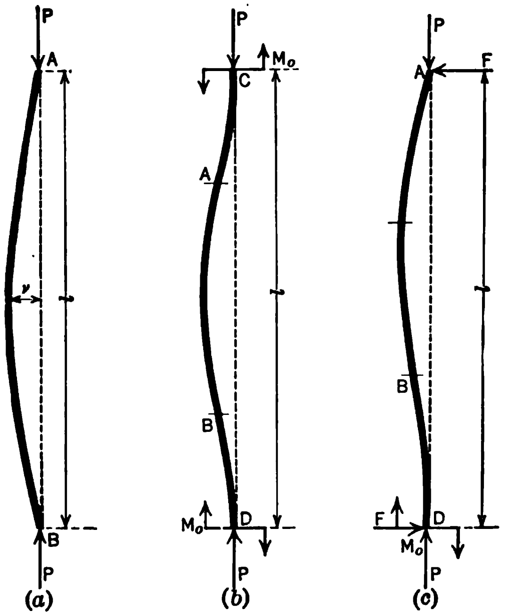

Fig. 215.

The expression for the ultimate strength in terms of these constants is evidently,

$$\frac{P}{A} = \frac{f_c}{1 + k\left(\frac{l}{\rho}\right)^2}. \quad \ldots \ldots \ldots \quad (5)$$

Case II. Column fixed in direction at the ends. — The deduction of the formula for this case is based on the assumptions made in Case I and the additional assumption that points of inflexion A and B are located half way between the center of the column and the ends (Fig. 215b). As soon as the column begins to deflect, couples will evidently be brought into action at the end sections, in addition to the centrally applied loads P. On this basis the portion of the column between A and B fulfills the conditions imposed in Case I and hence the expression for the ultimate strength may be obtained by substituting $\frac{l}{2}$ for l in equation (5), giving

$$\frac{P}{A} = \frac{f_o}{1 + \frac{k}{4}\left(\frac{l}{\rho}\right)^2}. \quad \ldots \ldots \ldots \quad (6)$$

Case III. Column fixed in direction at one end and free to turn at the other. — In this case the axis of the column is assumed to take the form indicated in Fig. (215c), with a point of inflexion B one third of the length of the column from the fixed end. If the portion between A and B is assumed to fulfill the conditions imposed in Case I, the expression for the ultimate strength may be determined by substituting $\frac{2}{3}\, l$ for l in equation (5), giving

$$\frac{P}{A} = \frac{f_c}{1 + \frac{4\,k}{9}\left(\frac{l}{\rho}\right)^2}. \quad \ldots \ldots \ldots \quad (7)$$

148. Rankine's Formulas. — The following method was proposed by Rankine for developing a formula which would give the ultimate load on a centrally loaded column of any length and free to turn at the ends (Fig. 214a).

Let $P_1 = f_o A$ equal the ultimate load for a short column and $P_2 = \pi^2 EA \left(\frac{\rho}{l}\right)^2$ equal the ultimate load for a long column, according to Euler's theory (Art. 146). Then, it was assumed by Rankine that, if P = the ultimate load for a column of any length, the equation

$$\frac{1}{P} = \frac{1}{P_1} + \frac{1}{P_2} \quad \ldots \ldots \ldots \quad (1)$$

would give a value of P which would hold for either a short or a long column; since, for a very short column $\frac{1}{P_2}$ would be so small as to be negligible and $P = P_1$, nearly, and for a long column the value of $\frac{1}{P_1}$ would be negligible when compared with $\frac{1}{P_2}$ and $P = P_2$, nearly.

Transposing and substituting the values of P_1 and P_2,

$$P = \frac{1}{\frac{1}{P_1} + \frac{1}{P_2}} = \frac{1}{\frac{1}{f_cA} + \frac{1}{\pi^2EA}\left(\frac{l}{\rho}\right)^2} = \frac{f_cA}{1 + \frac{f_c}{\pi^2E}\left(\frac{l}{\rho}\right)^2},$$

or,

$$\frac{P}{A} = \frac{f_c}{1 + \frac{f_c}{\pi^2E}\left(\frac{l}{\rho}\right)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This formula will evidently give the same value for the ultimate strength as Gordon's formula (equation 5, Art. 147), provided the constant

$$\frac{f_c}{\pi^2E} = k, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The values for f_c and k, given by Rankine for computing the ultimate loads on wrought-iron and cast-iron columns, were as follows:

Wrought-iron columns, $f_c = 36{,}000, \quad k = \frac{1}{9000}$;
Cast-iron columns, $f_c = 80{,}000, \quad k = \frac{1}{1600}$.

149. Empirical Formulas of the Gordon, or Rankine Type. — As stated previously, the assumption in regard to the deflection, made in the deduction of the Gordon formulas, is so inexact that the formulas must be treated as empirical; and the constants must be determined from the results of experiments covering the different types and sizes of columns to which the formulas are intended to apply.

Moreover, the conditions laid down in the theory in regard to the manner in which the ends of the column are held are rarely fulfilled in practice. For example a column may be built with flat ends, or bearing surfaces which, if perfect, would prevent the ends of the axis from turning but owing to lack of homogeneity and to deflects in construction, these conditions will be met only approximately. A column may also be built with pin supports through the ends of its axis and if such a column bends, the ends can evidently turn on the pins, but friction will prevent the ends from turning freely enough to fulfill the conditions imposed in the case of a column with rounded ends.

Therefore, if the formulas are to be of practical value, different sets of constants must be provided to meet the different conditions regarding the material, the type and size of the column and the method of supporting the ends.

Various sets of constants have been proposed for formulas of the Gordon and Rankine type, of which the following are given as examples. The formulas are proposed for determining the ultimate strengths of structural columns, of a medium grade of steel, constructed by riveting together plates and angles, or channels, of the ordinary standard sections.

Column with flat or "square" bearings,

$$\frac{P}{A} = \frac{50{,}000}{1 + \dfrac{l^2}{36{,}000\,\rho^2}} \qquad (1)$$

Column with pin bearings,

$$\frac{P}{A} = \frac{50{,}000}{1 + \dfrac{l^2}{18{,}000\,\rho^2}} \qquad (2)$$

Column with one pin bearing and one "square" bearing,

$$\frac{P}{A} = \frac{50{,}000}{1 + \dfrac{l^2}{24{,}000\,\rho^2}} \qquad (3)$$

The constants may be modified so that the formulas represent working strengths instead of ultimate strengths. For example, if a factor of safety of 4 is used, the expression for the working strength for a column with pin bearings, determined from equation (2), would evidently be

$$\frac{P}{A} = \frac{12{,}500}{1 + \dfrac{l^2}{18{,}000\,\rho^2}} \qquad (4)$$

Formulas for working strengths, similar to (4), but with somewhat different constants, can be found in the building laws of cities in the United States.

150. Parabolic Formulas. — It was proposed by Professor J. B. Johnson that if Gordon's formula

$$\frac{P}{A} = \frac{f_c}{1 + k\left(\dfrac{l}{\rho}\right)^2} \text{ (Art. 147),}$$

when written in the form

$$f_c = \frac{P}{A}\left[1 + k\left(\frac{l}{\rho}\right)^2\right],$$

were modified by substituting $k_1 = \frac{P}{A}k$ and transposing, the resulting equation of the parabola,

$$\frac{P}{A} = f_c - k_1\left(\frac{l}{\rho}\right)^2, \quad \dots\dots\dots \quad (1)$$

would represent the ultimate strengths of columns of moderate length provided the values of f_c and k_1 were properly chosen.

He proposed that for columns of mild steel with flat ends the formula be written

$$\frac{P}{A} = 42{,}000 - 0.62\left(\frac{l}{\rho}\right)^2, \quad \dots\dots\dots \quad (2)$$

its use to be limited to values of $\frac{l}{\rho} < 190$.

Similarly for mild steel columns with pin ends,

$$\frac{P}{A} = 42{,}000 - 0.97\left(\frac{l}{\rho}\right)^2, \quad \dots\dots\dots \quad (3)$$

for values of $\frac{l}{\rho} < 150$.

To determine the working strength in any case the ultimate strength, calculated from the equation, was to be divided by a proper factor of safety.

151. Straight-Line Formulas. — After a careful investigation of the results of tests which had been made on centrally loaded columns of various materials, supported in different ways at the ends, Mr. Thomas H. Johnson proposed that when plots of the ultimate strengths were made, with values of $\frac{P}{A}$ for ordinates and of $\frac{l}{\rho}$ for abscissæ, a straight line would fit the plot for columns of any given type and material as well as any curve.

Hence the equation of the straight line would represent the ultimate strengths of columns of any length, within specified limits, as well as any of the formulas for which a rational derivation had been attempted.

The general form of the equation of the straight line would be

$$\frac{P}{A} = f_c - k_o\left(\frac{l}{\rho}\right), \quad \text{. . . . . (1)}$$

where f_c and k_0 are constants, which can easily be obtained from a plot of the results of tests made as indicated above.

In the case of mild steel columns Johnson proposed the following formulas for determining the *ultimate strength* for values of $\frac{l}{\rho}$ within the limits indicated.

Mild steel columns with flat ends $\left(\frac{l}{\rho} < 195\right)$,

$$\frac{P}{A} = 52{,}500 - 179\frac{l}{\rho}. \quad \text{. . . . . (2)}$$

Mild steel columns with pin ends $\left(\frac{l}{\rho} < 159\right)$,

$$\frac{P}{A} = 52{,}500 - 220\frac{l}{\rho}. \quad \text{. . . . . (3)}$$

Evidently, the chief merit of formulas of this type is their simplicity. The results of experiments tend to show, however, that for some types of columns, at least, the ultimate strength for values of $\frac{l}{\rho}$ less than a given amount is nearly constant and that, for values of $\frac{l}{\rho}$ exceeding this amount, the ultimate strength can be represented by a formula of the straight-line type. Hence, if the constants in equation (1) are chosen to give the proper values of the ultimate strength for the longer columns, the formula will give values for very short columns which are too high. This difficulty is easily overcome, however, by imposing both minimum and maximum limits of $\frac{l}{\rho}$, between which the formula will hold; and adopting a constant value for the ultimate strength

$$\frac{P}{A} = f_c, \quad \text{. . . . . (4)}$$

when $\frac{l}{\rho}$ is less than the minimum limit.

It should also be noted that, owing to the friction on the bearings of a column supported on pins and the lack of homogeneity which usually exists in the ordinary column with either pin or flat ends, the results of experiments fail to show the difference between the ultimate strengths of columns with pin ends and of those with flat ends which the different theories would seem to indicate.

It is now customary, therefore, when the use of straight-line formulas is proposed to make no distinction between the constants for columns with flat ends and those for columns with pin ends, but to make one formula represent the ultimate strength of all columns of a given type, having values of $\frac{l}{\rho}$ between certain limits.

By introducing proper factors of safety the formulas can evidently be made to represent the working strength instead of the ultimate strength. Two of the accepted formulas of this type, representing the working strength of structural columns of mild steel, are the following:

$$\frac{P}{A} = 16{,}000 - 70\left(\frac{l}{\rho}\right), \text{ for values of } \frac{l}{\rho} > 30 \text{ and } < 120, \qquad (5)$$

with

$$\frac{P}{A} = 14{,}000, \text{ for values of } \frac{l}{\rho} < 30, \quad \dots\dots\dots\dots \qquad (6)$$

recommended by the American Railway Engineering and Maintenance of Way Association; and

$$\frac{P}{A} = 19{,}000 - 100\left(\frac{l}{\rho}\right), \text{ for values of } \frac{l}{\rho} > 60 \text{ and } < 120, \qquad (7)$$

with

$$\frac{P}{A} = 13{,}000, \text{ for values of } \frac{l}{\rho} < 60, \quad \dots\dots\dots\dots \qquad (8)$$

recommended by the American Bridge Company.

Final emphasis should be laid on the fact, mentioned heretofore, that no one of the column formulas given can be adapted to the determination of the ultimate strengths, or the working strengths, of all columns of any one material; but that the use of any one formula must be restricted to columns, not merely of the material, but of the type of cross section and ratio of $\frac{l}{\rho}$ within the limits of that of the columns from which the experimental data for determining the constants were obtained.

T. H. Johnson proposed that a straight-line formula with proper constants be used to represent the ultimate strength for values of $\frac{l}{\rho}$ below certain limits and that modified forms of Euler's formula (equation 21, Art. 146), with proper values of k, be used to represent the strengths of columns when $\frac{l}{\rho}$ exceeds the limits of the straight-line formulas. For mild steel columns with flat ends and $\frac{l}{\rho} > 195$ he proposed that Euler's formula with $k = 16$ be used; and for mild steel columns with pin ends and $\frac{l}{\rho} > 159$, $k = 25$.

By introducing proper factors of safety similar formulas for the working strengths of the short and the long columns could be obtained.

For purposes of comparison, graphs of the ultimate strengths of mild steel columns with flat ends and also with pin ends, given by the different formulas, are plotted in Fig. (216), the graphs being designated as follows:

(a) Euler's formula for columns with fixed ends — (16) (Art. 146), when $E = 28{,}000{,}000$ lbs. per sq. in.
(b) Empirical formula of the Gordon type for column with flat ends — (1) (Art. 149).
(b′) Empirical formula of the Gordon type for column with pin ends — (2) (Art. 149).
(c) Parabolic formula for column with flat ends — (2) (Art. 150).
(c′) Parabolic formula for column with pin ends — (3) (Art. 150).
(d) Straight-line formula for column with flat ends — (2) (Art. 151).
(d′) Straight-line formula for column with pin ends — (3) (Art. 151).
(e) Euler's formula — (21) (Art. 146), as modified for columns with flat ends, using $k = 16$, $E = 28{,}000{,}000$ lbs. per sq. in.
(e′) Euler's formula — (21) (Art. 146), as modified for columns with pin ends, using $k = 25$, $E = 28{,}000{,}000$ lbs. per sq. in.

Graphs of the formulas for working strengths are designated as follows:

(f) Formula (4) (Art. 149).
(g) Formulas (5) and (6) (Art. 151).
(h) Formulas (7) and (8) (Art. 151).

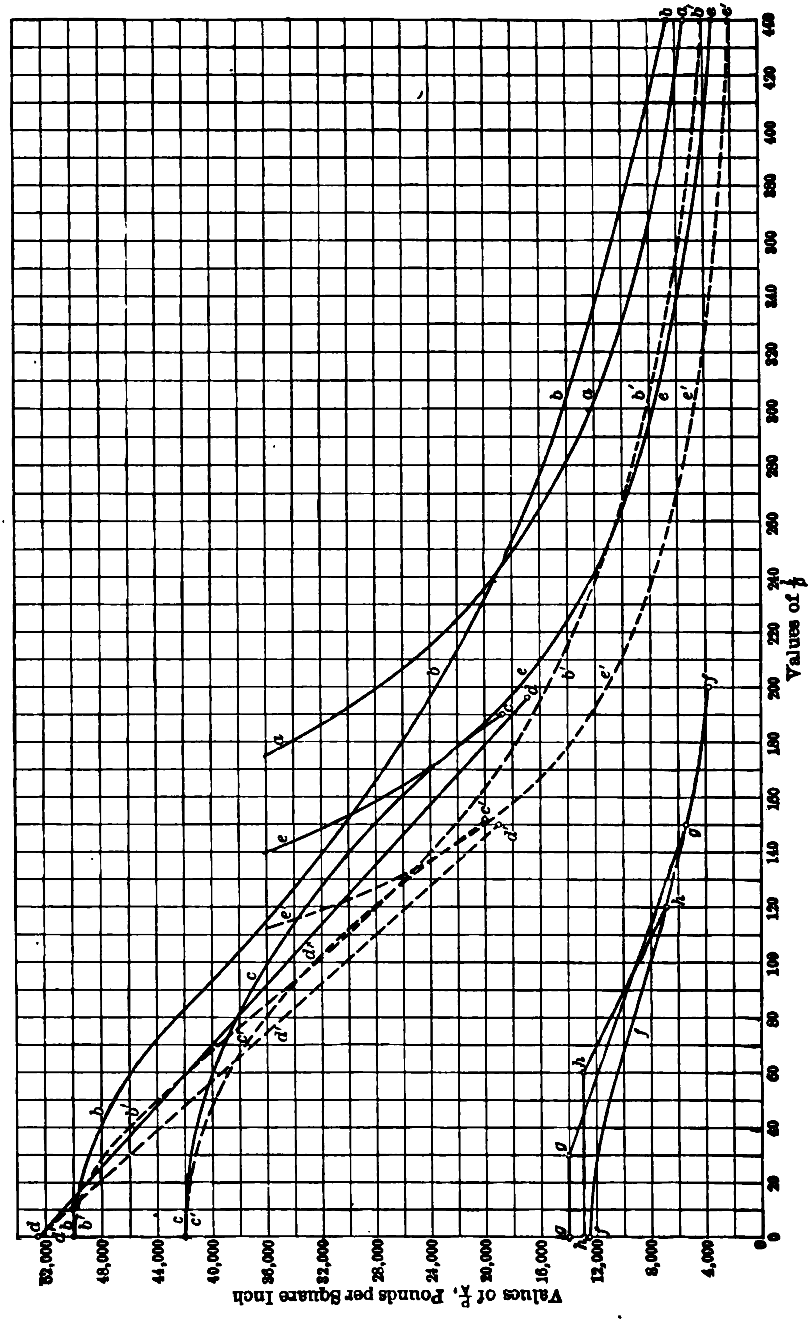
Values of $\frac{P}{A}$, Pounds per Square Inch
52,000
48,000
44,000
40,000
36,000
32,000
28,000
24,000
20,000
16,000
12,000
8,000
4,000
0
Values of $\frac{l}{b}$
0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
340
360
380
400
420
440
a
b
c
d
e
a'
b'
c'
d'
e'
f
g
h

FIG. 216.

152. Eccentric Loading. — Thus far in this chapter the discussion has been limited to the theories and the different types of formulas which have been proposed for determining the breaking strengths, or the working strengths, of centrally loaded columns.

When a column of ordinary dimensions is loaded eccentrically, that is, with forces acting parallel to the original axis of the column, the maximum stress intensity may be determined by the methods in Art. (126).

It is customary to limit the greatest stress intensity, determined by these methods, to the working strength for a centrally loaded column of the same dimensions. For example, if the straight-line formulas $\frac{P}{A} = 13{,}000$ and $\frac{P}{A} = 19{,}000 - 100\frac{l}{\rho}$ (Art. 151) are used to determine the working strengths of centrally loaded steel columns, the maximum stress intensities, when the columns are eccentrically loaded, must not exceed the values given by these formulas.

153. Strut or Tie Subjected to Combined Axial and Lateral Loading. — *Case I. Strut, fixed at one end, subjected to combined axial and uniform lateral loads.* Let OA represent the elastic curve formed by the axis of a horizontal strut of uniform section and material (Fig. 217) which is fixed in direction at A and subjected to a horizontal force P, acting through the center of gravity of the end section at O, combined with a uniformly distributed vertical load w per unit of length (for example, its own weight) the stress intensity throughout being less than the elastic limit.

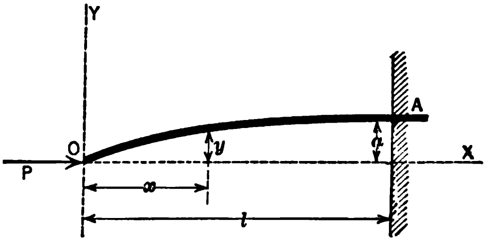

FIG. 217.

Let l = the length, A = the area and I the moment of inertia of any cross section, about a horizontal axis which is assumed to be a principal axis through the center of gravity. Refer OA to horizontal and vertical axes, with the origin at O, and let a = the vertical deflection of O. Let M_a = the bending moment at A and M = the bending moment at a cross section through any point (x, y), on the elastic curve.

An approximate solution for M_a and the greatest fiber stress can evidently be made, after the method suggested in Art. (126), by neglecting the lateral deflection due to the load P and calculating a from the equation for the greatest deflection of a uniformly loaded cantilever beam (Art. 98); in which case, if signs are taken positive,

$$a = \frac{wl^4}{8EI} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$M_a = \frac{wl^2}{2} + Pa = \frac{wl^2}{2} + \frac{Pwl^4}{8EI}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the greatest fiber stress

$$f = \frac{P}{A} + \frac{M_a c}{I} = \frac{P}{A} + \frac{c}{I}\left(\frac{wl^2}{2} + \frac{Pwl^4}{8EI}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For a more accurate solution, following the usual convention of signs for x, y and M,

$$M = -\frac{wx^2}{2} - Py \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and hence, according to the beam theory,

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{wx^2}{2EI} - \frac{Py}{EI},$$

or,

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = -\frac{wx^2}{2EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The solution of this equation may be made as follows: Let $y = u + v$. Then

$$\frac{d^2v}{dx^2} + \frac{Pv}{EI} = -\frac{wx^2}{2EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$\frac{d^2u}{dx^2} + \frac{Pu}{EI} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From (6) the value of the particular integral,

$$v = -\frac{wx^2}{2P} + \frac{wEI}{P^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

is readily obtained and from (7) the complementary function may be obtained as follows:

Multiplying (7) by $\frac{du}{dx}dx$ and converting it to the form,

$$\frac{EI}{P} \cdot \frac{du}{dx} d\left(\frac{du}{dx}\right) + u\,du = 0,$$

and integrating,

$$\frac{EI}{2P}\left(\frac{du}{dx}\right)^2 + \frac{u^2}{2} = \frac{c^2}{2},$$

where c = a constant. Solving for $\frac{du}{dx}$,

$$\frac{du}{dx} = \pm\sqrt{\frac{P}{EI}}\sqrt{c^2 - u^2},$$

or,

$$\frac{du}{\sqrt{c^2 - u^2}} = \pm\sqrt{\frac{P}{EI}}\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and integrating again,

$$\cos^{-1}\frac{u}{c} = \pm\left[x\sqrt{\frac{P}{EI}} + d\right],$$

or,

$$u = c\cos\left[x\sqrt{\frac{P}{EI}} + d\right]$$

$$= c\left[\cos x\sqrt{\frac{P}{EI}}\cos d - \sin x\sqrt{\frac{P}{EI}}\sin d\right], \quad . \quad . \quad . \qquad (10)$$

where d = a constant.

Letting the constants

$$c\cos d = C \quad \text{and} \quad -c\sin d = D$$

we obtain

$$u = C\cos x\sqrt{\frac{P}{EI}} + D\sin x\sqrt{\frac{P}{EI}}. \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

Therefore the complete solution of equation (5) is

$$y = -\frac{wx^2}{2P} + \frac{wEI}{P^2} + C\cos x\sqrt{\frac{P}{EI}} + D\sin x\sqrt{\frac{P}{EI}}. \quad . \quad . \qquad (12)$$

To determine C note that when $x = 0$, $y = 0$; and hence

$$C = -\frac{wEI}{P^2}.$$

To determine D note that when $x = l$, $\frac{dy}{dx} = 0$. Hence

$$\frac{dy}{dx} = -\frac{wx}{P} - C\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}} + D\sqrt{\frac{P}{EI}}\cos x\sqrt{\frac{P}{EI}} = 0,$$

when $x = l$, and

$$D = \frac{wEI}{P^2}\left[l\sqrt{\frac{P}{EI}}\sec l\sqrt{\frac{P}{EI}} - \tan l\sqrt{\frac{P}{EI}}\right].$$

Hence equation (12) reduces to

$$y = -\frac{wx^2}{2P} + \frac{wEI}{P^2}\left[1 - \cos x\sqrt{\frac{P}{EI}} + l\sqrt{\frac{P}{EI}}\sec l\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}}\right.$$

$$\left. - \tan l\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

When $x = l$, $y = a$, and hence

$$a = -\frac{wl^2}{2P} + \frac{wEI}{P^2}\left[1 - \sec l\sqrt{\frac{P}{EI}} + l\sqrt{\frac{P}{EI}}\tan l\sqrt{\frac{P}{EI}}\right]. \quad . \quad . \qquad (14)$$

and the greatest bending moment

$$M_a = -\frac{wl^2}{2} - Pa = -\frac{wEI}{P}\left[1 - \sec l\sqrt{\frac{P}{EI}} + l\sqrt{\frac{P}{EI}}\tan l\sqrt{\frac{P}{EI}}\right]. \quad . \quad . \qquad (15)$$

The greatest intensity of stress on the section through A will evidently be equal to

$$f = \frac{P}{A} + \frac{M_a c}{I}. \quad \ldots \ldots \ldots \quad (16)$$

If we let $\alpha = l\sqrt{\frac{P}{EI}}$ and apply the expansion

$$1 - \sec\alpha + \alpha\tan\alpha = \frac{\alpha^2}{2} + \frac{\alpha^4}{8} + \frac{7\,\alpha^6}{144} + \frac{113\,\alpha^8}{5760} + \cdots$$

to equation (15), we obtain

$$M_a = \frac{wEI}{P}\left[\frac{l^2}{2}\left(\frac{P}{EI}\right) + \frac{l^4}{8}\left(\frac{P}{EI}\right)^2 + \frac{7\,l^6}{144}\left(\frac{P}{EI}\right)^3 + \frac{113\,l^8}{5760}\left(\frac{P}{EI}\right)^4 + \cdots\right]$$

$$= \frac{wl^2}{2} + \frac{wl^4}{8\,EI}P\left[1 + 0.389\,l^2\left(\frac{P}{EI}\right) + 0.157\,l^4\left(\frac{P}{EI}\right)^2 + \cdots\right]. \quad \ldots \quad (17)$$

By comparing the bending moment obtained from (17) with that given by (2) for any specific case the error due to using the approximate solution given by the latter equation can be readily estimated.

Case II. Strut with hinged ends subjected to combined axial and uniform lateral loads. Let OAB represent the elastic curve formed by axis of a horizontal strut, held by frictionless hinges at O and B (Fig. 218), and subjected to a horizontal thrust P, through the center of gravity of each end section, combined with a uniformly distributed load w per unit of length, the greatest stress intensity in the strut being less than the elastic limit.

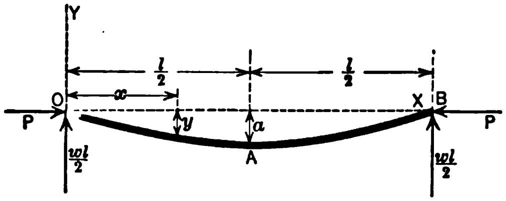

FIG. 218.

Let l = the length of the strut, M_a the bending moment at the middle section A and follow the notation adopted in Case I.

An approximate solution will evidently be, when signs are neglected,

$$a = \frac{5}{384}\frac{wl^4}{EI} \text{ (Art. 98)}, \quad \ldots \ldots \ldots \quad (18)$$

$$M_a = \frac{wl^2}{8} + Pa = \frac{wl^2}{8} + \frac{5\,Pwl^4}{384\,EI}. \quad \ldots \ldots \quad (19)$$

and the greatest fiber stress

$$f = \frac{P}{A} + \frac{c}{I}\left(\frac{wl^2}{8} + \frac{5\,Pwl^4}{384\,EI}\right). \quad \ldots \ldots \quad (20)$$

For a more exact solution, following the usual convention of signs for x, y and M

$$M = \frac{wl}{2}x - \frac{wx^2}{2} - Py, \quad \dots\dots\dots \quad (21)$$

and

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = -\frac{wx^2}{2\,EI} + \frac{wlx}{2\,EI}. \quad \dots\dots\dots \quad (22)$$

Proceeding in the same manner as in Case I, the solution of equation (22) is

$$y = -\frac{wx^2}{2\,P} + \frac{wlx}{2\,P} + \frac{wEI}{P^2} + C\cos x\sqrt{\frac{P}{EI}} + D\sin x\sqrt{\frac{P}{EI}}. \quad \dots \quad (23)$$

When $x = 0$, $y = 0$, and when $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$; and hence

$$C = -\frac{wEI}{P^2}, \quad D = -\frac{wEI}{P^2}\tan\frac{l}{2}\sqrt{\frac{P}{EI}}.$$

Hence equation (23) reduces to

$$y = -\frac{wx^2}{2\,P} + \frac{wlx}{2\,P} + \frac{wEI}{P^2}\left[1 - \cos x\sqrt{\frac{P}{EI}} - \tan\frac{l}{2}\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}}\right]. \quad \dots \quad (24)$$

When $x = \frac{l}{2}$, $y = a$; and hence

$$a = \frac{wl^2}{8\,P} + \frac{wEI}{P^2}\left[1 - \cos\frac{l}{2}\sqrt{\frac{P}{EI}} - \tan\frac{l}{2}\sqrt{\frac{P}{EI}}\sin\frac{l}{2}\sqrt{\frac{P}{EI}}\right]$$

$$= \frac{wl^2}{8\,P} + \frac{wEI}{P^2}\left[1 - \sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right]. \quad \dots\dots\dots\dots \quad (25)$$

The greatest bending moment

$$M_a = \frac{wl^2}{8} - Pa = -\frac{wEI}{P}\left[1 - \sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right]. \quad \dots\dots \quad (26)$$

and the greatest fiber stress

$$f = \frac{P}{A} + \frac{Ewc}{P}\left[\sec\frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right]. \quad \dots\dots\dots \quad (27)$$

If we let $\alpha = \frac{l}{2}\sqrt{\frac{P}{EI}}$ and apply the expansion

$$1 - \sec\alpha = -\frac{\alpha^2}{2} - \frac{5\,\alpha^4}{24} - \frac{61\,\alpha^6}{720} - \frac{1385\,\alpha^8}{40{,}320}\cdot\,\cdot$$

to equation (26), we obtain

$$M_a = \frac{wEI}{P}\left[\frac{l^2}{8}\left(\frac{P}{EI}\right) + \frac{5\,l^4}{384}\left(\frac{P}{EI}\right)^2 + 0.00132\,l^6\left(\frac{P}{EI}\right)^3 + 0.000134\,l^8\left(\frac{P}{EI}\right)^4 + \cdots\right]$$

$$= \frac{wl^2}{8} + \frac{5\,wl^4}{384\,EI}P\left[1 + 0.102\,l^2\left(\frac{P}{EI}\right) + 0.0103\,l^4\left(\frac{P}{EI}\right)^2 + \cdots\right]. \quad \dots \quad (28)$$

By comparing the bending moment obtained from (28) with that given by (19) for any specific case, the error due to using the approximate solution can be readily obtained.

Case III. Tie with hinged ends subjected to combined axial and lateral loads. By reversing the direction of the horizontal forces, Fig. (218) may be taken to represent a horizontal tie, subjected to an axial pull P combined with a uniformly distributed lateral load.

Assuming that the greatest stress intensity is below the elastic limit and following the notation of Case II an approximate solution may be made by the use of equation (20), the positive signs in this case representing tension instead of compression.

A more exact solution, using the same notation as before and the usual convention of signs for x, y and M, is the following:

$$M = \frac{wlx}{2} - \frac{wx^2}{2} + Py, \quad \dots \dots \dots \quad (29)$$

and hence

$$\frac{d^2y}{dx^2} - \frac{Py}{EI} = -\frac{wx^2}{2EI} + \frac{wlx}{2EI}. \quad \dots \dots \dots \quad (30)$$

The solution of (30) is made in the same manner as that of (5) and (22] and

$$y = \frac{wx^2}{2P} - \frac{wlx}{2P} + \frac{wEI}{P^2} + C\cosh x\sqrt{\frac{P}{EI}} + D\sinh x\sqrt{\frac{P}{EI}}. \quad . \quad . \quad (31)$$

The constants may be determined from the conditions

$y = 0$, when $x = 0$, and $\frac{dy}{dx} = 0$, when $x = \frac{l}{2}$, and the values are

$$C = -\frac{wEI}{P^2}, \quad D = \frac{wEI}{P^2}\tanh\frac{l}{2}\sqrt{\frac{P}{EI}}.$$

Substituting in (31),

$$y = \frac{wx^2}{2P} - \frac{wlx}{2P} + \frac{wEI}{P^2}\left[1 - \cosh x\sqrt{\frac{P}{EI}} + \tanh\frac{l}{2}\sqrt{\frac{P}{EI}}\sinh x\sqrt{\frac{P}{EI}}\right]. \quad (32)$$

When $x = \frac{l}{2}$, $y = a$, and

$$\begin{aligned} a &= -\frac{wl^2}{8P} + \frac{wEI}{P^2}\left[1 - \cosh\frac{l}{2}\sqrt{\frac{P}{EI}} + \tanh\frac{l}{2}\sqrt{\frac{P}{EI}}\sinh\frac{l}{2}\sqrt{\frac{P}{EI}}\right] \\ &= -\frac{wl^2}{8P} + \frac{wEI}{P^2}\left[1 - \operatorname{sech}\frac{l}{2}\sqrt{\frac{P}{EI}}\right]. \quad \dots \dots \dots \quad (33) \end{aligned}$$

The greatest bending moment

$$M_a = \frac{wl^2}{8} + Pa = \frac{wEI}{P}\left[1 - \operatorname{sech}\frac{l}{2}\sqrt{\frac{P}{EI}}\right] \quad . \quad . \quad . \quad . \quad (34)$$

and the greatest fiber stress

$$f = \frac{P}{A} + \frac{Ewc}{P}\left[1 - \operatorname{sech}\frac{l}{2}\sqrt{\frac{P}{EI}}\right], \quad . \quad . \quad . \quad . \quad . \quad (35)$$

the positive sign in this case indicating tension.

Solutions for other cases where struts are subjected to transverse loads may be obtained in a similar manner.*

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* See Paper by Arthur Morley in the Philosophical Magazine, June, 1908.

154. Long Column under Eccentric Load.—The method of determining the maximum stress intensity, due to eccentric loading on a column of ordinary length, has been discussed in Art. (152). The modifications in Euler's formulas for long columns, which result from an eccentricity of the load, are of some value.

Referring to Case I (Art. 146), assume that the load P at the end section of the column (Fig. 213) acts at a distance e from the axis, the plane containing the load and the axis of the column being perpendicular to a principal axis of any cross section. The bending moment at a section through any point (x, y) on the axis will then be equal to

$$M = P(a + e - x). \quad \text{. . . . .} \quad (1)$$

and equation (3) (Art. 146) becomes

$$\frac{d^2x}{dy^2} + \frac{Px}{EI} = \frac{P}{EI}(a + e). \quad \text{. . . . .} \quad (2)$$

The solution of this equation is evidently

$$x = (a + e)\left(1 - \cos y\sqrt{\frac{P}{EI}}\right) \text{ (equation 5, Art. 146). . .} \quad (3)$$

When $y = l$, $x = a$ and equation (3) becomes

$$a \cos l\sqrt{\frac{P}{EI}} = e\left(1 - \cos l\sqrt{\frac{P}{EI}}\right)$$

and hence

$$a = e\left(\sec l\sqrt{\frac{P}{EI}} - 1\right), \quad \text{. . . . .} \quad (4)$$

from which a value for a can be obtained.

The bending moment at the fixed end is equal to

$$M_o = P(a + e) = Pe\left(\sec l\sqrt{\frac{P}{EI}}\right) \quad \text{. . . . . .} \quad (5)$$

and the greatest intensity of the compressive stress

$$f_c = \frac{P}{A} + \frac{M_o c}{I} = P\left(\frac{1}{A} + \frac{ce}{I}\sec l\sqrt{\frac{P}{EI}}\right), \quad \text{. . . . .} \quad (6)$$

so long as the value of f_c does not exceed the elastic limit of the material. If tension exists on the section through O its greatest intensity will evidently be

$$f_t = P\left(\frac{ce}{I}\sec l\frac{P}{EI} - \frac{1}{A}\right). \quad \text{. . . . .} \quad (7)$$

In the case of the column hinged at both ends (Case II, Art. 146), the greatest stress intensity in tension and compression when the eccentricity of the load is equal to e, can evidently be obtained by putting $\frac{l}{2} = l$ in (6) and (7), which will give

$$f_c = P\left(\frac{1}{A} + \frac{ce}{I}\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right) \quad \text{. . . . .} \quad (8)$$

and

$$f_t = P\left(\frac{ce}{I}\sec\frac{l}{2}\sqrt{\frac{P}{EI}} - \frac{1}{A}\right) \quad \text{. . . . .} \quad (9)$$

155. Problems: Columns and Struts. —

Problem 1.

Given: A steel column with flat ends, having the cross section shown in Fig. (219), and subjected to an axial load. By use of the empirical formula (1) (Art. 149) of the Gordon type find the ultimate strength of the column:

(a) If the column is 25 ft. long;

(b) If the column is 12 ft. long.

Find the working loads for the columns of the two given lengths:

(a) By using a factor of safety of 4 with the above named formula;

(b) By using the straight line formula (5) or (6) (Art. 151);

(c) By using the straight line formula (7) or (8) (Art. 151).

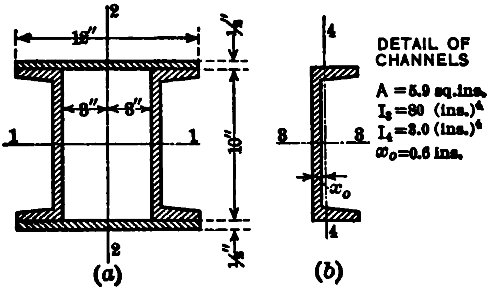

FIG. 219.

Problem 2.

Solve Problem (1), substituting the cross section shown in Fig. (220) for that shown in Fig. (219).

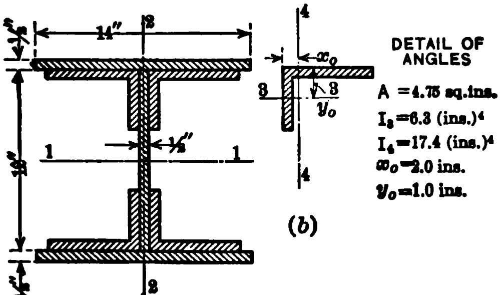

FIG. 220.

Problem 3.

Determine the ultimate strength of the columns given in Problems (1) and (2), assuming that the ends are supported on pins, the axes of which coincide in the principal axis (1-1), and using formula (2) (Art. 149).

Problem 4.

Determine the working load for a column having the cross section shown in Fig. (221) when subjected to an axial load, by use of formula (7) or (8) (Art. 151):

(a) If the column is 8 ft. long;
(b) If the column is 16 ft. long.

Problem 5.

Solve Problem (4), using formula (5) or (6) (Art. 151).

Problem 6.

Determine the ultimate strength of an axially loaded column, 25 ft. long, of the cross section shown Fig. (222), by use of Euler's formulas (Art. 146), using $E = 28{,}000{,}000$ lbs. per sq. in.

(a) Assume column with flat ends;
(b) Assume column with rounded ends.

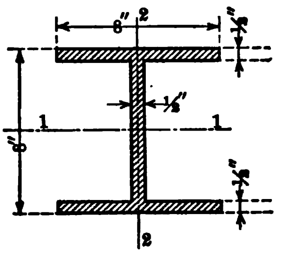

FIG. 221.

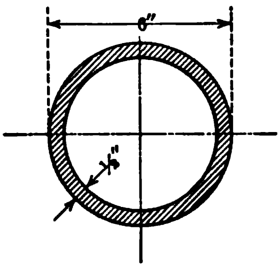

FIG. 222.

Problem 7.

Determine the ultimate strength of an axially loaded column, 40 ft. long, having the cross section shown in Fig. (221), by use of Euler's formula (Art. 146), using $E = 28{,}000{,}000$ lbs. per sq. in.

(a) Assume column with flat ends;
(b) Assume column with rounded ends.

Problem 8.

Determine the working load for an eccentrically loaded column, 25 ft. long, having the cross section shown in Fig. (219), assuming that the resultant load intersects the principal axis (2–2) at a distance 2″ from the center of gravity. Use formula (7) (Art. 151) to obtain the working strength.

Problem 9.

Solve Problem (7), assuming the resultant load acts through the principal axis (1–1) with an eccentricity of 2″.

Problem 10.

Solve Problem (8), substituting a column with the cross section shown in Fig. (220) for that shown in Fig. (219).

Problem 11.

Determine the greatest fiber stress in a column 30 ft. long, having the cross section shown in Fig. (222), when subjected to a load of 10,000 lbs. with an eccentricity of 2″. Assume that the column is supported on pins at each end and use the method of Art. (154), taking $E = 28{,}000{,}000$ lbs. per sq. in.

Problem 12.

Determine the greatest fiber stress in a horizontal strut 30 ft. long, having the cross section shown (Fig. 221), when subjected to a centrally applied force of 10,000 lbs. at each end. Assume that the ends of the strut are hinged in the direction of the axis (1–1), (web vertical), and that the weight of the material is 0.28 lb. per cu. in. Use the method of Art. (153), taking $E = 28{,}000{,}000$ lbs. per sq. in.

CHAPTER X.

SHAFTING AND SPRINGS.

156. Stress Due to Torsion in a Circular Shaft. — If a straight bar, or shaft, of uniform circular section and homogeneous material, is held in equilibrium under the action of two equal couples, of opposite sign, in planes perpendicular to its axis at the ends, the bar will undergo a distortion in shear and every cross section will be subjected to a shearing stress, the intensity of which at any point may be determined in the following manner.

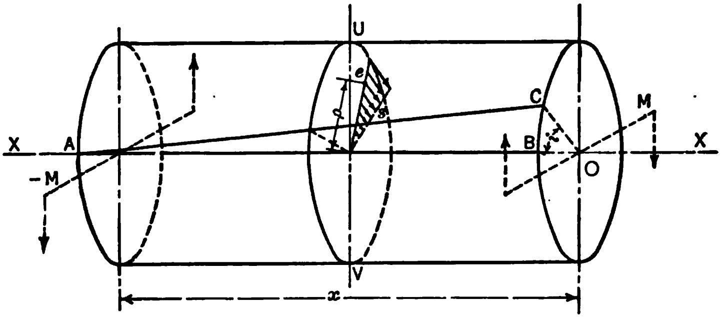

FIG. 223.

Let AB (Fig. 223) represent any line in the surface which is parallel to the axis of the bar before the couples M and $-M$ are applied at the ends. Let the end A be held stationary and the end B be free to rotate. The couples will twist the shaft about its axis so that any plane cross section will undergo a slight rotation, which is proportional to the distance of the section from the fixed end of the shaft, and hence the straight line AB will be distorted into a helix AC. It is assumed that any line of intersection of a cross section and a radial plane remains straight, undergoing an angular displacement only, during the distortion of the bar; the radial line OB, for example, being displaced to the position OC.

Let r = the radius and x = the length of the bar, and let i = the angle BOC. Since the material is homogeneous, the shearing strain at any point in the surface, in the directions of the tangent to the circumference of the cross section and the line parallel to the axis of the bar through any point, will be uniform for the entire bar and the measure of this strain will be the angle between the helix AC and any line in the surface parallel to the axis of the bar; the tangent of this angle evidently being equal to

$$\gamma = \frac{BC}{AB} = \frac{ri}{x}. \qquad (1)$$

The shearing strain, in directions parallel to the above, at any point e, within the bar at a distance ρ from the center, will evidently be equal to

$$\gamma' = \frac{\rho i}{x}. \qquad (2)$$

If G = the modulus of rigidity and the law of proportionality of shearing stress to strain holds for the material, the intensity of the shearing stress on the cross section UV, at the point e, will be equal to

$$s = G\gamma' = \frac{G\rho i}{x} \text{ (Art. 7)}. \qquad (3)$$

The direction of the shearing stress at e will evidently be at right angles to the radius through the point and, since $G\frac{i}{x}$ is a constant for all points in the cross section, the stress intensity at other points in the section will vary as their distances from the center of the shaft.

If we let the constant $G\frac{i}{x} = a$, the expression for the shearing stress intensity at e becomes

$$s = a\rho, \qquad (4)$$

the stress on an area dA at the point e will be equal to

$$s\,dA = a\rho\,dA,$$

and the moment of this stress about the center of the cross section will be equal to

$$\rho s\,dA = a\rho^2\,dA.$$

Therefore, since the resultant of the stress on UV must be a couple, equal in magnitude to the couple M, we shall obtain by integration

$$M = a\int \rho^2\,dA = aI_p, \quad . \; . \; . \; . \; . \; . \; . \quad (5)$$

where I_p = the polar moment of inertia of the cross section about its center. Hence $a = \frac{M}{I_p}$, and substituting this value in (4) we obtain

$$s = M\frac{\rho}{I_p}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

which is the formula for the stress intensity at any point in the cross section.

If we let f_s = the greatest intensity of the shearing stress on the cross section, it is evident that

$$f_s = M\frac{r}{I_p} = \frac{16\,M}{\pi d^3}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

where d = the diameter of the bar.

Twisting Moment, or Torque. — The couple M is called the *twisting moment*, or the *torque*. The equal and opposite couple formed by the shearing stress on the section is called the *moment of resistance in torsion.*

It is evident from the law of equality of shearing stresses on planes at right angles (Art. 24), that shearing stresses will exist on all longitudinal planes containing the axis of the shaft.

The direction of the shearing stress at any point in such a plane will be parallel to the axis, its intensity will be proportional to the distance of the point from the axis and its magnitude will be given by equation (6).

157. Angle of Torsion. — The angular displacement i of a radius at one end of the bar (Fig. 223), relatively to the radius at the other end which was originally in the same plane, is called the *angle of torsion, or the angle of twist in the length AB.* The value of i for any portion of the bar of length equal to x may be readily obtained by equating (3) and (6) (Art. 156) and reducing to the expression

$$i = \frac{Mx}{GI_p} = \frac{32\,Mx}{\pi\,d^4G}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

By eliminating M between this equation and (7) (Art. 156) we obtain an expression for the angle of torsion in terms of the greatest intensity of the shearing stress,

$$i = \frac{f_s x}{Gr}. \qquad (2)$$

Torsional Rigidity. — The quantity $\frac{i}{x}$, or the angle of twist per unit length of the bar, may be called the *twist;* and the ratio of the torque M to the twist, expressed by the formula

$$\frac{Mx}{i} = GI_p, \qquad (3)$$

may be taken as the measure of the torsional rigidity of the bar.

158. Elastic Limit in Torsion. — The relation between the angle of torsion and the torque for a bar of any material is very similar to that between the elongation and the load when the bar is subjected to tension. (Art. 8.)

If a bar, or shaft, of ductile steel is subjected to a gradually increasing twisting moment, the angle of twist will be found to increase very nearly in proportion to the torque until the elastic limit in torsion is reached, after which the rate of increase of the angle of twist will be much greater than that of the torque until fracture results.

If a line is plotted on rectangular coördinates, having values of the torque M for ordinates and corresponding values of the angle of torsion i for abscissæ, it will be similar in form to the diagrams for tension, shown in Figs. (6 and 7). Such a diagram may be called a *load-deformation diagram for torsion.* The line will be very nearly straight for values of M between zero and a value which may be called the *torque at the elastic limit,* and for values of M above this limit the line will be a curve with a continually decreasing angle of slope.

The maximum shearing stress intensity in a round bar at this limit, given by equation (7) (Art. 156), is called *the elastic limit in torsion.* If the twisting couple is removed before the elastic limit is reached the twist will disappear, but if the couple is removed after the elastic limit is reached the bar will retain a permanent twist, or *set in torsion.*

If the material is very ductile a *yield point in torsion* will appear at a torque somewhat higher than the elastic limit, similar to the yield point in tension.

Owing to the fact, however, that the stress intensity at points in the interior of the bar is less than the intensity at the outer layer, the torque at neither the elastic limit nor the yield point in the case of a solid bar is so definitely marked as the load at the elastic limit, or the yield point, for a tension bar. In the case of a hollow tube the analogy between the stress strain relations in torsion and tension is much closer, since the stress intensity in all parts of the tube is very nearly uniform for either tension or torsion.

For other materials than mild steel the load-deformation diagrams in torsion and tension will also be found to be approximately similar in form.

159. Stress Beyond the Elastic Limit in Torsion. — It will be evident from the assumptions in the theory of torsion (Art. 156) that when the stress intensity in a round bar exceeds the elastic limit, the formula

$$f_s = \frac{Mr}{I_p}$$

will cease to give the true value of the maximum shearing stress.

The value of f_s obtained by substituting for M, in the above formula, the twisting moment required to fracture the shaft may be called the *modulus of rupture in torsion*. By applying proper factors of safety this quantity may be used in determining values of the working strength in torsion, similar to determining the working fiber stress for bending from the *transverse modulus of rupture* (Art. 81).

A hardening effect, similar to that produced by applying a stress beyond the elastic limit in tension (Art. 14), will be produced by applying stress beyond the elastic limit in torsion, the modulus of rupture and the elastic limit in torsion being increased and the ductility diminished thereby.

Ductility in torsion. The number of turns per unit of length which can be produced in a bar before fracture may be taken as a measure of the ductility of the material in torsion.

160. Transmission of Power by Shafting. — When a shaft is used to transmit power, the relation between the torque and the power transmitted is expressed by the following simple equation,

$$M = \frac{\text{h.p.} \times 33{,}000 \times 12}{2\pi N}, \quad \ldots \ldots \quad (1)$$

where M = the torque, expressed in in. lbs., h.p. = the horse power, N = the number of revolutions of the shaft per minute. Substituting this value of M in equation (7) (Art. 156) we obtain

$$f_s = \frac{12 \times 33{,}000 \times \text{h.p.} \times r}{2\pi N I_p}. \quad \ldots \ldots \quad (2)$$

If the shaft is solid and d = its diameter, equation (2) reduces to

$$f_s = \frac{3{,}168{,}000 \text{ h.p.}}{\pi^2 N d^3}, \quad \ldots \ldots \ldots \quad (3)$$

from which

$$d = \sqrt[3]{\frac{3{,}168{,}000 \text{ h.p.}}{f_s \pi^2 N}} = K \sqrt[3]{\frac{\text{h.p.}}{N}}, \quad \ldots \ldots \quad (4)$$

where

$$K = \sqrt[3]{\frac{3{,}168{,}000}{\pi^2 f_s}} = \frac{68.47}{\sqrt[3]{f_s}}. \quad \ldots \ldots \ldots \quad (5)$$

Equation (4) gives in simple terms the diameter of a shaft required to transmit a given amount of power, at a given speed in revolutions per minute, when the value of the constant K for the material in the shaft has been determined.

161. Hollow Circular Shafts. — The assumptions made in the theory and the formulas deduced thereby for determining the intensity of the shearing stress and the angle of torsion will apply equally as well to a hollow shaft as to the solid bar. If r_1 = the inside radius and r = the outside radius of a hollow shaft, the polar moment of inertia will be given by the formula

$$I_p = \frac{\pi}{2}(r^4 - r_1^4),$$

and by substituting its value in equations (7) (Art. 156) and (1) (Art. 157) the values of the maximum shearing stress intensity and the angle of torsion for any given torque are obtained.

For any maximum shearing stress intensity the ratio of the moment of resistance in torsion of a hollow shaft to that of a solid shaft of the same outside diameter will be equal to the ratio of the two polar moments of inertia; for, if M and M' are the moments of resistance in torsion and I_p and I_p' are the polar moments of inertia of the hollow and solid shafts, respectively,

$$M = \frac{f I_p}{r} = \frac{f\pi}{2r}(r^4 - r_1^4) \quad \ldots \ldots \ldots \quad (1)$$

and

$$M' = \frac{f I_p'}{r} = \frac{f\pi}{2r} r^4, \quad \ldots \ldots \ldots \ldots \quad (2)$$

and hence

$$\frac{M}{M'} = \frac{I_p}{I_p'} = \frac{r^4 - r_1^4}{r^4} = 1 - \left(\frac{r_1}{r}\right)^4. \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

The above equation will also represent the ratio of the torsional rigidity of the hollow shaft to that of the solid one.

The weights of any given length of the two shafts will be proportioned to the areas of the two cross sections and, if W and W' are the weights of the hollow and solid shafts, respectively,

$$\frac{W}{W'} = \frac{r^2 - r_1^2}{r^2} = 1 - \left(\frac{r_1}{r}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

162. Torsion in Bars of Non-Circular Section. — When a straight bar, which is not circular in section, is subjected to torsion by the application of couples at the ends, the cross sections do not remain plane, as in the round bar (Art. 156), but are distorted into curved surfaces. The intensities of the shearing stress at all points in a cross section are not proportional to their distances from the axis of the bar, the points of maximum intensity being the points in the perimeter of the section which are nearest the axis of the bar.

Saint-Venant was the first to deduce correct expressions for the intensity of the shearing stress due to torsion and the angle of torsion in prismatic bars of non-circular cross section. The theory is difficult, however, and the resulting formulas only will be quoted here. Previous to Saint-Venant's work it had been customary to use the formulas for the round bar (Arts. 156–157) for determining the maximum stress intensity and the angle of torsion in a bar of any other shape; substituting for I the polar moment of inertia of the cross section, about its center of gravity, and for r, the distance from the center of gravity to the most distant point in the perimeter.

The magnitude of the error in the results obtained in this way may be readily found by comparison with the results given by the use of the following equations.

In each case the notation previously adopted is followed, namely:

x = the length of the bar;
M = the twisting moment, or torque;
f_s = the greatest intensity of shearing stress;
i = the angle of torsion in the length x;
G = the modulus of rigidity.

Elliptical Cross Section. — Let $2a$ = the major axis and $2b$ = the minor axis of the ellipse.

$$f_s = \frac{2M}{\pi ab^2}; \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$i = \frac{(a^2 + b^2)}{\pi a^3 b^3} \frac{Mx}{G}, \quad \ldots \ldots \ldots \quad (2)$$

or,

$$i = \frac{4\pi^2 I_p}{A^4} \frac{Mx}{G}, \quad \ldots \ldots \ldots \ldots \quad (3)$$

where A = the area and I_p = the polar moment of inertia of the ellipse about its center.

The maximum stress intensity given by equation (1) occurs at the ends of the minor axis b; and it may be noted that if a and b are interchanged and made to represent the minor and major axes, respectively, equation (1) will give the stress intensity at the ends of the major axis. It will be observed that the value of i given by (2) or (3) will be the same in either case.

When $a = b$ the formulas reduce to the equations for the bar of circular section (Arts. 156–157).

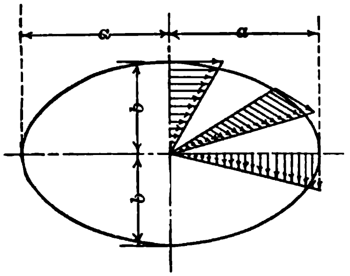

FIG. 224.

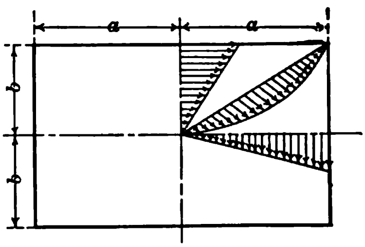

FIG. 225.

The stress intensities at points on any radius vector of the ellipse are directly proportional to the distances of the points from the center, and the direction of the stress at any point is parallel to the tangent to the ellipse at the end of the radius vector (Fig. 224).

Rectangular Cross Section. — Let $2a$ = the length of the longer side and $2b$ = the length of the shorter side of the rectangle.

The theoretical equations for determining the maximum intensity of stress and the angle of torsion are very complex: but

Saint-Venant suggested the use of the following empirical equations which give results for all ratios of $\frac{a}{b}$ with an error of less than *four per cent*, in comparison with the purely theoretical values.*

$$f_s = \frac{M\,(15\,a + 9\,b)}{40\,a^2b^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (4)$$

$$i = \frac{1}{\left[\frac{16}{3} - 3.36\frac{b}{a}\left\{1 - \frac{1}{12}\left(\frac{b}{a}\right)^4\right\}\right]ab^3}\,\frac{Mx}{G} \; . \; . \; . \qquad (5)$$

Saint-Venant also found that the equation for the angle of torsion in the bar of elliptical section (equation 3) might be used to determine approximately the angle of torsion in a bar of rectangular cross section. If we substitute 40 as being nearly equal to $4\,\pi^2$ equation (3) will become

$$i = \frac{40\,I_p}{A^4}\,\frac{Mx}{G} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (6)$$

To determine the angle of torsion substitute for A the area and for I_p the polar moment of inertia of the rectangle about its center.

The error in the results obtained from (6) is small when $a > 2\,b$. When $a < 2\,b$ more accurate results can be obtained by using 42 instead of 40 for the constant.

The maximum intensity of the shearing stress (equation 4) occurs at the middle point of the side $2\,a$. The variation of the stress intensity along the principal axes and the diagonal is indicated in Fig. (225).

Square Cross Section. — The equations for the bar of rectangular section will apply to a bar of square cross section. We will let $2\,a$ = the side of the square. Then by putting $b = a$ in equations (4) and (5) and reducing we obtain

$$f_s = \frac{3\,M}{5\,a^3} = 0.6\frac{M}{a^3} \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (7)$$

and

$$i = \frac{1}{2.25\,a^4}\,\frac{Mx}{G} = \frac{0.4444}{a^4}\,\frac{Mx}{G} \; . \; . \; . \; . \; . \; . \qquad (8)$$

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* A History of the Theory of Elasticity and of the Strength of Materials — Todhunter and Pearson.

The value of i may be obtained with a comparatively small error by using equation (6), substituting 42 for 40.

Torsional Rigidity. — Formulas for the torsional rigidity of each of the foregoing sections can be readily obtained by solving equations (2) or (3), (5) or (6), and (8) or (6) as modified, for the value of $\frac{Mx}{i}$ (Art. 157).

It will be observed that for each of the sections considered the formula for the angle of twist has the form

$$i = \frac{Mx}{GC}, \qquad (9)$$

where C = a constant depending on the shape of the section, and that expression for the torsional rigidity will take the form

$$\frac{Mx}{i} = GC. \qquad (10)$$

Equations (9) and (10) are in the same form as the equations in Art. (157), the constant C replacing the constant I_p in the expressions for the angle of torsion and for the torsional rigidity in the circular shaft.

Summary of Values of C. —

$\frac{\pi a^3 b^3}{a^2 + b^2} = \frac{A^4}{4\pi^2 I_p}$. (Elliptical section, $2a$ = major axis, $2b$ = minor axis.)

$\frac{\pi d^4}{32} = \frac{A^4}{4\pi^2 I_p}$. (Circular section, d = diameter.)

$\left[\frac{16}{3} - 3.36\frac{b}{a}\left\{1 - \frac{1}{12}\left(\frac{b}{a}\right)^4\right\}\right]ab^3 = \frac{A^4}{40 I_p}$ (nearly). (Rectangular section, $2a$ = long dimension, $2b$ = short dimension.)

$2.25a^4 = \frac{A^4}{42 I_p}$ (nearly). (Square section, $2a$ = side of square.)

In each case A = area of cross section, I_p = polar moment of inertia about center of section.

163. Resilience in Torsion. — A bar under stress in torsion will evidently possess a certain amount of *strain energy*, or *resilience in torsion* (Art. 15). If the material is elastic and follows the law of proportionality of stress intensity to strain stated in

Art. (156), the resilience of a round bar of length x, following the notation previously adopted, will evidently be equal to

$$R = \frac{M}{2} i = \frac{M^2 x}{2 G I_p} = \frac{f_s^2 I_p x}{2 G r^2}. \quad \ldots \ldots \quad (1)$$

If the bar is solid, equation (1) reduces to the form

$$R = \frac{f_s^2 \pi r^2 x}{4 G} = \frac{f_s^2 V}{4 G}, \quad \ldots \ldots \quad (2)$$

where V = the volume of the bar.

Similarly, the torsional resilience of a bar of elliptical section (Art. 162) will be equal to

$$R = \frac{M}{2} i = \frac{(a^2 + b^2)}{\pi a^3 b^3} \frac{M^2 x}{2 G} = \frac{(a^2 + b^2)}{8 a^2} \frac{f_s^2 V}{G}; \quad \ldots \quad (3)$$

and, approximately, that of the bar of rectangular section (Art. 162) will be equal to

$$R = \frac{M}{2} i = \frac{40 I_p}{A^4} \frac{M^2 x}{2 G} = \frac{125 (a^2 + b^2)}{3 (15 a + 9 b^2)} \frac{f_s^2 V}{G} \quad \ldots \quad (4)$$

and that of the square bar (Art. 162) will be equal to

$$R = \frac{M}{2} i = \frac{0.4444}{a^4} \frac{M^2 x}{2 G} = 0.154 \frac{f_s^2 V}{G}. \quad \ldots \ldots \quad (5)$$

164. Combined Torsion and Bending. — In the preceding Articles the stresses and strains produced in straight bars, or shafts, by the action of twisting couples only, have been considered. Usually the stress in a shaft is affected by bending, due to its own weight, the weights and pulls of pulleys and belts, the thrust of cranks, etc., and sometimes by an end thrust in the direction of the axis.

When the transverse forces are known the bending moment at any cross section can be found by the method in Art. (129) and, when the shaft is circular, the neutral axis of the bending stress will be perpendicular to the plane of the resultant bending couple acting at the section.

We will let M_b = the resultant bending moment and M_t = the twisting moment at any cross section of a circular bar, r = the radius, I = the moment of inertia of the section about its diameter and I_p = its polar moment of inertia about the center.

The stress intensity at any point in the section will be the resultant of a normal component due to bending,

$$f = \frac{M_b y}{I} \text{ (Art. 69)}, \quad \ldots \ldots \quad (1)$$

and two shearing components, namely: the component due to torsion,

$$s = \frac{M_t \rho}{I_p} = \frac{M_t \rho}{2\,I} \text{ (Art. 156)}, \quad \ldots \ldots \quad (2)$$

in the direction perpendicular to the radius through the point, and the component due to bending,

$$s_1 = \frac{SQ}{bI} \text{ (Art. 89)}, \quad \ldots \ldots \quad (3)$$

in the direction perpendicular to the neutral axis. At points farthest from the neutral axis equation (1) becomes

$$f = \pm \frac{M_b r}{I}, \quad \ldots \ldots \quad (4)$$

while at these points equation (3) becomes

$$s_1 = 0. \quad \ldots \ldots \quad (5)$$

At any point on the circumference of the shaft the shearing stress intensity, due to torsion, becomes

$$s = \frac{M_t r}{I_p} = \frac{M_t r}{2\,I}, \quad \ldots \ldots \quad (6)$$

acting in the direction of the tangent to the circle at that point.

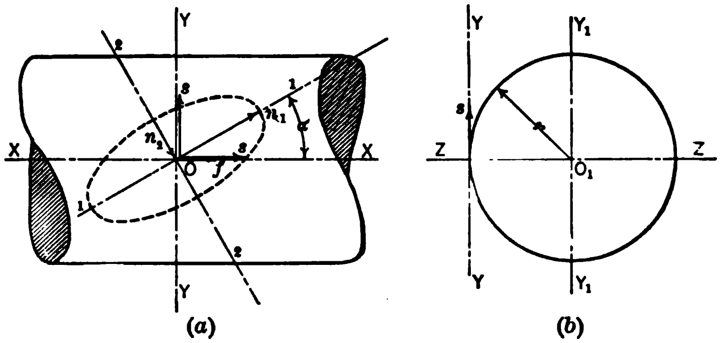
Hence, if we let O (Fig. 226a) represent the point in the cross section YOY at which the tensile stress intensity due to bending is a maximum, the stress intensity on the X plane through this point will be the resultant of a normal component f (equation 4) and a shearing component s (equation 6). The axis $Y_1O_1Y_1$ (Fig. 226b) will be the neutral axis of the bending stress. On the Y plane through O there will be a shearing stress only, of intensity s (Art. 24), parallel to the axis of the shaft.

At the point O, therefore, we have a state of plane stress (Art. 23) and the resultant stresses at O on all planes will be parallel to the Z plane. Let $O1$ and $O2$ represent the normals to the principal planes of stress through O (Art. 28) and α = the angle $XO1$. We shall then have

$$\tan 2\,\alpha = \frac{2\,s}{f} \text{ (Art. 27)} \quad \ldots \ldots \quad (7)$$

and, substituting the values of f and s from equations (4) and (6) and reducing,

$$\tan 2\alpha = \frac{M_t}{M_b}. \quad \dots\dots\dots \quad (8)$$



(a) (b)

FIG. 226.

It is evident from (8) that α will have values from 0° to ±45° for all possible values of M_t and M_b: the X and Y planes being the principal planes of stress when $M_t = 0$; and the principal planes of stress making angles of ±45° with the X and Y planes when $M_b = 0$.

If we let n_1 and n_2 represent the principal stress intensities we shall have

$$n_1 = \frac{f}{2} + \frac{1}{2}\sqrt{f^2 + 4s^2} \quad \dots\dots\dots \quad (9)$$

and

$$n_2 = \frac{f}{2} - \frac{1}{2}\sqrt{f^2 + 4s^2} \text{ (Art. 32).} \quad \dots\dots \quad (10)$$

It is evident that the foregoing analysis will apply equally as well to the point on the circumference, diametrically opposite the point O, at which the compressive stress intensity due to bending is a maximum; and, by introducing plus and minus signs to indicate tension and compression, respectively, equations (9) and (10) will represent the principal stresses at either point.

The ellipse of stress for the point O is indicated (Fig. 226a) and it should be observed that for all values of f and s the sign of n_2 will be opposite to that of n_1.

Substituting in (9) the values of f and s from (4) and (6), we obtain

$$n_1 = \frac{M_b r}{2I} + \frac{1}{2}\sqrt{\left(\frac{M_b r}{I}\right)^2 + 4\left(\frac{M_t r}{2I}\right)^2}$$

$$= \frac{r}{2I}\left[M_b + \sqrt{M_b^2 + M_t^2}\right] = \frac{16}{\pi d^3}\left[M_b + \sqrt{M_b^2 + M_t^2}\right], \quad . \quad (11)$$

where d = the diameter of the shaft.

Similarly,

$$n_2 = \frac{r}{2I}\left[M_b - \sqrt{M_b^2 + M_t^2}\right] = \frac{16}{\pi d^3}\left[M_b - \sqrt{M_b^2 + M_t^2}\right]. \quad . \quad (12)$$

It becomes evident that the greatest stress intensity in the entire shaft will occur at a point in the cross section at which the quantity

$$\tfrac{1}{2}\left[M_b + \sqrt{M_b^2 + M_t^2}\right]$$

is the greatest. This quantity is sometimes called the equivalent bending moment, since it is equal to the bending moment which would produce a maximum stress intensity equal to n_1 on a cross section perpendicular to the axis of the shaft. It should be remembered, however, that n_1 is actually the stress intensity on an oblique section through the shaft.

The greatest intensity of shearing stress in the shaft will occur on planes at 45° with the principal planes at O (Art. 31) and will be equal to

$$s_0 = \frac{n_1 - n_2}{2} = \frac{r}{2I}\sqrt{M_b^2 + M_t^2} = \frac{16}{\pi d^3}\sqrt{M_b^2 + M_t^2}. \quad . \quad (13)$$

Hollow Shaft. — The principal stress intensities and the maximum shearing stress intensity in a hollow shaft, subject to combined twisting and bending, can be readily obtained by substituting for I in the general formulas (11), (12) and (13), its value in terms of r, the outside radius and r_1, the inside radius of the shaft, namely,

$$I = \frac{\pi}{4}(r^4 - r_1^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Principal Strains. — Having the principal stresses, the principal strains at O are given by the equations

$$e_1 = \frac{n_1}{E} - \frac{n_2}{mE}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$e_2 = \frac{n_2}{E} - \frac{n_1}{mE}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$e_3 = -\frac{n_1}{mE} - \frac{n_2}{mE} \text{ (Art. 48)}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

By transposing E and substituting for n_1 and n_2 the values expressed by (11) and (12), equation (15) becomes

$$Ee_1 = \frac{r}{2I}\left[(M_b + \sqrt{M_b^2 + M_t^2}) - \frac{1}{m}(M_b - \sqrt{M_b^2 + M_t^2})\right]$$

$$= \frac{r}{2I}\left[\frac{m-1}{m} M_b + \frac{m+1}{m}\sqrt{M_b^2 + M_t^2}\right]. \quad . \quad . \quad . \quad . \quad (18)$$

Equation (18) represents the product of the greatest extension at O and the modulus of elasticity. For an iron or steel shaft it is usually quoted in the simplified form, obtained by substituting for m an approximate value 4,* in which case

$$Ee_1 = \frac{r}{8I}[3\,M_b + 5\sqrt{M_b^2 + M_t^2}]. \quad . \quad . \quad . \quad . \quad (19)$$

It should be observed that the product Ee_1 is not equal to a stress intensity, as would be the case if the stress at O were a simple tension or compression, its value in the case of shafting being always greater than n_1. The formula is frequently used in place of the formula for greatest stress intensity (equation 11) in the design of shafting.

In a similar manner the value of the product

$$Ee_2 = \frac{r}{8I}[3\,M_b - 5\sqrt{M_b^2 + M_t^2}] \quad . \quad . \quad . \quad . \quad (20)$$

is readily obtained.

Lateral Deflection. — The lateral deflection in a line of shafting when the bending is in one plane only can be easily estimated by use of the deflection formulas in the common beam theory. When the bending is not in one plane, plots of the lateral deflections, due to the component bending moments in two planes at right angles, can be made and from these the maximum resultant deflection in the shaft can be estimated. In a rotating shaft the maximum lateral deflection should not exceed a certain limit, even when the greatest stress intensity is within the working strength of the material.

A rule, frequently quoted, is that the greatest lateral deflection of a shaft due to the transverse loads upon it shall not exceed $\frac{1}{100}$ inch per foot of span.

165. Combined Torsion and End Pressure, or Tension. — A shaft may be subjected to torsion combined with a thrust, or a pull, such as the thrust of a propeller, or the pressure, or tension, due to a weight on a vertical shaft. In such a case the stress on any cross section will be the resultant of a uniform normal stress and the shearing stress due to torsion.

---

* A more nearly correct value in the case of steel is $m = 3.6$.

If the shaft is circular in section and we let P = the axial thrust, or pull, A = the area of the cross section, and follow otherwise the previous notation, the principal stress intensities at any point in the circumference of a given cross section can be found by substituting

$$f = \frac{P}{A} \quad \dots\dots\dots\dots\dots \quad (1)$$

and

$$s = \frac{M_t r}{2I} = \frac{2M_t}{Ar} \quad \dots\dots\dots \quad (2)$$

in equations (9) and (10) (Art. 164), which will give

$$n_1 = \frac{P}{2A} + \frac{1}{2}\sqrt{\left(\frac{P}{A}\right)^2 + 4\left(\frac{2M_t}{Ar}\right)^2}$$

$$= \frac{1}{2A}\left[P + \sqrt{P^2 + \left(\frac{4M_t}{r}\right)^2}\right] \quad \dots\dots \quad (3)$$

and

$$n_2 = \frac{1}{2A}\left[P - \sqrt{P^2 + \left(\frac{4M_t}{r}\right)^2}\right] \dots\dots \quad (4)$$

The angle α between the normals $O1$ and OX can be found from the equation

$$\tan 2\alpha = \frac{2s}{f} = \frac{4M_t}{Pr} \dots\dots\dots\dots\dots \quad (5)$$

The principal strains can be found by substituting the above values of n_1 and n_2 in equations (15) and (16) (Art. 164).

If we let $m = 4$, the greater of the products of the principal strains and the modulus of elasticity will be equal to

$$Ee_1 = \frac{1}{8A}\left[3P + 5\sqrt{P^2 + \left(\frac{4M_t}{r}\right)^2}\right]. \quad \dots\dots \quad (6)$$

If both the torque and the thrust are uniform throughout the length of the shaft, the principal stresses and strains for every point in the surface will evidently be the same.

To determine magnitudes only, P may evidently be taken positive for either tension or compression. To determine the direction of the principal planes at any point the proper signs for tension, or compression, should be used.

166. Combined Torsion, Bending and End Pressure, or Tension. — When the cross section of the bar is circular, if we follow the notation previously adopted, the principal stress intensities at any point in the surface of the bar, which lies in the plane of bending, can be found by substituting the values of the maximum normal and shearing stress intensities on the cross section through the point, which will evidently be equal to

$$f = \frac{P}{A} + \frac{M_b r}{I} \quad \ldots\ldots\ldots \quad (1)$$

and

$$s = \frac{M_t r}{2I}, \quad \ldots\ldots\ldots\ldots \quad (2)$$

in equations (9) and (10) (Art. 164).

The directions of the principal planes can be found by substituting the values of f and s in equation (7) and the principal strains can be found by substituting the values of the principal stresses in equations (15) and (16) (Art. 164).

The solution will be simplified by solving (1) and (2), calling a tensile stress plus and a compressive stress minus, and substituting the numerical values in the above-mentioned equations for principal stresses and strains.

Non-circular Sections. — If a bar of elliptical, or rectangular, cross section is subjected to torsion combined with bending, or an end thrust, or with both together, the principal stress intensities may be estimated by the method indicated above for the round bar.

The intensity of the normal stress f at the end of either principal axis of the section, due to the bending and end thrust, can be found by the method of Art. (129). The intensity of the shearing stress s at either of these points, due to twisting, can be found by use of the equations in Art. (162). By substituting the values of f and s, determined in this manner, in equations (9) and (10) (Art. 164) the principal stress intensities at the points on the surface of the bar, which are located at the ends of either principal axis of a cross section, can be obtained.

Unless the bending is in a plane of symmetry, however, the value of the maximum principal stress intensity, obtained by the above method, is not necessarily the greatest stress intensity in the bar; since the greatest intensity of the normal stress f, when the bending is not in a plane of symmetry, is located at a point which is not on a principal axis of the cross section (Art. 129). It is possible, therefore, that in such a case the greatest principal stress intensity may be located at some other point than that at which the shearing stress intensity due to torsion is a maximum.

167. Helical Spring Subjected to an Axial Load. — A helical, or coiled, spring is formed by wrapping a rod of uniform section around a cylinder, maintaining a fixed distance between the coils so that the central line, or axis, of the rod forms a helix. For convenience, we shall call the rod, or wire, forming the spring, the *wire* simply and will consider at first the spring made up of a solid round wire only.

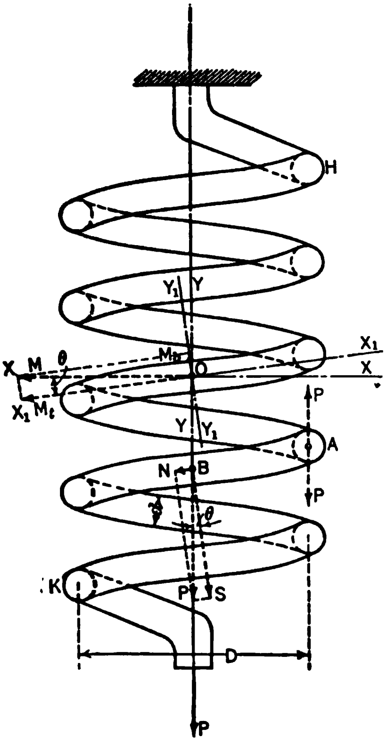


Fig. 227.

The *axis of the coil* is the axis of the cylinder on which the helix, formed by the central line or axis of the wire, lies and the diameter of this cylinder is called the *diameter of the coil.*

We shall let Fig. (227) represent such a spring, which is formed at the ends in such a manner that one end of the axis of the coil can be fixed in position and a load acting along the axis can be applied at the free end. In the following analysis, the helical portion only will be considered, the ends of the spring beyond the sections H and K in the sketch being assumed to be rigid.

Let D = the diameter of the coil, d = the diameter of the wire, I_p = the polar moment of inertia of the cross section of the wire about its center, n = the number of turns in the helix, l = the total length of the helix, θ = the angle of inclination of the helix with any plane at right angles to the axis of the coil, and P = the resultant axial load, which is represented as causing an extension of the coil, although it will be evident that the following analysis would apply equally well if the load acted in the opposite direction, compressing the spring.

Under the action of the load P, the stress on every cross section of the wire in the coil will evidently be the same and the axis of the wire will form a helix, which will have a different slope and will be found to lie on a cylinder having a slightly different diameter from that of the original helix.

The stress on any cross section of the wire, taken normal to the helix, such as the section at A for example, will be the resultant of a stress due to a couple, which is equal to

$$M = \frac{PD}{2}. \qquad (1)$$

and a stress due to a force acting through the center of the section, which is equal and parallel to the axial load P.

The couple M can be resolved into a couple

$$M_t = M \cos \theta, \qquad (2)$$

causing a twist in the wire about an axis OX_1, tangent to the helix, and a couple

$$M_b = M \sin \theta, \qquad (3)$$

causing a bending of the wire about an axis OY_1, perpendicular to OX_1.

Since the stress on every cross section of the wire is the same, the moment axes of the resultant couple M and its components M_t and M_b are shown, for the sake of clearness, at the cross section through O. It will be observed that these moment axes lie in the vertical plane, which is tangent to the cylinder on which the helix, formed by the axis of the wire, is located.

The component force P, acting through the center of the cross section, can be resolved into a normal component,

$$N = P \sin \theta, \qquad (4)$$

and a shearing component,

$$S = P \cos \theta. \qquad (5)$$

For the sake of clearness the resolution is indicated at the cross section through B.

Close Coiled Spring.—When the distance between the coils of the spring is small, the angle θ is so small that M_b is negligible and $M_t = M$ (very nearly).

Likewise N is negligible and $S = P$ (very nearly). For a coiled spring of ordinary proportions, the stress and the distortion due to the component S is comparatively small and it is customary to neglect this component in deducing the formulas for the stress intensity and for the distortion in the coil.

Hence for this case the maximum stress intensity in the wire is very nearly the same as that in a straight rod of the same cross section, subjected to a torque M_t (Art. 156), and will be expressed by the equation

$$f_s = \frac{M_t r}{I_p} = \frac{PD}{2}\,\frac{16}{\pi d^3} = \frac{8\,PD}{\pi d^3}; \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

and the angle of torsion for the entire length of the wire in the coil will be very nearly equal to that for a straight rod of the same dimensions, subjected to a torque M_t (Art. 157), and will be represented by the equation

$$i = \frac{M_t l}{GI_p} = \frac{2 f_s l}{dG}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

The total vertical displacement of the free end of the spring, or the extension of the coil, will be equal to

$$\delta = \frac{D}{2} i, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

evidently being the same as the displacement of the end of an arm of length $\frac{D}{2}$, attached to the end of a straight rod of the same length and diameter and subjected to the same uniform twisting moment as the wire in the coil.

Combining (8) and (7), substituting the value of I_p and reducing, we obtain

$$\delta = \frac{8PD^2 l}{\pi\, d^4 G} = \frac{f_s D l}{dG}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

where $l = n\pi D$ (very nearly).

It is evident from (9) and (6) that both the extension of the coil and maximum stress intensity are directly proportional to the axial load.

Hence the resilience of the coil will be represented by the expression

$$R = \frac{P}{2}\,\delta = \frac{4\,P^2 D^2 l}{\pi\, d^4 G} = \frac{M_t^2 l}{2\,GI_p}, \quad . \; . \; . \; . \; . \; . \; . \quad (10)$$

or, by substituting the value of P obtained from (6) and the value of δ from (9),

$$R = \frac{f_s \pi d^3}{16\,D}\,\frac{f_s D l}{dG} = \frac{f_s^2 \pi\, d^2 l}{16\,G} = \frac{f_s^2}{4\,G} V, \quad . \; . \; . \; . \; . \; . \; . \quad (11)$$

where V = the volume of the wire in the coil. The above formula is the same as equation (2) (Art. 163). Evidently the value of δ (equation 9) could have been obtained by solving the following equation between the work done by the load during the displacement of the end of the coil and the resilience of the wire in the coil, as given in the above-mentioned article;

$$\frac{P}{2}\,\delta = \frac{f_s^2}{4\,G} V. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

Open Coiled Spring. — In this case, the value of θ is so large that the bending moment M_b will have an appreciable effect on stress and the distortion in the coil. The stress and the distortion due the forces N and S will be comparatively small and, as in the previous case, it is customary to neglect the effect of these components.

On any cross section of the wire there will be a shearing stress due to torsion, the maximum intensity of which is

$$f_s = \frac{M_t r}{I_p} = \frac{8\,PD\cos\theta}{\pi d^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where $M_t = \frac{PD}{2}\cos\theta$ (equation 2), and a stress due to bending, the maximum intensity of which is

$$f = \frac{M_b r}{I} = \frac{16\,PD\sin\theta}{\pi d^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $\qquad M_b = \frac{PD}{2}\sin\theta$ (equation 3) and $I = \frac{I_p}{2} = \frac{\pi d^4}{64}\cdot$

The moment axis M_b (Fig. 227), of the couple producing the bending stress, coincides with the line of intersection of the cross section and the vertical plane through its center. Hence, the points of maximum stress intensity are at the ends of the horizontal diameter, and the stress is tension at the inside and compression at the outside of the coil.

Substituting in equation (9) (Art. 164) and reducing, we obtain for the greatest principal stress intensity,

$$n_1 = \frac{r}{2I}\left[M_b + \sqrt{M_b^2 + M_t^2}\right]$$
$$= \frac{8\,PD}{\pi d^3}\left[\sin\theta + \sqrt{\sin^2\theta + \cos^2\theta}\right] = \frac{8\,PD}{\pi d^3}(\sin\theta + 1), \quad . \quad . \quad . \quad (15)$$

which reduces to the form of equation (6) when $\theta = 0$.

Similarly, from (10) (Art. 164) we obtain

$$n_2 = \frac{8\,PD}{\pi d^3}(\sin\theta - 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The greatest shearing stress intensity will be equal to

$$s_0 = \frac{n_1 - n_2}{2} = \frac{8\,PD}{\pi d^3} \text{ (Art. 31)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

which is the same as the intensity on the cross section normal to the axis of the wire in the closely coiled spring.

The bending moment M_b will produce a change in the curvature of the wire in the coil and, if we assume that the relation between the change in curvature and the bending moment is the same as for a straight bar of the same cross section, we shall have

$$\frac{1}{r} - \frac{1}{r_1} = \frac{M_b}{EI} \text{ (Art. 97)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where r_1 = the initial radius of curvature and r = the radius of curvature after bending. This equation is nearly correct when r_1 is several times as large as the radius of the wire.

If we let di_1 represent the change, due to the bending, in the slope of the tangent at one end, relatively to that at the other, for a very short length of the wire $dl = r_1\,d\alpha$ (Fig. 228), we shall have

$$\frac{1}{r_1} = \frac{d\alpha}{dl} \quad \text{and} \quad \frac{1}{r} = \frac{d\alpha + di_1}{dl};$$

and hence equation (18) will take the same form as that for the straight beam, namely,

$$\frac{1}{r} - \frac{1}{r_1} = \frac{di_1}{dl} = \frac{M_b}{EI}, \quad \ldots\ldots\ldots\ldots \quad (19)$$

where M_b is called positive when the bending tends to increase the curvature.

A positive bending moment, therefore, will decrease the diameter of the coil and increase the number of turns which, when the coil is right-handed as shown in Fig. (227), will result in a right-handed rotation of the free end about its axis, as seen from the fixed end of the coil. The reverse will evidently be true when the bending moment is negative. Owing to the slope of the helix, the bending couple will also produce a change in the length of the coil.

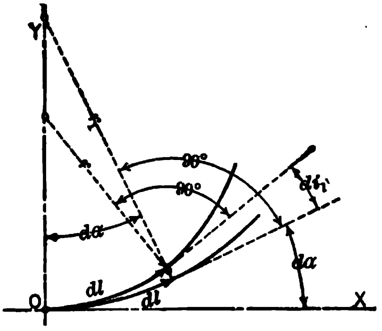


FIG. 228.

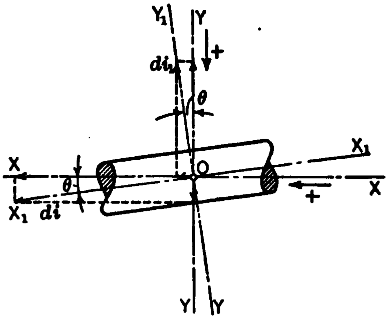


FIG. 229.

The twisting couple M_t will produce a twist in the wire, which will result in a rotation of the free end of the coil about the axis, in addition to an elongation of the coil. If di = the angle of torsion in the length of wire dl,

$$\frac{di}{dl} = \frac{M_t}{GI_p} \quad \text{(Art. 157)}. \quad \ldots\ldots\ldots\ldots \quad (20)$$

The total rotation and the total axial displacement at the free end of the coil, due to the bending and twisting couples combined, can be determined in the following manner. For the sake of clearness the axes OX, OY and OX_1, OY_1 (Fig. 227) are reproduced in Fig. (229). The rotations di and di_1 about the axes OX_1 and OY_1 can be resolved into components about the axes OX and OY and by adding these components the resultant rotations about OX and OY in a length dl of the wire can be found.

Let $d\phi$ = the resultant rotation in the length dl about the axis of OY and

$d\omega$ = the resultant rotation in this length about the axis OX. Taking the positive directions as indicated (Fig. 229),

$$d\phi = \sin\theta\, di - \cos\theta\, di_1 \quad \ldots\ldots\ldots\ldots \quad (21)$$

and

$$d\omega = \cos\theta\, di + \sin\theta\, di_1. \quad \ldots\ldots\ldots\ldots \quad (22)$$

Substituting the values of di and di_1 from (19) and (20),

$$d\phi = \frac{M_t \sin\theta}{GI_p}\, dl - \frac{M_b \cos\theta}{EI}\, dl \quad \ldots\ldots\ldots \quad (23)$$

and

$$d\omega = \frac{M_t \cos\theta}{GI_p}\, dl + \frac{M_b \sin\theta}{EI}\, dl. \quad \ldots\ldots\ldots \quad (24)$$

Since these component rotations are uniform for the entire length of the spring, the rotation of the free end about the axis of the coil, if we substitute the values of M_t and M_b from (2) and (3) and reduce, will be equal to

$$\begin{aligned} \phi &= l\left(\frac{M_t \sin\theta}{GI_p} - \frac{M_b \cos\theta}{EI}\right) \\ &= Ml \sin\theta \cos\theta \left(\frac{1}{GI_p} - \frac{1}{EI}\right) \\ &= \frac{8\,PDl \sin 2\theta}{\pi d^4}\left(\frac{1}{G} - \frac{2}{E}\right) = \frac{8\,PDl \sin 2\theta}{5\,\pi\, d^4 G}, \quad \ldots\ldots \quad (25) \end{aligned}$$

if we substitute $E = \frac{5}{2} G$ (Art. 7).

Hence, for the above relation between E and G, the rotation at the free end of the spring is right-handed as seen from the fixed end and, for any given load P and length of wire l, the rotation is a maximum when $\theta = 45°$ and approaches zero as θ approaches zero.

In a similar manner, the component rotation ω for the entire coil becomes

$$\omega = l\left(\frac{M_t \cos\theta}{GI_p} + \frac{M_b \sin\theta}{EI}\right), \quad \ldots\ldots\ldots \quad (26)$$

and hence, for the axial displacement of the free end, we obtain

$$\begin{aligned} \delta &= \frac{D}{2}\omega = \frac{Dl}{2}\left(\frac{M_t \cos\theta}{GI_p} + \frac{M_b \sin\theta}{EI}\right) = \frac{8\,PD^2 l}{\pi d^4}\left(\frac{\cos^2\theta}{G} + \frac{2\sin^2\theta}{E}\right) \\ &= \frac{8\,PD^2 l}{\pi\, d^4 G}\left(1 - \frac{1}{5}\sin^2\theta\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (27) \end{aligned}$$

when $E = \frac{5}{2} G$. The value of l in the preceding formulas will evidently be equal to

$$l = n\pi D \sec\theta. \quad \ldots\ldots\ldots\ldots\ldots \quad (28)$$

It becomes evident from (27) that when $E = \frac{5}{2} G$ the extension of the spring for any given length of wire l and load P decreases as θ increases and that the extension approaches the value given by equation (9) as θ approaches zero.

Equation (27) might have been obtained by equating the work done by the load P during the displacement to the resilience due to the torsion produced by the couple M_t (equation 10), plus the resilience due to bending under the uniform bending moment M_b (Art. 109, equation 8) and solving for δ; the strain energy equation being

$$P\frac{\delta}{2} = \frac{M_t^2 l}{2\,GI_p} + \frac{M_b^2 l}{2\,EI}. \quad \ldots\ldots\ldots\ldots \quad (29)$$

It will be observed that as a spring elongates, whether it is open or close coiled, the angle of inclination θ changes and that in order to obtain correct results from the foregoing equations the final rather than the initial value of θ should be used. For a spring of ordinary dimensions, however, the change in θ is so small that sufficiently accurate results are obtained by using its initial value. For a very flexible spring the accuracy of the results can be increased by making trial solutions with estimated final values of θ until the calculated values agree closely enough with the estimated values.

As previously stated, the foregoing equations apply equally well to a spring subjected to an axial load in compression, the axial twist and the change in length being opposite in direction simply to the twist and extension of the tension spring.

While it is customary to neglect the effect of the normal and shearing components N and S (equations 4 and 5) on the stress and deformation in a spring, an estimate of the stress intensity and the total elongation due to shear in the wire in the coil can easily be made on the basis of the assumption that the stress on any cross section, due to N or S, is uniformly distributed. On this basis the stress intensity due to the component S on a cross section of a close coiled spring, having the dimension $D = 5\,d$ and subjected to an axial load P, would be about 10 per cent of the value of f_s due to the twist in the wire. The effect on the deformation is much less; the elongation in a coil, having $D = 5\,d$, due to a distortion in shear figured on the above assumption, being about 2 per cent of the elongation due to the twist in the wire. The stress intensity and the elongation of the coil due to the component S become relatively less as the ratio $\frac{D}{d}$ increases.

168. Helical Spring Subjected to an Axial Twist. — If a helical spring is held at one end and subjected to a twisting couple only, in a plane perpendicular to the axis of the coil at the other, the wire in the coil will be subjected to uniform bending, which will result in a change in the diameter and also in the length of the coil.

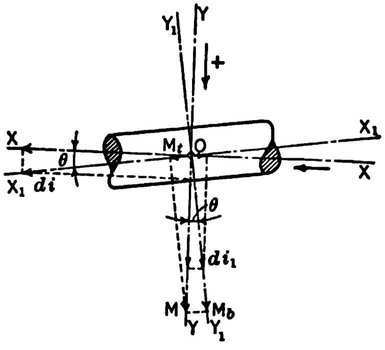


FIG. 230.

We will follow the notation of Art. (167) in representing the dimensions of the spring and will let M = the moment of the couple and OY and OX (Fig. 230) represent the pair of coordinate axes through the center of gravity of any cross section of the wire, such as O (Fig. 227), which are respectively parallel to the axis of the coil and tangent to the cylinder on which the helix lies; also let OX_1 and OY_1 represent the axes in the plane XOY which are respectively tangent and normal to the helix at O. The resultant of the stress on the cross section of the wire at O must evidently be equal to the couple M and, if the bending is positive, causing an increase in the curvature of the wire, the moment axis of the couple will be represented by the vector OM (Fig. 230).

The couple M can be resolved into a couple

$$M_b = M \cos\theta, \quad \ldots\ldots\ldots\ldots \quad (1)$$

causing bending in the wire about OY_1 as an axis, and a couple

$$M_t = M \sin\theta, \quad \ldots\ldots\ldots\ldots \quad (2)$$

causing a twist in the wire about its axis.

Close Coiled Spring. — If the coil is closely wound, M_t will be so small as to be negligible and $M_b = M$, very nearly.

The greatest stress intensity in the wire due to the bending will be equal to

$$f = \frac{Mr}{I} = \frac{32\,M}{\pi d^3}. \quad \ldots\ldots\ldots\ldots \quad (3)$$

If we assume, as in Art. (167), that the change in curvature is equal to $\frac{M_b}{EI}$ and let ϕ = the angular displacement of the free end of the coil, we shall have

$$\frac{d\phi}{dl} = \frac{M}{EI} = \frac{64\,M}{\pi d^4 E}, \quad \ldots\ldots\ldots\ldots \quad (4)$$

and hence, since the curvature is uniform throughout the length of the coil,

$$\phi = \frac{Ml}{EI} = \frac{64\,Ml}{\pi d^4 E}. \quad \ldots\ldots\ldots\ldots \quad (5)$$

This result might have been obtained by equating the work done by the twisting couple to the resilience due uniform bending in a wire of length l (Art. 109),

$$\frac{M}{2}\phi = \frac{M^2 l}{2\,EI}, \quad \ldots\ldots\ldots\ldots \quad (6)$$

and solving for the value of ϕ.

The value of l in the above equations will be equal to $l = n\pi D$ (very nearly).

Open Coiled Spring. — In this case, the twist of the wire about its axis must be taken into account and the greatest principal stress intensity can be found by substituting the values of M_b and M_t in equation (11) (Art. 164) giving

$$n_1 = \frac{r}{2I}\left[M_b + \sqrt{M_b^2 + M_t^2}\right] = \frac{16}{\pi d^3}\left[M\cos\theta + \sqrt{(M\cos\theta)^2 + (M\sin\theta)^2}\right]$$

$$= \frac{16\,M}{\pi d^3}(\cos\theta + 1), \quad \ldots\ldots\ldots\ldots \quad (7)$$

which approaches the value given by (3), for the close coiled spring, as θ approaches zero.

Following the notation of Art. (167) the rotation about the axis OY (Fig. 230), in a length dl, will be equal to

$$d\phi = \sin\theta\, di + \cos\theta\, di_1 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

and, similarly the rotation about OX will be equal to

$$d\omega = \cos\theta\, di - \sin\theta\, di_1. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

Substituting the values of di_1 and di from equations (19) and (20) (Art. 167) and reducing, under the condition that the component rotations are constant throughout the length of the coil, we obtain

$$\begin{aligned}\phi &= l\left(\frac{M_t \sin\theta}{GI_p} + \frac{M_b \cos\theta}{EI}\right) = Ml\left(\frac{\sin^2\theta}{GI_p} + \frac{\cos^2\theta}{EI}\right)\\ &= \frac{32\,Ml}{\pi d^4}\left(\frac{\sin^2\theta}{G} + \frac{2\cos^2\theta}{E}\right) = \frac{16\,Ml}{\pi d^4 E}(4 + \sin^2\theta), \quad . \; . \; . \quad (10)\end{aligned}$$

when $G = \frac{2}{5}E$.

Similarly,

$$\omega = l\left(\frac{M_t \cos\theta}{GI_p} - \frac{M_b \sin\theta}{EI}\right), \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

and hence the total change in the length of coil will be equal to

$$\begin{aligned}\delta &= \frac{D}{2}\omega = \frac{Dl}{2}\left(\frac{M_t \cos\theta}{GI_p} - \frac{M_b \sin\theta}{EI}\right) = \frac{8\,MDl \sin 2\theta}{\pi d^4}\left(\frac{1}{G} - \frac{2}{E}\right)\\ &= \frac{4\,MDl \sin 2\theta}{\pi d^4 E}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)\end{aligned}$$

when $G = \frac{2}{5}E$. The value of l in this case will be equal to

$$l = n\pi D \sec\theta. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

It is evident that when the twist is in the opposite direction to that assumed (Fig. 230), the magnitude of the angle of twist and the axial displacement at the free end can be determined directly from the foregoing equations, since the signs for both of the component couples and both of the component rotations will be reversed.

169. Helical Spring Subjected to a Combined Axial Load and Twist.

— If a helical spring is subjected to an axial load, combined with a twisting couple in a plane perpendicular to the axis of the coil, the stress intensity and the component displacements can be found by resolving both the couple due to the axial load and the twisting couple into components M_t and M_b, as shown in Articles (167) and (168), and combining to obtain a resultant couple ΣM_t producing a twist in the wire about its axis, and a resultant bending couple ΣM_b.

The maximum stress intensity and the angle of twist and the axial displacement, at the free end of the coil, can then be found by substituting the values of ΣM_t and ΣM_b for M_t and M_b in the general formulas (15), (25) and (27), (Art. 167), for the open coiled spring, taking proper account of signs. If the spring is close coiled the solution can be simplified by putting $\theta = 0$ in the above-mentioned equations.

170. Spring of Hollow Circular Section. — If the helical spring, in any of the preceding cases (Arts. 167–169), were made of a hollow circular tube, the maximum stress intensity, the angle of twist and the axial displacement at the free end could evidently be found by substituting the values of I_p and I, for the hollow section, in the general equations and reducing, in the same manner as for the solid circular section.

171. Spring of Non-Circular Section. — If a helical spring in any of the cases given (Arts. 167–169) is made up of wire having a square, rectangular or elliptical section, the twisting and bending couples acting at any cross section of the coil can be found in the same manner as in the case of a spring of circular section.

The maximum stress intensity can be estimated by calculating the greatest normal stress intensity by the theory of simple bending (Art. 66) and the greatest shearing stress intensity from one of the equations in Art. (162) and substituting these values in equation (9) (Art. 164); determining the maximum principal stress intensity in the same manner as in the case of the non-circular bar subjected to torsion and bending (Art. 166).

The angle of twist and the axial displacement at the free end of the coil, due to any combination of axial loads and twisting couples, can be estimated by substituting in the general equations for a spring of circular section, subjected to the same combination of loads (Arts. 167–169), the value of C (Art. 162), for I_p; and by substituting for I the value of the moment of inertia of the cross section about the principal axis which is parallel to the axis of the coil.

172. Time of Oscillation of a Spring. — If a weight W is attached to the free end of a helical spring and the spring is set in vibration under this load by extending the coil in the direction of the axis and then releasing it, the spring will oscillate for a time before coming to rest.

Since the extension of the coil is proportional to the force applied, the motion of the weight W will be harmonic and the time of a complete oscillation can be determined in the following manner.

We will restrict the analysis to the close coiled spring and let

P_1 = the unbalanced force exerted by the spring on the weight when it is displaced a distance δ_1 from its mean position.

The time of a complete oscillation will then be equal to

$$T = 2\pi\sqrt{\frac{\delta_1}{a}} = 2\pi\sqrt{\frac{W\delta_1}{gP_1}} \text{ (Art. 159, Vol. I)}, \quad . \quad . \quad (1)$$

where $a = \frac{F}{m} = \frac{P_1}{W}g.$

The value of the ratio $\frac{P_1}{\delta_1}$ for a spring of any given dimensions can easily be determined from equation (9) (Art. 167).

When the axis of the coil is vertical and the weight W is suspended from the free end, equation (1) can be simplified as follows:

Let δ = the vertical displacement of the free end under a static load W, which will evidently be the mean displacement of the weight W when the spring is vibrating.

Then

$$\frac{W + P_1}{\delta + \delta_1} = \frac{W}{\delta} = \frac{P_1}{\delta_1},$$

and hence

$$\delta = \frac{W\delta_1}{P_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting this value in (1),

$$T = 2\pi\sqrt{\frac{\delta}{g}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the above analysis, the weight of the spring has been neglected, it being assumed to be small compared with the weight W.

Similarly, if the close coiled helical spring, with a weight W attached to the free end, is set to oscillating about its axis in torsion, the angular displacement of the free end will be proportional to the twisting couple and, by analogy, the time of a complete oscillation will be equal to

$$T = 2\pi\sqrt{\frac{\phi_1}{\alpha}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\alpha = \frac{M_1}{I_w}g$, M_1 = the torque exerted by the spring when ϕ_1 = the angular displacement of the weight and I_w = the moment of inertia of the weight about the axis of the coil, which in this case must be a principal axis of the weight.

Hence

$$T = 2\pi\sqrt{\frac{I_w\phi_1}{gM_1}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where the ratio $\frac{M_1}{\phi_1}$ can be readily obtained from equation (5) (Art. 168). As before, the weight of the spring is assumed to be negligible compared to the weight W.

173. Problems — Shafting and Springs.

Problem 1.

Find the twisting moment, or torque, which will produce a maximum shearing stress of 9000 lbs. per sq. in. in a solid shaft 4″ diameter. What is the angle of torsion in a length of 10 ft., if $G = 12{,}000{,}000$ lbs. per sq. in.?

Solution. — Transposing equation (7) (Art. 156) and substituting the numerical values, we obtain

$$M = \frac{f_s I_p}{r} = \frac{9000\pi\,8}{2} = 113{,}100 \text{ in. lbs.} = 9425 \text{ ft. lbs.}$$

Substituting in equation (1) (Art. 157), we obtain

$$i = \frac{Mx}{GI_p} = \frac{113{,}100 \times 120 \times 2}{12{,}000{,}000 \times \pi \times 16} = 0.045 \text{ rad.} = 2.58°,$$

or by substituting in (2) (Art. 157),

$$i = \frac{f_s x}{Gr} = \frac{9000 \times 120}{12{,}000{,}000 \times 2} = 0.045 \text{ rad.} = 2.58°.$$

Problem 2.

A pulley is keyed to a solid shaft 4″ diameter. If the moment of inertia of the pulley and shaft about the axis of the shaft is 30,000 lbs. $(\text{ft.})^2$, find what twisting moment applied to the shaft is necessary to impart to the pulley a speed of 300 revolutions per minute in 10 seconds. Find the greatest intensity of the shearing stress in the shaft due to this twisting moment.

Problem 3.

Find the diameter of a solid shaft required to fulfil the conditions that the maximum shearing stress due to torsion = 8000 lbs. per sq. in. and the angle of torsion in a length of 10 ft. is 2.5°. Assume $G = 12{,}000{,}000$ lbs. per sq. in.

Problem 4.

In a test made on a piece of solid Tobin Bronze shafting 1¼″ diameter the increase in the angle of torsion in a length of 25″, between a torque of 1920 in. lbs. and a torque of 4450 in. lbs., was 4.17°. Find the modulus of rigidity.

Problem 5.

Solve Problem 4 for a shaft 1¾″ diameter and an angle of torsion, between a torque of 2250 in. lbs. and a torque of 13,500 in. lbs., of 2.22°.

Problem 6.

The following data were obtained from a torsion test of a piece of hot rolled steel shafting:

Length of specimen between jaws of testing machine.	48 ins.
Diameter of cross section	1.63 ins.
Length in which the angle of torsion was measured	30 ins.
Twisting moment at elastic limit	20,400 in. lbs.
Maximum twisting moment before fracture	67,320 in. lbs.
Angle of torsion, between twisting moments of 1800 in. lbs. and 19,800 in. lbs.	3.95°
Number of twists of specimen, between the jaws of testing machine, at fracture	13.3

Find:

(a) The elastic limit in torsion,
(b) The modulus of rupture in torsion,
(c) The modulus of rigidity,
(d) The number of turns per foot of length at fracture.

Problem 7.

Find the greatest allowable torque that may be applied to a hollow shaft 16″ outside diameter and 10″ inside diameter, provided the greatest intensity of shearing stress is 8000 lbs. per sq. in. Find the angle of torsion in a length of 10 ft., assuming $G = 12{,}000{,}000$ lbs. per sq. in.

Problem 8.

Determine the percentage increase in the torque and also in the weight per unit length of a solid shaft having the same outside diameter as that of the hollow shaft in Problem (7). Compare the torsional rigidity of the solid shaft with that of the hollow shaft.

Problem 9.

Determine the allowable torque for a solid shaft of the same material and having the same weight per unit length as the hollow shaft in Problem (7). Compare the torsional rigidities of the two shafts.

Problem 10.

Determine the resilience in torsion of the hollow shaft in Problem (7) when the greatest shearing stress intensity is 8000 lbs. per sq. in. Compare this result with the resilience of the solid shaft, having the same outside diameter, under the same maximum stress intensity.

Problem 11.

Find the horse-power that a solid shaft 3″ diameter will transmit, at a speed of 150 revolutions per minute, provided the greatest allowable shearing stress intensity is 9000 lbs. per sq. in. Find the angle of torsion in a length of 20 feet. $G = 12{,}000{,}000$ lbs. per sq. in.

Problem 12.

Find the horse-power that a hollow shaft 12" outside diameter and 8" inside diameter will transmit at a speed of 100 revolutions per minute, provided the maximum allowable shearing stress intensity is 8000 lbs. per sq. in. Find the angle of torsion in a length of 50 ft. $G = 12{,}000{,}000$ lbs. per sq. in.

Problem 13.

Assuming the maximum allowable intensity of shearing stress is 8000 lbs. per sq. in., deduce the value of K in the formula

$$d = K\sqrt[3]{\frac{\text{h.p.}}{N}},$$

where d = the diameter of the solid shaft required to transmit a given horse-power. By use of the above formula determine the diameter of a shaft required to transmit 60 horse-power at 300 revolutions per minute.

Problem 14.

A length of shaft, 3" diameter, supported in hangers 6 ft. from center to center (Fig. 231), transmits 40 horse-power from the belt on pulley No. 1 to that on pulley No. 2, at a speed of 300 revolutions per minute. The tight and loose sides of the belt on each pulley are parallel and the belt on No. 1 is horizontal and that on No. 2 runs at 45° with the horizontal, as indicated.

Pulley No. 1 is 42" diameter and weighs 100 lbs. and pulley No. 2 is 48" diameter and weighs 150 lbs.

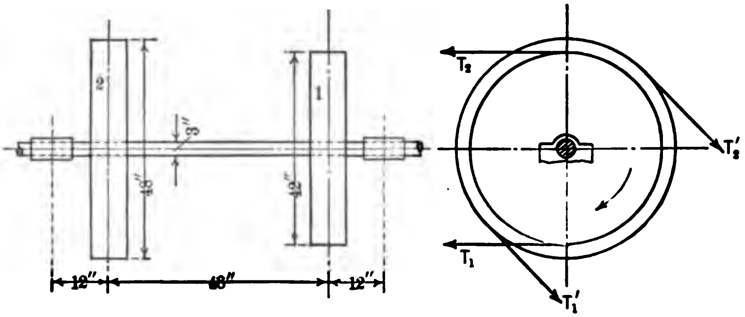


FIG. 231.

Determine the greatest stress intensity in the shaft, neglecting the weight of the shaft, the effect of centrifugal force in the belt (Art. 191, Vol. I) and assuming that the sum of the tensions on the tight and loose sides is 2.5 times the effective pull, or $T_1 + T_2 = 2.5\,(T_1 - T_2)$. Determine the greatest value of the product Ee_1, of the strain and modulus of elasticity.

Solution. — The torque

$$M_t = \frac{40 \times 33{,}000}{2\,\pi\,300} = 700 \text{ ft. lbs.} = 8400 \text{ in. lbs.}$$

Hence

$$T_1 - T_2 = \frac{8400}{21} = 400 \text{ lbs. and } T_1 + T_2 = 2.5 \times 400 = 1000 \text{ lbs.},$$

$$T_1' - T_2' = \frac{8400}{24} = 350 \text{ lbs. and } T_1' + T_2' = 2.5 \times 350 = 875 \text{ lbs.}$$

Resolving the forces acting on the shaft into H and V components and calculating the reactions at the end bearings, we obtain the system of vertical components, shown in Fig. (232 *a*), and the system of horizontal components, shown in Fig. (232 *b*). The bending moments at the pulleys, due to the vertical forces will be

$$M_1 = 12 \times 211.5 = 2538 \text{ in. lbs.},$$

$$M_2 = 12 \times 657.5 = 7890 \text{ in. lbs.};$$

and the bending moments due to the horizontal forces will be

$$M_1' = 12 \times 730 = 8760 \text{ in. lbs.},$$

$$M_2' = 12 \times 349 = 4188 \text{ in. lbs.},$$

the bending moment diagrams being indicated in the sketches.

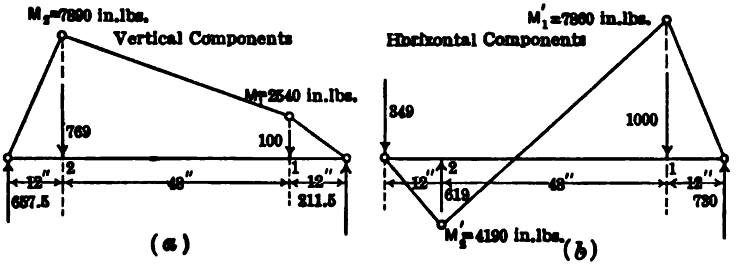


FIG. 232.

The resultant bending moment at pulley No. 1 will be equal to

$$\sqrt{(M_1)^2 + (M_1')^2} = 9120 \text{ in. lbs.},$$

and the resultant bending moment at pulley No. 2 will be equal to

$$\sqrt{(M_2)^2 + (M_2')^2} = 8930 \text{ in. lbs.}$$

Hence the greatest bending moment in the shaft is at pulley No. 2 and

$$M_b = 9120 \text{ in. lbs.}$$

Substituting in equation (12) (Art. 164) we obtain, for the greatest normal stress intensity,

$$n_1 = \frac{16}{\pi 3^3} [9120 + \sqrt{(9120)^2 + (8400)^2}]$$

$$= \frac{16}{27 \pi} (9120 + 12{,}400) = 4060 \text{ lbs. per sq. in.};$$

and, by substituting in equation (19) (Art. 164), we obtain, for the greatest value of Ee_1,

$$Ee_1 = \frac{4}{\pi 3^3}(3 \times 9120 + 5 \times 12{,}400) = 4210 \text{ lbs. per sq. in.}$$

The angle between the normal to the principal plane and the normal to the cross section can be found by substituting in equation (8) (Art. 164), which will give

$$\tan 2\alpha = \frac{8400}{9120} = 0.921,$$

$$2\alpha = 42.6^\circ \quad \text{and} \quad \alpha = 21.3^\circ.$$

Problem 15.

Determine the diameter of the shaft required in Problem (14) to satisfy the condition that the greatest principal stress intensity shall not exceed 7000 lbs. per sq. in. Determine the diameter required to satisfy the condition that the maximum value of Ee_1 shall not exceed 7000 lbs. per sq. in.

Problem 16.

A length of shafting, 4″ diameter, supported at the ends by hangers 8 ft. apart from center to center, transmits 60 h.p. from a pulley 5 ft. diameter, weighing 150 lbs., placed with its center 2 ft. from the center of one hanger, to a pulley 4 ft. diameter, weighing 100 lbs., placed with its center 2 ft. from the center of the other hanger. The belt on the first pulley is horizontal and on the other vertical. If the speed of the shaft is 360 revolutions per minute and it is assumed that sum of the tensions of the tight and loose sides of the belt on each pulley is equal to 2.5 times the difference of the tensions, find the greatest principal stress intensity in the shaft, neglecting the weight of the shaft. Find the angles which the principal planes of stress at the point of maximum stress intensity make with the axis of the shaft.

Problem 17.

Find the product Ee_1 of the modulus of elasticity and the greatest principal strain in the shaft in Problem (16), assuming $m = 4$.

Problem 18.

A solid shaft, 6″ diameter, is subjected to a maximum bending moment of 200,000 in. lbs. If the shaft turns at a speed of 100 revolutions per minute what horse-power can be transmitted, provided the greatest intensity of stress in the material does not exceed 8000 lbs. per sq. in.

Problem 19.

A solid propeller shaft, 6″ diameter, transmits 400 horse-power at a speed of 100 revolutions per minute. If the thrust of the screw is 6000 lbs., find the maximum intensity of compressive stress in the shaft when the bending stresses are negligible.

Problem 20.

Solve Problem (19) when there is a bending moment of 20,000 in. lbs. in addition to the thrust and the twisting moment.

Problem 21.

The horizontal bar AB, 4" diameter, is fixed at the end A and is subjected to a vertical load supported on the end of the horizontal arm BC, which is perpendicular to the axis of the bar (Fig. 233). Find the magnitude of P, on the assumption that the greatest principal stress intensity $n_1 = 8000$ lbs. per sq. in.

Problem 22.

Find the magnitude of the load P in Problem (21) on the assumption that the product $Ee_1 = 8000$ lbs. per sq. in.

Problem 23.

If the bar in Fig. (233) is hollow and d = the outside diameter and d_1 = the inside diameter, determine the diameter of the bar required to support a load of 3000 lbs. at C under the conditions of Problem (21), assuming $d_1 = \frac{3}{4} d$.

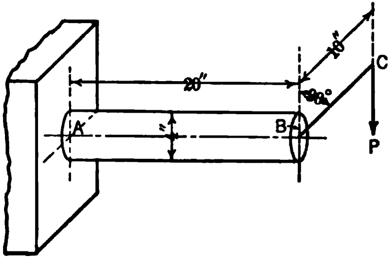


FIG. 233.

Problem 24.

If the force P, acting on the bar in Problem (21), were replaced by a force of 2000 lbs., acting in the vertical plane through C, which is parallel to the axis of the bar, and making an angle of 30° with the horizontal plane through AB and BC, determine the greatest stress intensity in the bar.

Problem 25.

Calculate the moment of resistance in torsion of a straight bar of elliptical cross section 4" × 2", assuming $f_s = 8000$ lbs. per sq. in. Determine the angle of torsion in a length of 50" under a torque equal to the above moment of resistance, assuming $G = 12,000,000$ lbs. per sq. in.

Problem 26.

Solve Problem (25) for a bar having a rectangular cross section 4" × 2".

Problem 27.

Solve Problem (25) for a bar having a cross section 2" square.

Problem 28.

Solve Problem (21), replacing the round bar with a bar 4" square.

Problem 29.

A close coiled helical spring, having the dimensions $D = 2''$, $d = \frac{1}{4}''$, $n = 12$ (Art. 167), is subjected to an axial load P in tension. Find the allowable load on the spring and the elongation of the coil, assuming $f_s = 30,000$ lbs. per sq. in., $G = 12,000,000$ lbs. per sq. in.

Note. — Solve by using equations (6) and (9) (Art. 167).

Problem 30.

An open coiled spring, having the dimensions $D = 2''$, $d = \frac{1}{4}''$, $\theta = 30°$, the number of turns being the same as for the spring in Problem (29), is subjected to an axial load P in tension. Find the allowable load on the spring and the elongation of the coil, assuming $n_1 = 30{,}000$ lbs. per sq. in., $G = 12{,}000{,}000$ lbs. per sq. in.

Find the angle of twist at the free end of the coil.

Note. — Solve by use of equations (15), (28), (27) and (25) (Art. 167) and compare results with those obtained in Problem (29).

Problem 31.

Find the suitable dimensions for a helical spring of round wire to support an axial load of 2000 lbs. in compression, with an axial displacement of 2'', assuming $f_s = 50{,}000$ lbs. per sq. in., $G = 12{,}000{,}000$ lbs. per sq. in.

Note. — Use the formulas for the close coiled spring and determine first the value of the ratio $\frac{D}{d^3}$. Having this ratio, decide on the values of d and D which will give a spring of correct proportions. Then determine the length of wire, number of coils and initial height of spring necessary to give the required displacement.

Problem 32.

A close coiled helical spring, having the dimensions $D = 3''$, $d = \frac{1}{2}''$ $n = 10$, is fixed at one end and subjected to a twisting couple of 400 in. lbs. at the other, in a plane perpendicular to the axis of the coil. Determine the greatest stress intensity and the angle of twist at the free end of the coil.

Note. — Use equations (3) and (5) (Art. 168).

Problem 33.

An open coiled helical spring, having the dimensions $D = 3$, $d = \frac{1}{2}''$, $\theta = 30°$, and the same length of wire in the coil as the spring in Problem (32), is fixed at one end and subjected to a torque of 400 in. lbs. about the axis of the coil at the other. Determine the greatest stress intensity, the angle of twist and the axial displacement at the free end.

Note. — Use equations (7), (13), (12) and (10), (Art. 168) and compare results with those obtained in Problem (32).

Problem 34.

Solve Problem (29), substituting a wire $\frac{1}{4}''$ square for the round wire. (See Art. 171.)

Problem 35.

Solve Problem (32) substituting a wire $\frac{1}{2}''$ square for the round wire. (See Art. 171.)

Problem 36.

Solve Problem (31) substituting a square wire for the round wire.

CHAPTER XI.

CURVED BARS.

174. Definition and Limitations. — A *curved beam* may be defined as a bar subjected to bending by the action of external forces, the central axis of which in the unstrained state is a plane curve. The stresses and strains due to external forces acting upon a straight beam, under certain limitations, have been determined in Chapters IV and V. Similar limitations will be imposed in developing the theory for curved beams, namely: (*a*) the material is homogeneous, (*b*) the central axis passing through the center of gravity of every cross section is a plane curve, (*c*) every cross section is symmetrical with respect to its line of intersection with the plane of the central axis, the term *cross section* being used to indicate a section normal to the central axis; (*d*) the external forces are in equilibrium and act in the plane of the central curve. Under these conditions the elastic curve (Art. 95) formed by the central axis of the beam after bending will be a plane curve located in the plane of the original central curve.

175. Curved Bar Subjected to Uniform Bending. — If a curved bar is subjected to bending, under the limitations in Art. (174), by equal and opposite couples acting at the ends, the resultant of the stress on any cross section of the bar will evidently be a couple, equal in magnitude to the terminal couples and the bending moment will be uniform throughout the bar.

If the three assumptions, which were made in the theory of bending of straight bars (Art. 66) are held to be true, however, the stress on a cross section will not be uniformly varying and the line of zero intensity, or the neutral axis of the stress, will not pass through the center of gravity of the section. Hence the formulas for the stress intensity and deflection in a straight bar will not give correct values for a bar having an initial curvature.

Let Fig. (234 *a*) represent a curved bar in the unstrained state and *AB* and *GH* two radial cross sections intersecting the central axis of the bar at N and T, the portion of the axis NT being so small that it may be considered to be the arc of a circle. Let r_1 = the initial radius of curvature of the arc NT and l = its length; let l_1 = the length RS, intercepted by the cross sections AB and GH on a layer at any distance y from the central axis, and let ϕ = the angle between AB and GH. Under the action of two equal and opposite terminal couples M (Fig. 234 *b*), the bar will undergo a change in curvature and layers, or fibers, parallel to the central axis and lying on the side of an unstrained neutral layer opposite to the center of curvature, will be elongated, while those on the same side as the center of curvature are shortened. The planes of the cross sections AB and GH will intersect in a straight line passing through the center of curvature of NT, the angle between them being changed a small amount $\delta\phi$, where δ represents the ratio of the increment in ϕ to its original value, analogous to the longitudinal strain e (Art. 4). The bending moment M will be called *plus* when it tends to *increase* the *curvature* and *minus* when it tends to *decrease* the *curvature.*

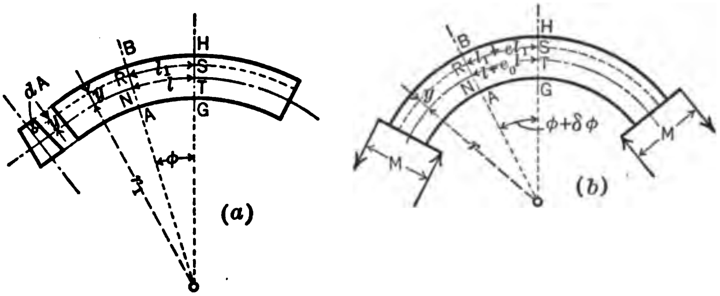


FIG. 234.

We will let r = the final radius of curvature of the arc NT, e_0 = the strain, or extension, in the central axis and e = the extension in the layer at the distance y from the central axis, the change in y, due to the deformation, being neglected. The sign of y will be *plus* when it is measured away from the center of curvature and *minus* when it is measured toward the center, and *tensile stresses* and *strains* will be called *plus* and *compressive stresses* and *strains, minus.* The length of the portion of the central axis NT, between the cross sections AB and GH, after bending, will then be equal to $l + e_0l$ and the length of the layer RS will be equal to $l_1 + el_1$.

It will follow from the assumption that plane cross sections remain plane after bending that

$$l = r_1\phi, \qquad (1)$$

$$l + e_0 l = r(\phi + \delta\phi), \qquad (2)$$

$$l_1 = (r_1 + y)\phi \qquad (3)$$

and

$$l_1 + el_1 = (r + y)(\phi + \delta\phi). \qquad (4)$$

Dividing (4) by (3) and transposing,

$$e = \frac{(r + y)(\phi + \delta\phi)}{(r_1 + y)\phi} - 1, \qquad (5)$$

and dividing (2) by (1),

$$1 + e_0 = \frac{r(\phi + \delta\phi)}{r_1\phi}. \qquad (6)$$

Hence,

$$\frac{(\phi + \delta\phi)}{\phi} = (1 + e_0)\frac{r_1}{r};$$

and substituting in (5),

$$\begin{aligned} e &= \frac{r_1(r + y)(1 + e_0)}{r(r_1 + y)} - 1 \\ &= \frac{\left(1 + \frac{y}{r}\right)(1 + e_0)}{1 + \frac{y}{r_1}} - \frac{\left(1 + \frac{y}{r_1}\right)(1 + e_0)}{1 + \frac{y}{r_1}} + e_0 \\ &= \frac{r_1 y}{r_1 + y}\left(\frac{1}{r} - \frac{1}{r_1}\right)(1 + e_0) + e_0. \qquad (7) \end{aligned}$$

Since the resultant of the stress on the cross section AB is a couple, if we let $f =$ the intensity of the normal stress at any point at a distance y from the central layer and observe that under the assumptions of the theory

$$f = Ee,$$

we shall have

$$\int f\,dA = E\int e\,dA = Er_1(1 + e_0)\left(\frac{1}{r} - \frac{1}{r_1}\right)\int \frac{y}{r_1 + y}dA + Ee_0\int dA = 0;$$

and hence

$$r_1(1 + e_0)\left(\frac{1}{r} - \frac{1}{r_1}\right)Q_1 + e_0 A = 0, \qquad (8)$$

where A = the area of the cross section and

$$Q_1 = \int \frac{y}{r_1 + y}\, dA,$$

which may be called a modified moment of the cross section about the axis through its center of gravity. It is evident that the value of Q_1 approaches zero as r_1 increases.

The moment of the couple formed by the stress on AB will be equal to

$$M = \int fy\, dA = E\int ey\, dA = Er_1(1 + e_0)\left(\frac{1}{r} - \frac{1}{r_1}\right)\int \frac{y^2}{r_1 + y}\, dA + Ee_0\int y\, dA. \qquad (9)$$

But $\int y\, dA = 0$, and hence

$$\frac{M}{E} = r_1(1 + e_0)\left(\frac{1}{r} - \frac{1}{r_1}\right)\int \frac{y^2}{r_1 + y}\, dA = r_1(1 + e_0)\left(\frac{1}{r} - \frac{1}{r_1}\right)Q_2, \qquad (10)$$

where $Q_2 = \int \frac{y^2}{r_1 + y}\, dA$ may be called a modified moment of inertia of the cross section. It is evident that as r_1 increases Q_2 approaches the value $\frac{I}{r_1}$, the moment of inertia of the cross section, about the axis through its center of gravity, divided by the radius of curvature.

From the simultaneous solution of (8) and (10) we obtain

$$e_0 = -\frac{MQ_1}{EAQ_2} \qquad (11)$$

and, by substituting this value in (10) and transposing,

$$\frac{1}{r} - \frac{1}{r_1} = \frac{M}{Er_1Q_2\left(1 - \frac{MQ_1}{EAQ_2}\right)} = \frac{MA}{r_1(EAQ_2 - MQ_1)}. \qquad (12)$$

Substituting the values of e_0 and $\left(\frac{1}{r} - \frac{1}{r_1}\right)$ in (7), we obtain

$$e = \frac{r_1 y}{r_1 + y}\left(\frac{MA}{r_1(EAQ_2 - MQ_1)}\right)\left(1 - \frac{MQ_1}{EAQ_2}\right) - \frac{MQ_1}{EAQ_2} = \frac{M}{EQ_2}\left(\frac{y}{r_1 + y} - \frac{Q_1}{A}\right). \qquad (13)$$

Hence

$$f = Ee = \frac{M}{Q_2}\left(\frac{y}{r_1 + y} - \frac{Q_1}{A}\right). \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The values of Q_1 and Q_2 can be conveniently determined by writing

$$Q_1 = \int \frac{y}{r_1 + y} dA = \int \left(1 - \frac{r_1}{r_1 + y}\right) dA = A - B, \quad . \quad (15)$$

where $B = r_1 \int \frac{dA}{r_1 + y}$, ... (16)

and $Q_2 = \int \frac{y^2}{r_1 + y} dA = \int \left(y - r_1 + \frac{r_1^2}{r_1 + y}\right) dA$

$$= 0 - r_1 A + r_1 B = -r_1 (A - B). \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Substituting the values of Q_1 and Q_2 in (14)

$$f = \frac{M}{r_1 (B - A)}\left(\frac{y}{r_1 + y} - \frac{A - B}{A}\right)$$

$$= \frac{M}{r_1 (B - A)}\left(\frac{B}{A} - \frac{r_1}{r_1 + y}\right). \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If we let y_0 = the distance from the center of gravity to the neutral axis of the stress, we shall obtain, by substituting y_0 for y and putting (18) equal to zero,

$$\frac{B}{A} = \frac{r_1}{r_1 + y_0};$$

and hence

$$y_0 = \frac{Ar_1}{B} - r_1 = \frac{A - B}{B} r_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Since B will be greater than A, y_0 will be negative; which indicates that the neutral axis lies on the same side of the center gravity as the center of curvature, or *inside* of the central layer of the beam.

An inspection of (18) will show that the greatest stress intensity will occur either in the inside or the outside layer of the bar and that the distribution of the stress over any cross section will depend on the value of B for the section.

If the curvature of the bar is constant, that is, if the central axis is a circle, the value of B will be constant and the maximum stress intensity and the distribution of the stress on every cross section will be the same; but, if the curvature varies, B will vary and hence the maximum stress intensity will vary from section to section, although both the bending moment and cross section are uniform throughout the bar.

It should be observed that, as r_1 increases, equation (14) approaches the form of the equation for the normal stress intensity in the straight beam,

$$f = \frac{My}{I}; \quad \ldots\ldots\ldots\ldots \quad (20)$$

and it will be found that, in cases where the radius of curvature is large compared with the dimensions of the cross section, sufficiently accurate results can be obtained by using (20) for calculating the stress intensity in place of the more complex formulas (14) or (18).

176. Curved Bar Subjected to Ordinary Bending. — When a curved bar is subjected to ordinary bending by any balanced system of forces, acting in the plane of curvature as indicated in Fig. (235), the forces acting on the part of the bar on either side of a cross section AB can be resolved into a set of normal components, acting along the axis OX, through O the center of gravity of the section, a set of shearing components, acting along OY, and a system of couples. By combining these components the system can be reduced to a single normal force

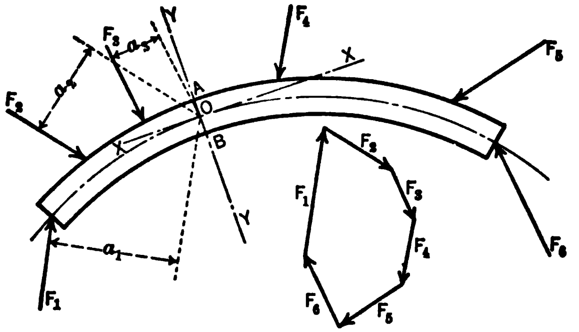


FIG. 235.

$$P = \Sigma X, \quad \ldots\ldots\ldots\ldots \quad (1)$$

acting through O, a shearing force

$$S = \Sigma Y, \quad \ldots\ldots\ldots\ldots \quad (2)$$

acting along AB, and couple

$$M = \Sigma Fa, \quad \ldots\ldots\ldots\ldots \quad (3)$$

where F represents any force acting on the beam and a represents the distance from O to its line of action. The normal component P will produce a uniform stress over the cross section, the intensity of which will be equal to

$$f_1 = \frac{P}{A}, \quad \ldots\ldots\ldots\ldots \quad (4)$$

and the couple M will produce a varying stress, the intensity of which at any point in the cross section will be represented by the equation

$$f_2 = \frac{M}{r_1(B-A)}\left(\frac{B}{A} - \frac{r_1}{r_1+y}\right) \text{ (Art. 175).} \quad \ldots \quad (5)$$

Hence the resultant intensity of the normal stress at any point in the cross section will be equal to

$$f = f_1 + f_2. \quad \ldots\ldots\ldots\ldots \quad (6)$$

The neutral axis of the stress due to the bending couple will be located at a distance

$$y_0 = \frac{A-B}{B} r_1 \text{ (Art. 175)} \quad \ldots\ldots\ldots \quad (7)$$

from the center of gravity, and the distance between the center of gravity and the neutral axes of the combined stress can be found by placing (6) equal to zero and solving for y.

As in the case of the bar subjected to uniform bending (Art. 175), when the radius of curvature r_1 is large compared with the dimensions of the cross section, sufficiently accurate results can be obtained by the use of the equations for the straight bar (Art. 126) in place of the more complex equations.

No special formula for the intensity s of the shearing stress on a cross section, or longitudinal layer, will be deduced, the ordinary formula

$$s = \frac{SQ}{bI} \text{ (Art. 88)} \quad \ldots\ldots\ldots \quad (8)$$

being sufficiently accurate when the radius of curvature is large; and, when the radius of curvature is small compared with the dimensions of the cross section, an accurate determination of the shearing stress intensity is of little value. If desired it may be approximately estimated by the use of equation (8).

It should be observed that in a bar of large curvature the proportion between the length and the dimensions of the cross section is liable *to be such that, unless the distribution of the stress over the ends of the bar is similar to that called for by the theory, the actual distribution of the stress on any cross section will deviate from the theoretical (Art. 53). Hence for such cases the foregoing formulas must be regarded as empirical and the values of the working stress intensities for use in the formulas in any particular case must be determined from the results of experiments made on bars of the same material and of similar dimensions loaded under similar conditions.*

177. Graphical Representation of Normal Stress. — In order to determine the normal stress intensity f at any point in a given cross section, it is necessary to have the values of B and A for the section. The deduction of the equations for B and plots showing the variation in f for a few typical cross sections follow.

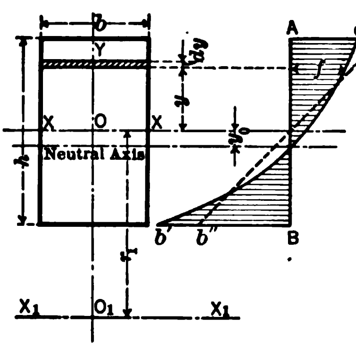


FIG. 236.

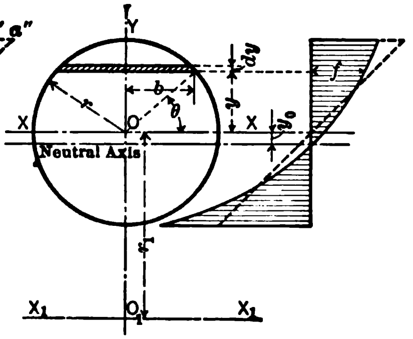


FIG. 237.

Rectangular Section.—Let b and h represent the dimensions of the section (Fig. 236) and let X_1X_1 represent the axis through the center of curvature O_1 of the bar. Let r_1 = the distance from X_1X_1 to the center of gravity O of the cross section and take for dA a strip, of width dy, at a distance y from O. Then

$$B = r_1\int\frac{dA}{r_1+y} = r_1b\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{dy}{r_1+y} = r_1b\log_e(r_1+y)\Big]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$= r_1b\log_e\frac{2r_1+h}{2r_1-h} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

and

$$A = bh. \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

The distance y_0, from the center of gravity to the neutral axis of the stress due to bending, can then be found from equation (19) (Art. 175) and the stress intensity f at any point, due to any bending couple M, can be found from equation (18) (Art. 175). The variation in the stress intensity across the section is indicated by the curve $a'b'$, the intensity Bb' at the inside of the section, nearest the center of curvature, being much greater than the intensity Aa' at the outside.

The form of the curve $a'b'$ and the distance y_0 will change as r_1 changes, the neutral axis approaching the center of gravity and the stress intensity curve $a'b'$ approaching the straight line $a''b''$ as r_1 increases.

Circular Section. — Let r = the radius of the circle (Fig. 237) and $2\,b$ = the length of a strip at a distance y from the center of gravity. Then, if we follow the previous notation,

$$B = r_1 \int \frac{2\,b\,dy}{r_1 + y}.$$

But $b = r\cos\theta$, $y = r\sin\theta$, $dy = r\,d\theta\cos\theta$ and, therefore,

$$B = 2\,r_1 r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\theta\,d\theta}{r_1 + r\sin\theta} = 2\,r_1 r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1 - \sin^2\theta)\,d\theta}{r_1 + r\sin\theta}$$

$$= 2\,r_1 r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-\frac{1}{r}\sin\theta + \frac{r_1}{r^2} - \frac{r_1^2 - r^2}{r^2\,(r_1 + r\sin\theta)}\right] d\theta$$

$$= 2\,r_1 r^2 \left[\frac{1}{r}\cos\theta + \frac{r_1}{r^2}\theta - \frac{r_1^2 - r^2}{r^2} \times \frac{2}{\sqrt{r_1^2 - r^2}} \tan^{-1} \frac{r_1 \tan\frac{\theta}{2} + r}{\sqrt{r_1^2 - r^2}}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2\,r_1 r^2 \left(0 + \pi\frac{r_1}{r^2} - \frac{2\sqrt{r_1^2 - r^2}}{r^2} \times \frac{\pi}{2}\right).$$

$$= 2\,\pi r_1\,(r_1 - \sqrt{r_1^2 - r^2}) \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

and $A = \pi r^2$. . . . . (4)

The distance y_0 and the stress intensity f at any point can be determined by equations (19) and (18), (Art. 175) as before, the variation in stress intensity being indicated by the plot (Fig. 237).

Any Cross Section. — The value of B for any cross section can be determined with sufficient accuracy by dividing the area into narrow strips, of width Δy and area ΔA, parallel to the principal axis through the center of gravity, calculating the value of $\frac{r_1}{r_1 + y}\,\Delta A$ for each strip and adding together, y being the distance from the center of gravity to the center of the strip in each case. The result obtained by this process is represented by the expression

$$B = \Sigma \frac{r_1}{r_1 + y}\Delta A. \quad \ldots\ldots\ldots\ldots \quad (5)$$

Having B and A the values of y_0 and the stress intensity f can be calculated in the same manner as for the other sections. The variation of the stress intensity, obtained in the above manner, is indicated for the sections shown in Figs. (238) and (239). So long as the width of the strips used in making the calculation is less than $\frac{1}{10}$ the total depth of the section, the accuracy of the results will be sufficient for ordinary purposes.

Variation in Maximum Stress Intensity with Radius of Curvature. — It will be observed that, when $y = -c_1$, the distance from the center of gravity to the side of the cross section nearest the center of curvature, the formula for the maximum normal stress intensity due to bending may be written in the form

$$f = M\frac{1}{S''}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where the quantity

$$\frac{1}{S''} = \left[\frac{1}{r_1(B-A)}\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad (7)$$

corresponds to the reciprocal of the section modulus $S' = \frac{I}{c}$ in the formula

$$f = M\frac{1}{S'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

for the straight beam.

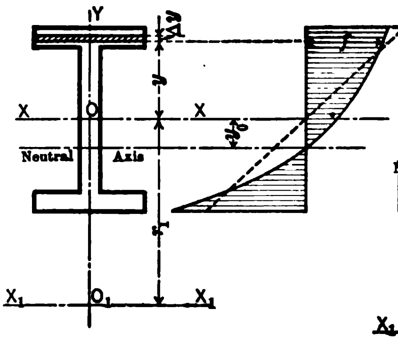


FIG. 238.

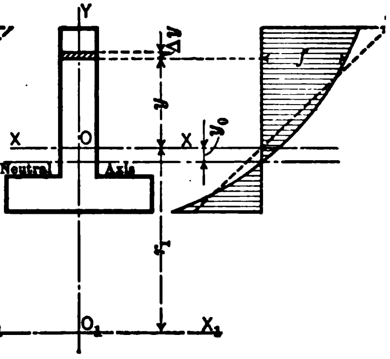


FIG. 239.

A comparison of the results given by equations (6) and (8) for a beam of any given curvature can be made by calculating the values of the coefficients $\frac{1}{S''}$ and $\frac{1}{S'}$.

The plot (Fig. 240) represents the variation in the value of $\frac{1}{S''}$ with the radius of curvature in the case of a curved bar of circular cross section of unit radius, the ordinates representing values of $\frac{1}{S''}$ and the abscissæ, values of r_1 = the radius of curvature of the central axis. For the sake of comparison, the value of $\frac{1}{S'}$ is shown by the straight line. It will be seen, for example, that when $r_1 = 2$ the value of f given by (8) is about 38 per cent smaller than the value given by (6) and that when $r_1 = 5$ the error is about 15 per cent, the value given by (6) gradually approaching the value given by (8) as r_1 increases.

Similarly, when $y = c_2$, the distance from the center of gravity to the side of the cross section farthest from the center of curvature, the formula for the normal stress intensity may be written

$$f = M\left[\frac{1}{r_1(B-A)}\left(\frac{B}{A} - \frac{r_1}{r_1 + c_2}\right)\right] = M\frac{1}{S'''}. \quad . \quad . \quad . \quad . \quad (9)$$

The variation of $\frac{1}{S'''}$ with the value of r_1 for the circular cross section of unit radius is also indicated in the plot (Fig. 240), the value of f given by (8) when $r_1 = 2$ being in this case about 42 per cent larger than the value given by (9) and about 15½ per cent larger when $r_1 = 5$.

Similar results would be obtained for bars with other cross sections.

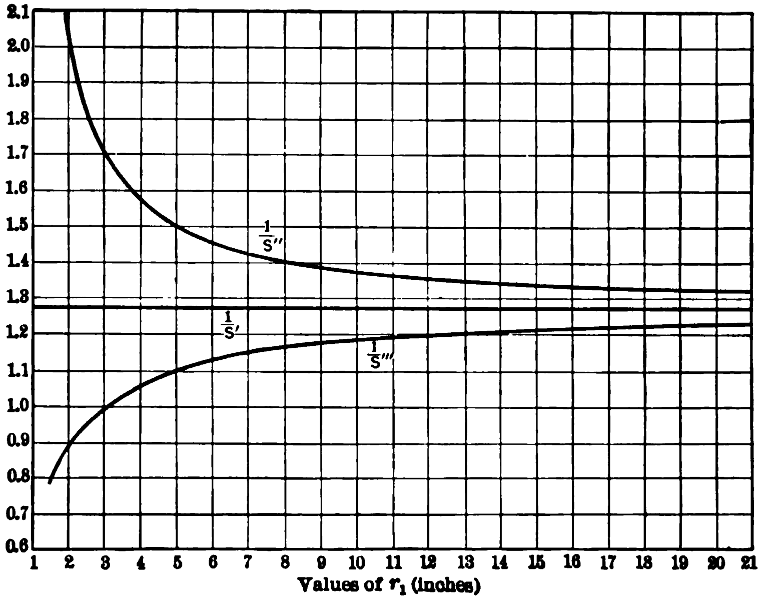


FIG. 240.

178. Flexure of Curved Bars. — In the derivation of the formulas for the stress intensity in the curved bar subjected to bending it has been shown that the central axis undergoes a change both in curvature and length and hence the displacement, due to bending, of any point on the axis will, in general, be the resultant of two component displacements in the directions of any pair of coördinate axes which may be chosen.

Let ODC (Fig. 241) represent the central axis of a curved bar before bending. The components of the displacement of any point O relative to the point C can be determined in the following manner. Let (x_1, y_1) be the coördinates of C with respect to any pair of coördinate axes with the origin at O. Let δx = the displacement of O, relative to C, in the direction OX, δy = its displacement in the direction OY and $\delta\alpha$ = the change in the angle between the tangents at O and C, due to bending the bar. Let D be the center of gravity of any cross section of the bar, whose coördinates are (x, y), α = the angle between the tangent at D and OX, M = the bending moment, r_1 = the initial radius of curvature and r = the final radius of curvature of the central axis at the point D. Following the notation in the preceding articles and combining equation (12) (Art. 175) with equation (19) (Art. 167), we obtain the following expression for the change in curvature at A,

$$\frac{di}{ds} = \frac{1}{r} - \frac{1}{r_1} = \frac{MA}{r_1\,(EAQ_2 - MQ_1)}, \quad \ldots \ldots \quad (1)$$

where di = the change in $d\alpha$, the difference of the slopes of the tangents at the ends of the length ds of the central curve at D.

We will let OF represent the resultant displacement and OE and EF, the component displacements in the directions OX and OY, of the point O, due to the change in curvature in the length ds only, while the curvature of the remainder of the central axis is considered as remaining unchanged.

Then, if we let OD = the length of the chord from O to the point (x, y) we shall have, neglecting signs,

$$\begin{aligned} OE &= d\,(\delta x) = OF \cdot \cos EOF = OD \cdot di \cdot \cos EOF \\ &= OD \cdot \cos ODK \cdot di = y\,di; \quad \ldots \ldots \ldots \quad (2) \end{aligned}$$

and

$$\begin{aligned} EF &= d\,(\delta y) = OF \cdot \sin EOF = OD \cdot di \cdot \sin EOF \\ &= OD \cdot \sin ODK \cdot di = x\,di. \quad \ldots \ldots \ldots \quad (3) \end{aligned}$$

Substituting the value of di from (1) in equations (2) and (3) and integrating, we obtain

$$\delta x = \int y\,di = \int_{x=0}^{x=x_1} \frac{MA}{r_1\,(EAQ_2 - MQ_1)}\,y\,ds, \quad \ldots \quad (4)$$

$$\delta y = -\int x\,di = -\int_{x=0}^{x=x_1} \frac{MA}{r_1\,(EAQ_2 - MQ_1)}\,x\,ds. \quad . \quad (5)$$

For the change $\delta\alpha$, in the angle $(\alpha_0 - \alpha_1)$ between the tangents at O and C, we have

$$\delta\alpha = \int d\,(\delta\alpha) = \int di = \int_{x=0}^{x=x_1} \frac{MA}{r_1\,(EAQ_2 - MQ_1)}\,ds. \quad . \quad (6)$$

It will be observed that when the central axis is concave toward OX and M is positive, as indicated (Fig. 241), δx and $\delta\alpha$ are positive and δy is negative.

When the bending moment is negative, or when the curvature of the central axis relative to OX is reversed, the signs of the displacements can be easily determined. When the curvature is comparatively small, that is, the radius of curvature of the central axis is several times the dimensions of the cross section of the bar,

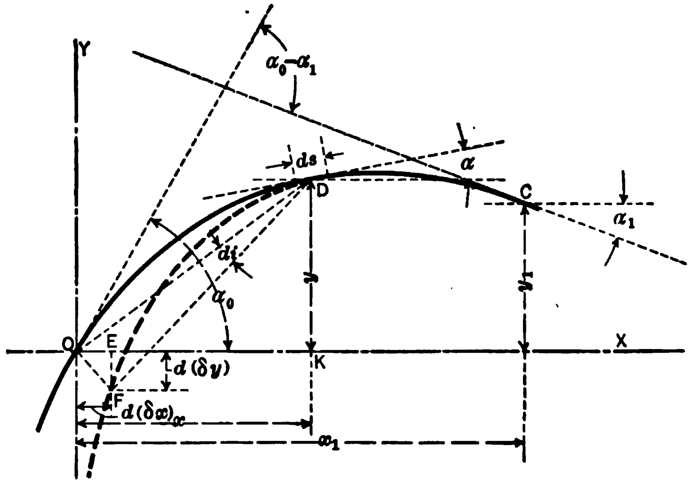


FIG. 241.

$$Q_1 = 0 \text{ (nearly) and } Q_2 = \frac{I}{r_1} \text{ (nearly) (Art. 175)}$$

and equations (1), (4), (5) and (6) reduce to

$$\frac{di}{ds} = \frac{1}{r} - \frac{1}{r_1} = \frac{M}{EI}, \quad \ldots \ldots \ldots \quad (7)$$

$$\delta x = \int_{x=0}^{x=x_1} \frac{M}{EI} y\, ds, \quad \ldots \ldots \ldots \quad (8)$$

$$\delta y = -\int_{x=0}^{x=x_1} \frac{M}{EI} x\, ds, \quad \ldots \ldots \ldots \quad (9)$$

$$\delta \alpha = \int_{x=0}^{x=x_1} \frac{M}{EI} ds. \quad \ldots \ldots \ldots \quad (10)$$

The above equations evidently reduce to the equations for the straight beam (Arts. 95–96) when $\frac{1}{r_1} = 0$ and the axis OX is taken to coincide with the central axis of the beam.

In the deduction of the foregoing equations the effect of the component displacements due to the change in length of the axis of the bar has been neglected, as these are small compared with the displacements due to the bending and in many cases will have no appreciable effect on the results.

The expressions for the component displacements $\delta x'$ and $\delta y'$, due to the change in length of the axis, may be determined as follows: Let P = the resultant normal force acting on the section at D (Fig. 241) which is assumed to be positive, if tension, and negative, if compression. The strain in the central axis will then be the resultant of the strain due to the normal force P and that due to the bending moment M (equation 11, Art. 175) or,

$$e_0 = \frac{P}{AE} - \frac{MQ_1}{EAQ_2} = \frac{P}{AE} + \frac{M}{EAr_1}. \quad . \quad . \quad . \quad . \quad (11)$$

If the expansion, or contraction, due to temperature change is to be allowed for, we should add to the value of e_0 given by (11) the quantity ϵT, where ϵ = the coefficient of linear expansion of the material in the bar and T = the temperature change, in which case

$$e_0 = \frac{P}{AE} + \frac{M}{EAr_1} + \epsilon T, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

the value of T being plus when the temperature increases.

The change in the length ds, due to the strain e_0, will be $e_0\,ds$ and the component displacements at O in the directions OX and OY, due to the change in the length ds alone, will be respectively equal to

$$-e_0 \cos \alpha\, ds = -\, e_0 \frac{dx}{ds} ds = -\, e_0\, dx$$

and

$$-\, e_0 \sin \alpha\, ds = -\, e_0 \frac{dy}{ds} ds = -\, e_0\, dy,$$

the negative signs indicating that for the bar shown the displacements along OX and OY are negative when e_0 is positive.

Hence the total displacements in the directions OX and OY due to the total change in length of the portion OC of the central axis will be respectively equal to

$$\delta x' = -\int_{x=0}^{x=x_1} e_0\, dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\delta y' = -\int_{x=0}^{x=x_1} e_0\, dy; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and hence the resultant displacements, due the combined bending, axial thrust and temperature changes, will be equal to

$$\delta x_1 = \delta x + \delta x' = \int_{x=0}^{x=x_1} \frac{MA}{r_1\,(EAQ_2 - MQ_1)}\, y\, ds$$
$$- \int_{x=0}^{x=x_1} \left[\frac{P}{AE} + \epsilon T + \frac{M}{EAr_1}\right] dx \quad . \quad . \qquad (15)$$

and

$$\delta y_1 = \delta y + \delta y' = - \int_{x=0}^{x=x_1} \frac{MA}{r_1\,(EAQ_2 - MQ_1)}\, x\, ds$$
$$- \int_{x=0}^{x=x_1} \left[\frac{P}{AE} + \epsilon T + \frac{M}{EAr_1}\right] dy. \qquad (16)$$

The change in $(\alpha_0 - \alpha_1)$, due to the strain e_0 in the length ds alone, will be $\frac{e_0\, ds}{r_1}$ and the total increment in $(\alpha_0 - \alpha_1)$, due to the total change in length of OC, will be

$$\delta\alpha' = \int_{x=0}^{x=x_1} \frac{e_0}{r_1}\, ds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (17)$$

Hence the total change in $(\alpha_0 - \alpha_1)$, due to combined bending, thrust and temperature changes, will be

$$\delta\alpha_1 = \delta\alpha + \delta\alpha' = \int_{x=0}^{x=x_1} \frac{MA}{r_1\,(EAQ_2 - MQ_1)}\, ds$$
$$+ \int_{x=0}^{x=x_1} \frac{1}{r_1}\left(\frac{P}{AE} + \epsilon T + \frac{M}{EAr_1}\right) ds$$
$$= - \int_{x=0}^{x=x_1} \left(\frac{MA}{EAQ_2 - MQ_1} + \frac{P}{AE} + \epsilon T + \frac{M}{EAr_1}\right) d\alpha. \quad . \qquad (18)$$

In ordinary cases the above equations are too complex to be of practical value and, except in cases in which the curvature is large compared with the cross section of the bar, the following approximate forms are sufficiently accurate:

$$\delta x_1 = \int_{x=0}^{x=x_1} \frac{M}{EI}\, y\, ds - \int_{x=0}^{x=x_1} \left(\frac{P}{AE} + \epsilon T\right) dx, \quad . \quad . \qquad (19)$$

$$\delta y_1 = - \int_{x=0}^{x=x_1} \frac{M}{EI}\, x\, ds - \int_{x=0}^{x=x_1} \left(\frac{P}{AE} + \epsilon T\right) dy, \quad . \qquad (20)$$

$$\delta\alpha_1 = - \int_{x=0}^{x=x_1} \left(\frac{Mr_1}{EI} + \frac{P}{AE} + \epsilon T\right) d\alpha. \quad . \quad . \quad . \quad . \quad . \qquad (21)$$

When the normal thrust P and the temperature change T are small compared with the bending moment M, the terms $\frac{P}{AE}$ and

ϵT in equations (19), (20) and (21), can be omitted without causing a serious error and the equations reduce to the forms in equations (8), (9) and (10). When the bar is of uniform cross section all of the foregoing equations are simplified by making E, A and I, or Q_1 and Q_2, constant and, when the bending is uniform, the equations are simplified by making M constant.

179. The Hook. — An approximate method of determining the stresses in a hook, or an open link, by use of the straight beam formula has been discussed in Art. (133). Owing to the curvature, the results obtained by this method are evidently incorrect and a solution by the method of Art. (176) will give much more accurate results.

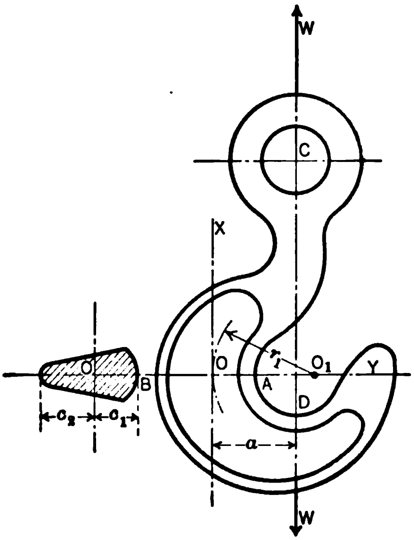


FIG. 242.

Let the hook (Fig. 242) be subjected to a pull W along the line CD. Let a = the distance from the line of action of W to the center of gravity O, of the cross section AB which is perpendicular to CD. Let r_1 = the radius of curvature of the central axis of the hook at O. Then from Art. (176) we obtain for the stress intensity at any point in AB,

$$f = f_1 + f_2 = \frac{P}{A} + \frac{M}{r_1(B - A)}\left(\frac{B}{A} - \frac{r_1}{r_1 + y}\right). \quad . \quad . \quad . \quad . \quad (1)$$

It should be observed that in this case $P = W$, $M = -Wa$ and the greatest stress intensity will occur at the point A for which $y = -c_1$.

180. The Circular Ring. — If a circular ring of uniform cross section and material is subjected to a pull by two equal and opposite forces, acting along a diameter, the greatest stress intensity and the deformation in the ring can be determined in the following manner:

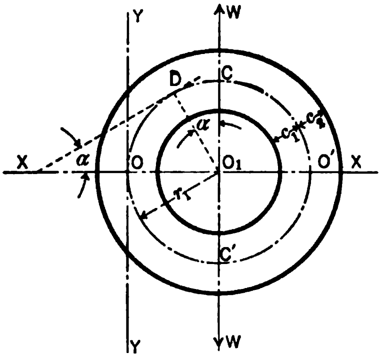


FIG. 243.

First Solution. — Let r_1 = the radius of the central axis of the ring (Fig. 243) before the load is applied and let W = the load, acting along the diameter CC'. Let M_c = the bending moment at the cross section at C, M_0 = the bending moment at the cross section at O on the diameter OX, perpendicular to CC', M = the bending moment at any cross section D and α = the angle between the tangent to the central axis at D and OX.

The stress on the cross section at O will be the resultant of a uniform normal stress $P_0 = \frac{W}{2}$ and a varying stress, whose resultant is the couple M_0.

The stress on the cross section D will be the resultant of a uniform normal stress,

$$P = P_0 \sin\alpha = \frac{W}{2}\sin\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

a shearing stress,

$$S = P_0 \cos\alpha = \frac{W}{2}\cos\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and a varying stress, whose resultant is the couple

$$M = M_0 + P_0 (r_1 - r_1 \sin\alpha) = M_0 + \frac{Wr_1}{2}(1 - \sin\alpha). \quad . \quad . \quad . \quad (3)$$

The stress on the cross section at C to the left of W, will be the resultant of a shearing stress,

$$S_c = \frac{W}{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and a varying stress, whose resultant is the couple

$$M_c = M_0 + \frac{W}{2} r_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

An approximate solution for the value of M_0 can be made as follows: It will be observed that under the load W the diameter CC' will increase and the diameter OO' will diminish and that the change in the angle between the tangents to the central axis at the points O and C, due to the distortion, will be equal to zero.

Hence, by substituting the above value of M in equation (10) (Art. 178), the coördinates of the point C being (x_1, y_1), we obtain

$$\delta\alpha = \frac{1}{EI}\int_{x=0}^{x=x_1}\left[M_0 + \frac{Wr_1}{2}(1 - \sin\alpha)\right]ds = 0. \quad . \quad . \quad . \quad . \qquad (6)$$

But $ds = -r_1\,d\alpha$ and hence

$$\begin{aligned}\delta\alpha &= -\frac{1}{EI}\int_{\frac{\pi}{2}}^{0}\left[M_0 + \frac{Wr_1}{2}(1 - \sin\alpha)\right]r_1\,d\alpha \\ &= -\frac{1}{EI}\left[M_0 r_1\alpha + \frac{Wr_1^2}{2}(\alpha + \cos\alpha)\right]_{\frac{\pi}{2}}^{0} \\ &= \frac{1}{EI}\left(-\frac{Wr_1^2}{2} + M_0 r_1\frac{\pi}{2} + \frac{Wr_1^2}{2}\frac{\pi}{2}\right) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)\end{aligned}$$

and solving for M_0,

$$M_0 = \frac{Wr_1}{\pi}\left(1 - \frac{\pi}{2}\right) = Wr_1\left(\frac{1}{\pi} - \frac{1}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Substituting the value of M_0 in (3), we obtain

$$M = \frac{Wr_1}{\pi}\left(1 - \frac{\pi}{2}\sin\alpha\right) = Wr_1\left(\frac{1}{\pi} - \frac{1}{2}\sin\alpha\right); \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and from (5) we obtain

$$M_c = \frac{Wr_1}{\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

which is the greatest bending moment in the ring. It will be observed that the bending moment M_c is positive and that M_0 is negative. There will be a point N, between O and C, at which the bending is equal to zero, which may be located by putting (9) equal to zero and solving for α', the angle between the tangent at N and OX, which will give,

$$\alpha' = \sin^{-1}\frac{2}{\pi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

The greatest normal stress intensity on any cross section D (Fig. 243) will be located at the inside of the section and will be represented by equation (6) (Art. 176) or, if we substitute the values of f_1 and f_2 in terms of the normal stress (equation 1) and the bending moment (equation 9),

$$f = \frac{W}{2A}\sin\alpha + \frac{W\left(\frac{1}{\pi} - \frac{1}{2}\sin\alpha\right)}{(B - A)}\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right), \quad . \quad . \quad . \qquad (12)$$

where c_1 = the distance of the inside of the cross section from the central axis OC.

At the cross section O equation (12) reduces to

$$f = \frac{W}{2A} + \frac{W\left(\frac{1}{\pi} - \frac{1}{2}\right)}{(B - A)}\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right), \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

which is the maximum tensile stress intensity in the ring; and at the cross section C equation (12) reduces to

$$f = \frac{W}{\pi(B - A)}\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

which is a compressive stress and is the maximum intensity of stress in the ring.

The normal stress intensity at the outside of the ring can be readily obtained by substituting c_2, the distance from the center of gravity to the outside of the cross section, for $-c_1$ in equations (12), (13) and (14).

The change in the length of the diameters OO' and CC' can be approximately determined by substituting the values of $ds = -r_1\,d\alpha$, $y = r_1 \cos\alpha$, $x = r_1(1 - \sin\alpha)$ and the value of M from equation (9) in equations (8) and (9) (Art. 178) and obtaining the displacements of O, relative to C, in the directions OX and OY, as follows:

$$\delta x = -\frac{1}{EI}\int_{\frac{\pi}{2}}^{0} W r_1\left(\frac{1}{\pi} - \frac{1}{2}\sin\alpha\right) r_1^2 \cos\alpha\, d\alpha$$

$$= -\frac{W r_1^3}{EI}\left[\frac{1}{\pi}\sin\alpha - \frac{1}{4}\sin^2\alpha\right]_{\frac{\pi}{2}}^{0}$$

$$= \frac{W r_1^3}{EI}\left(\frac{1}{\pi} - \frac{1}{4}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (15)$$

and

$$\delta y = \frac{1}{EI}\int_{\frac{\pi}{2}}^{0} W r_1\left(\frac{1}{\pi} - \frac{1}{2}\sin\alpha\right) r_1^2 (1 - \sin\alpha)\, d\alpha$$

$$= \frac{W r_1^3}{EI}\left[\frac{\alpha}{\pi} + \left(\frac{1}{2} + \frac{1}{\pi}\right)\cos\alpha + \frac{\alpha}{4} - \frac{1}{8}\sin 2\alpha\right]_{\frac{\pi}{2}}^{0}$$

$$= \frac{W r_1^3}{EI}\left[\frac{1}{\pi} - \frac{\pi}{8}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

Since δx is positive the diameter OO' will be shortened and the total decrease will be equal to

$$\delta\,(OO') = \frac{2 W r_1^3}{EI}\left(\frac{1}{\pi} - \frac{1}{4}\right) \quad \ldots\ldots\ldots\ldots \quad (17)$$

and, since δy is negative, the diameter CC' will be lengthened and the total increase will be equal to

$$\delta\,(CC') = \frac{2 W r_1^3}{EI}\left(\frac{\pi}{8} - \frac{1}{\pi}\right). \quad \ldots\ldots\ldots\ldots \quad (18)$$

Second Solution. — In the preceding solution, by use of the approximate equations (7–9) (Art. 178), the stress due to the normal force P and the effect of the curvature on the distribution of the stress due to bending have been neglected. A more accurate solution can be made by use of the more exact forms of the equations for the displacements as follows:

By substituting the values $Q_1 = (A - B)$ and $Q_2 = -r_1(A - B)$ (Art. 175) in equation (18) (Art. 178), we obtain

$$\delta\alpha_1 = \int_{x=0}^{x=x_1} -\left[\frac{A}{B - A}\left(\frac{M}{EAr_1 + M}\right) + \frac{P}{AE} + \epsilon T + \frac{M}{EAr_1}\right] d\alpha. \quad (19)$$

and, assuming no temperature change and that $\frac{M}{EAr_1 + M} = \frac{M}{EAr_1}$ (very nearly) for any allowable bending moment, equation (19) will reduce to

$$\delta\alpha_1 = \int_{x=0}^{x=x_1} -\left[\frac{B}{B - A}\left(\frac{M}{EAr_1}\right) + \frac{P}{AE}\right] d\alpha. \quad \ldots\ldots \quad (20)$$

Substituting the values of M and P, from equations (3) and (1), in equation (20) and observing that the change in the angle between the tangents at O and C, due to the distortion of the ring (Fig. 243), is zero, we have

$$\delta\alpha_1 = \int_{\frac{\pi}{2}}^{0} - \left[\frac{B}{(B-A)\,2\,EA}\left\{\frac{2\,M_0}{r_1} + W\,(1-\sin\alpha)\right\} + \frac{W\sin\alpha}{2\,EA}\right]d\alpha = 0. \quad (21).$$

Reducing (21), we obtain

$$\int_{\frac{\pi}{2}}^{0}\left[B\left(\frac{2\,M_0}{r_1} + W\right) - AW\sin\alpha\right]d\alpha = 0; \quad \ldots\ldots \quad (22)$$

and integrating,

$$-B\left(\frac{2\,M_0}{r_1} + W\right)\frac{\pi}{2} + AW = 0;$$

and solving for M_0, we have

$$M_0 = \left(\frac{2\,W}{\pi}\,\frac{A}{B} - W\right)\frac{r_1}{2} = \frac{Wr_1}{\pi}\left(\frac{A}{B} - \frac{\pi}{2}\right) = Wr_1\left(\frac{A}{\pi B} - \frac{1}{2}\right). \quad . \quad (23)$$

Substituting the value of M_0 in (3), we have

$$M = \frac{Wr_1}{\pi}\left(\frac{A}{B} - \frac{\pi}{2}\sin\alpha\right) = Wr_1\left(\frac{A}{B\pi} - \frac{1}{2}\sin\alpha\right); \quad . \; . \; . \quad (24)$$

and, by substituting in (5), we obtain

$$M_c = \frac{Wr_1A}{\pi B} \quad \ldots\ldots\ldots\ldots\ldots \quad (25)$$

The point N, at which the bending moment is zero, may be located by placing (24) equal to zero and solving for α', the angle between the tangent at N and OX, which will give

$$\alpha' = \sin^{-1}\frac{2\,A}{\pi B}. \quad \ldots\ldots\ldots\ldots \quad (26)$$

For the greatest normal stress intensity at any cross section D, we have

$$f = \frac{W}{2\,A}\sin\alpha + \frac{W\left(\frac{A}{B\pi} - \frac{1}{2}\sin\alpha\right)}{(B-A)}\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right) \quad . \; . \; . \; . \quad (27)$$

and for the cross section at O the greatest intensity will be

$$f = \frac{W}{2\,A} + \frac{W\left(\frac{A}{B\pi} - \frac{1}{2}\right)}{(B-A)}\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right) \quad . \; . \; . \; . \; . \quad (28)$$

and for the cross section at C,

$$f = \frac{WA}{\pi B\,(B-A)}\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right), \quad . \; . \; . \; . \; . \; . \; . \quad (29)$$

which is the greatest normal stress intensity in the ring. The equations for the stress intensity at the outside of the ring at the cross sections D, O and C can be obtained by substituting c_2 for $-c_1$ in equations (27), (28) and (29).

The variation in the bending moment and the stress intensities at the inside, or *intrados*, and at the outside, or *extrados*, of the ring is represented by the diagram (Fig. 244).

The variation in M for a ring of circular cross section, of radius $r = 1$ and having the proportions $r_1 = 3\,r$, is shown by the curve on the left of the central axis CC', which is constructed by plotting the values of M at the different cross sections radially from the central axis of the ring as a base line, positive values of M being measured outward and negative values inward from the base line.

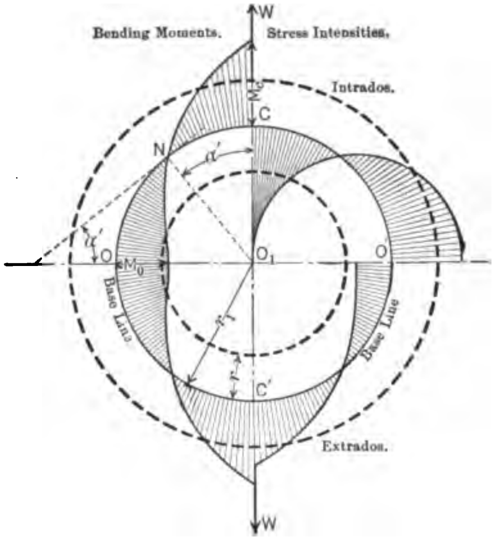


FIG. 244.

The variation in the stress intensity at the intrados of the ring is shown by the curve in the upper right-hand quadrant, tensile stress intensities being plotted radially outward and compressive stress intensities radially inward from the central axis as a base line. Similarly, the diagram in the lower right-hand quadrant represents the variation in the stress intensity at the extrados of the ring, tensile stresses being measured outward and compressive stress intensities inward from the central axis.

It will be observed that the greatest tensile stress intensity is located at the intrados, at the sections O and O', and that the greatest compressive stress intensity is located at the intrados, at the sections C and C', and that the latter is the maximum stress intensity for the entire ring.

To determine the change in the diameters OO' and CC' we have, by substituting the values of Q_1 and Q_2 in (1) (Art. 178),

$$\frac{MA}{r_1\,(AEQ_2 - MQ_1)} = \frac{MA}{(B - A)\,r_1\,(EAr_1 + M)} = \frac{M}{(B - A)\,Er_1^2}\ \text{(very nearly)};$$

and, assuming that there is no change in temperature, equation (15) (Art. 178), for the displacement of O relative to C in the direction OX, will reduce to

$$\delta x_1 = \int_{x=0}^{x=x_1} \frac{M}{(B-A)Er_1^2}\, y\, ds - \int_{x=0}^{x=x_1}\left(\frac{P}{AE} + \frac{M}{AEr_1}\right)dx; \quad . \quad . \quad (30)$$

and equation (16) (Art. 178), for the displacement of O relative to C in the direction OY, will become

$$\delta y_1 = -\int_{x=0}^{x=x_1} \frac{M}{(B-A)Er_1^2}\, x\, ds - \int_{x=0}^{x=x_1}\left(\frac{P}{AE} + \frac{M}{AEr_1}\right)dy. \quad . \quad (31)$$

Substituting the values $x = r_1(1 - \sin\alpha)$, $y = r_1 \cos\alpha$, $ds = -r_1\, d\alpha$, $dx = -r_1 \cos\alpha\, d\alpha$, $dy = -r_1 \sin\alpha\, d\alpha$, $P = \frac{W}{2}\sin\alpha$ and $M = Wr_1\left(\frac{A}{B\pi} - \frac{1}{2}\sin\alpha\right)$ in the above equations and observing that when $x = 0$, $\alpha = \frac{\pi}{2}$ and when $x = x_1$, $\alpha = 0$, we obtain

$$\begin{aligned}\delta x_1 &= \int_{\frac{\pi}{2}}^{0} - \frac{Wr_1}{(B-A)E}\left(\frac{A}{B\pi} - \frac{1}{2}\sin\alpha\right)\cos\alpha\, d\alpha \\ &\quad + \int_{\frac{\pi}{2}}^{0}\left[\frac{Wr_1}{2AE}\sin\alpha + \frac{Wr_1}{AE}\left(\frac{A}{B\pi} - \frac{1}{2}\sin\alpha\right)\right]\cos\alpha\, d\alpha \\ &= \frac{Wr_1}{(B-A)E}\left(\frac{A}{B\pi} - \frac{1}{4}\right) - \frac{Wr_1}{\pi BE}; \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (32)\end{aligned}$$

and

$$\begin{aligned}\delta y_1 &= \int_{\frac{\pi}{2}}^{0} \frac{Wr_1}{(B-A)E}\left(\frac{A}{B\pi} - \frac{1}{2}\sin\alpha\right)(1 - \sin\alpha)\, d\alpha \\ &\quad + \int_{\frac{\pi}{2}}^{0}\left[\frac{Wr_1}{2AE}\sin\alpha + \frac{Wr_1}{AE}\left(\frac{A}{B\pi} - \frac{1}{2}\sin\alpha\right)\right]\sin\alpha\, d\alpha \\ &= \frac{Wr_1}{(B-A)E}\left[\frac{A}{B\pi} - \frac{\pi}{8} + \frac{1}{2}\left(\frac{B-A}{B}\right)\right] - \frac{Wr_1}{\pi BE}. \; . \; . \; . \; . \quad (33)\end{aligned}$$

Since δx_1 is positive, the diameter OO' will be shortened and the total decrease will be equal to

$$\delta(OO') = \frac{2Wr_1}{(B-A)E}\left(\frac{A}{B\pi} - \frac{1}{4}\right) - \frac{2Wr_1}{\pi BE}, \quad . \; . \; . \; . \; . \quad (34)$$

and, as δy_1 is negative, the diameter CC' will be lengthened and the formula for the total increase may be written

$$\delta(CC') = \frac{2Wr_1}{(B-A)E}\left[\frac{\pi}{8} - \frac{A}{B\pi} - \frac{1}{2}\left(\frac{B-A}{B}\right)\right] + \frac{2Wr_1}{\pi BE}. \quad . \quad (35)$$

The following approximate expressions for the change in diameters, as accurate as the conditions of loading in ordinary cases will warrant the use of, can be obtained by combining the approximate values for the change due to bending only, given by (17) and (18), with the change due to the normal component P, represented by the last term in (34) or (35):

$$\delta\,(OO') = \frac{2\,Wr_1^3}{EI}\left(\frac{1}{\pi} - \frac{1}{4}\right) - \frac{2\,Wr_1}{\pi AE}; \quad \ldots\ldots \quad (36)$$

$$\delta\,(CC') = \frac{2\,Wr_1^3}{EI}\left(\frac{\pi}{8} - \frac{1}{\pi}\right) + \frac{2\,Wr_1}{\pi AE}. \quad \ldots\ldots\ldots \quad (37)$$

181. Chain Links. — The simplest form of a chain link is made up of semicircular ends and straight sides as indicated in Fig. (245), the pull being applied along the central axis CC'. If the theories for determining the stresses in curved bars and in straight bars were rigidly adhered to, there would be a sharp change in the distribution of the stress at the cross sections O, O', etc., where the curved and straight parts join. Assuming that this change takes place in a very short distance in the neighborhood of these cross sections, an approximate solution for the stresses may be obtained as follows:

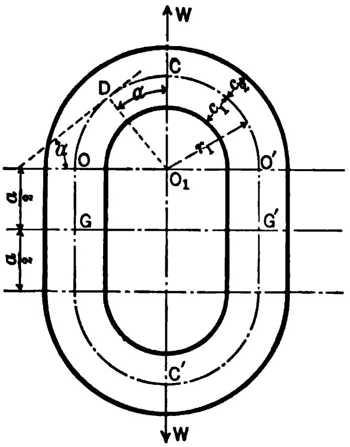


Fig. 245.

Letting W = the load and M_0, M_c and M represent the bending moments at O, C and D, as before, (Art. 180), we have

$$M = M_0 + \frac{Wr_1}{2}(1 - \sin\alpha). \quad \ldots\ldots\ldots \quad (1)$$

The change in the angle between the tangents at G and C due to the load W will be zero and, if there is no temperature change and we assume that the change in the angles between the tangents at O and G, for the straight portion OG, is equal to

$$\frac{M_0 a}{2\,EI} \text{ (equation 10, Art. 97),}$$

and that the change in the angle between the tangents at O and C, for the curved portion OC, is equal to

$$\frac{1}{EI}\left(-\frac{Wr_1^2}{2} + M_0 r_1 \frac{\pi}{2} + \frac{Wr_1^2}{2}\,\frac{\pi}{2}\right) \text{ (equation 7, Art. 180),}$$

we have

$$\frac{1}{EI}\left[M_0 r_1 \frac{\pi}{2} - \frac{W r_1^2}{2}\left(1 - \frac{\pi}{2}\right) + \frac{M_0 a}{2}\right] = 0 \quad \ldots\ldots \quad (2)$$

and solving for M_0,

$$M_0 = \frac{W r_1^2}{2}\left(\frac{2 - \pi}{a + \pi r_1}\right) \quad \ldots\ldots\ldots\ldots \quad (3)$$

Substituting in (1) and reducing we obtain

$$\begin{aligned} M &= \frac{W r_1^2}{2}\left(\frac{2 - \pi}{a + \pi r_1}\right) + \frac{W r_1}{2}(1 - \sin\alpha) \\ &= \frac{W r_1}{2}\left(\frac{a + 2 r_1}{a + \pi r_1} - \sin\alpha\right); \quad \ldots\ldots\ldots\ldots \quad (4) \end{aligned}$$

and at C, where $\alpha = 0$,

$$M_c = \frac{W r_1}{2}\left(\frac{a + 2 r_1}{a + \pi r_1}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Substituting in the equation for the greatest normal stress intensity at any cross section D, we have

$$f = \frac{W \sin\alpha}{2A} + \frac{W}{2(B - A)}\left(\frac{a + 2 r_1}{a + \pi r_1} - \sin\alpha\right)\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right); \ldots \quad (6)$$

for a cross section just to the left of C, the greatest normal stress intensity will be equal to

$$f = \frac{W}{2(B - A)}\left(\frac{a + 2 r_1}{a + \pi r_1}\right)\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right); \quad \ldots\ldots \quad (7)$$

for a cross section at O, assuming the distribution of stress to be that in the curved bar, the greatest normal stress intensity will be equal to

$$f = \frac{W}{2A} + \frac{W}{2(B - A)}\left(\frac{a + 2 r_1}{a + \pi r_1} - 1\right)\left(\frac{B}{A} - \frac{r_1}{r_1 - c_1}\right); \ldots \quad (8)$$

and for any cross section between O and G, assuming the distribution of stress to be that in the straight bar, the greatest normal stress intensity will be equal to

$$f = \frac{W}{2A} - \frac{W r_1 c_1}{2I}\left(\frac{a + 2 r_1}{a + \pi r_1} - 1\right) = \frac{W}{2A} - \frac{W r_1^2 c_1}{2I}\left(\frac{2 - \pi}{a + \pi r_1}\right). \quad . \quad (9)$$

The preceding equations give the stress intensities at the inside of the link, where $y = -c_1$. The expressions for the stress intensities at the outside of the link, at the cross sections D, C, O and G, can be obtained by substituting c_2 for $-c_1$ in equations (6), (7), (8) and (9).

A comparison of results will show that in a link of circular cross section the greatest intensity of the tension on the cross section at C will exceed that on the cross section at G, calculated from equation (9); and that the greatest intensity of the normal stress in the link will be the compressive stress at the inside of the cross section at C.

182. The Spiral Spring. — The flat spiral spring of the type used for driving clocks and other mechanisms is usually made up of a strip of metal of rectangular cross section, fastened to a pin at the center O and held by a hinge connection at the end A (Fig. 246). Usually the hinge A is stationary and the spring is wound up by turning the pin O, but obviously the stress and the distortion in the spring would be the same if the pin O were fixed and the winding were done by pulling the end A around the central axis.

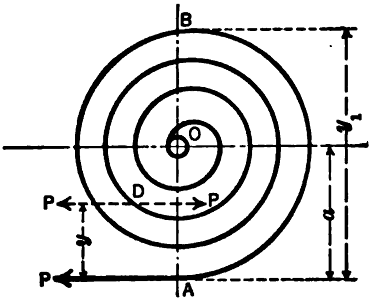


FIG. 246.

If P = the force applied at A, in the direction of the tangent to the spiral, the stress at any section D will be the resultant of a force equal and parallel to P, acting through the center of gravity of the section, and a couple $M = Py$. For different sections of the spiral, the couple M will have values varying from 0 to Py_1 and the average value will be nearly equal to

$$M = Pa. \quad \text{. . . . .} \quad (1)$$

The force P, acting at the center of gravity of any section, may be resolved into a normal and shearing component and the values of each of these components will vary, through the length of the spiral, from a minimum zero to a maximum P.

Except possibly in the portion close to the center, the radius of curvature will be so large, compared with the depth of the cross section, that the formulas for the straight beam will give accurate values for the stress intensity.

The greatest bending moment occurs at B and is equal to

$$M_0 = Py_1 = 2\,Pa \text{ (nearly)}; \quad \text{. . . . .} \quad (2)$$

and hence the greatest stress intensity due to bending will be equal to

$$f = \frac{M_0 c}{I}, \quad \text{. . . . .} \quad (3)$$

where $\frac{I}{c}$ = the section modulus of the spring, this stress intensity being greater than that near the center, where the curvature is greater but the bending moment is only half as large. The stress intensity $\frac{P}{A}$, due to the normal component P at the section B, will evidently be compression but will be so small as to be negligible.

The angular displacement α, of the point A relative to O, may be estimated by substituting the average value of M (equation 1) in equation (10) (Art. 178) and integrating over the entire length of the spiral; this angular displacement evidently being equal to the change in the angle between the tangents to the curve at A and O. Hence

$$\alpha = \frac{M}{EI}\int_{s=0}^{s=l} ds = \frac{Ml}{EI} = \frac{Pal}{EI} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

The resilience of the spring in terms of the above value of α will be equal to

$$R = \frac{M}{2}\alpha = \frac{P^2a^2l}{2EI} = \frac{f^2\rho^2}{2Ec^2}V, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

where f = the greatest intensity of stress, ρ = the radius of gyration of the cross section and V = the volume of the spring.

If the section is rectangular

$$R = \frac{f^2}{6E}V. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

183. The Carriage Spring. — The ordinary type of carriage spring is made up of curved bars, or *leaves*, clamped together at the center in the manner indicated in Fig. (247).

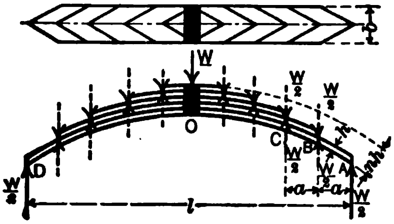


FIG. 247.

The flexibility of the spring is increased, without diminishing the strength, by shortening the successive leaves as shown in the sketch. The curvature of the leaves in the initial state, before clamping together, varies; the curvature of the shortest leaf being the greatest and that of the others being diminished as the length is increased.

The object of this is to attain the condition that the pressure between any two successive leaves of the built-up spring shall be concentrated, as nearly as possible, at the ends of the shorter of the two leaves.

Assuming that this condition is realized, the pressure at B between the first and second leaves of the spring, due to a load W at the center C, will be equal to $\frac{W}{2}$ and the bending moment in the portion OB of the first leaf will be uniform and equal to

$$M = \frac{W}{2}a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where a = the horizontal distance between A and B.

Therefore, if the portion of the leaf BO is of uniform section and the end BA is tapered, by varying the breadth alone as indicated, or in any other way, so as to form as nearly as possible a beam of uniform strength (Art. 85) the greatest fiber stress throughout the leaf will be constant; and hence, since the radius of curvature is large compared with the depth of a cross section, the greatest intensity of stress will be given by the formula

$$f = \frac{Mc}{I} = \frac{Wa}{2}\frac{c}{I}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Similarly, the pressure between the second and third leaves will be $\frac{W}{2}$ and, if the horizontal distance between B and C is equal to a, the bending moment in the portion OC of the second leaf will be uniform and equal to that in (1). Hence, if the cross section of the portion OC is the same as that in the first leaf and the end CB is tapered in the same manner as the end BA, the maximum stress intensity throughout the second leaf will be the same as in the first leaf.

It will follow, if the successive leaves are designed with the same tapered ends and the central portions with the same cross section, that the maximum stress intensity throughout the spring will be the same.

The deflection of the spring may be estimated as follows: Since the straight portions of all the leaves are assumed to be subjected to the same uniform bending moment the change in curvature of all will be the same and, since the initial curvature is small, will be equal in magnitude to

$$\frac{1}{r} - \frac{1}{r_1} = \frac{M}{EI} = -\frac{Wa}{2\,EI} \text{ (equation 7, Art. 178)}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the ends are tapered, as shown in the sketch, the change in curvature of the end BA will also be equal to the value represented by (3) (see equation 2, Case *a*, Art. 103).

Hence, for the magnitude of the deflection of O relative to the supports A and D, we have

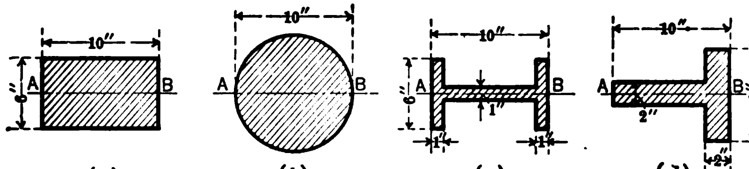
$$v_0 = \frac{Ml^2}{8\,EI} = \frac{Wal^2}{16\,EI} \text{ (Case } a\text{, Art. 97)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The actual deflection in the spring will differ from this, owing to the effect the friction between the leaves and the fact that the ideal conditions imposed in the theory are not realized in the actual spring.

184. Problems — Curved Bars.

Problem 1.

Find the moment of resistance of each of the cross sections shown in Fig. (248) assuming the central axis of the member in each case to be a curve lying in the plane of symmetry *AB* and that the radius of curvature of the central axis is 20″ and the greatest normal stress intensity is 5000 lbs. per sq. in



(*a*) (*b*) (*c*) (*d*)

Fig. 248.

Problem 2.

Solve Problem (35) (Art. 134), using the formula for the curved bar.

Problem 3.

Find the load that may be carried by a hook, having the cross section shown in Fig. (249), the center of curvature of the central axis being in the line *AB* produced and 3″ to the right of *B*. Assume $f = 8000$ lbs. per sq. in. and that the line of action of the resultant load passes through the center of curvature.

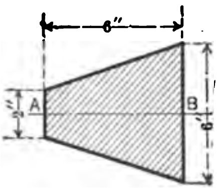


Fig. 249.

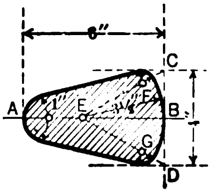


Fig. 250.

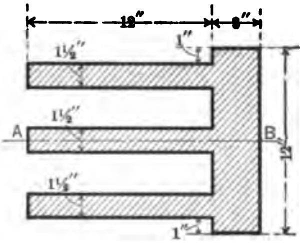


Fig. 251.

Problem 4.

Find the greatest intensity of the tensile and compressive stress in a hook, having the cross section shown in Fig. (250), the center of curvature of the central axis being in the axis of symmetry *AB* and 3″ to the right of *B*. The load is 20,000 lbs. and its line of action passes through the center of curvature.

Problem 5.

Find the pressure that may be exerted by a curved rocker arm, having the cross section shown in Fig. (251), the center of curvature of the central axis being in the axis of symmetry *AB* and 25″ to the right of *B*. The line of action of the resultant pressure, exerted by the arm is 30″ to the right of *B*. The greatest allowable intensity of the normal stress is 6000 lbs. per sq. in.

Problem 6.

Find the stress intensities at the intrados and the extrados at the cross sections *O* and *C* of the circular ring (Fig. 243), due to a pull of 6000 lbs. through the center of the ring, assuming the cross section to be 1¼″ diameter and the radius of the central axis to be equal to 3″:

(a) By the first method (Art. 180);

(b) By the second method (Art. 180).

Problem 7.

Find the decrease in the diameter *OO′* and the increase in the diameter *CC′* of the ring given in Problem (6):

(a) By the first method (Art. 180);

(b) By the second method (Art. 180).

Problem 8.

Find the allowable load *W* for a chain link (Fig. 245) having a cross section ¾″ diameter, the radius of the central axis at the ends being 1¼″ and the length of the straight portion 1½″. The greatest allowable intensity of normal stress is 8000 lbs. per sq. in. Find the greatest intensity of the tensile stress in the straight portions of the link.

CHAPTER XII.

ARCHES AND CATENARIES.

185. The Arch. — An arch may be defined as a member, or structure, whose central axis is a plane curve which is attached through hinges, or otherwise, to fixed or unyielding supports and is usually designed in such a manner that the bending moments due to transverse loading are offset, as largely as possible, by the moments of the reactions at the supports. In very special cases bending may be eliminated entirely, the resultant of the stress at every cross section coinciding with the central axis. The arch may be *solid*, having a cross section similar to that of a beam or built-up girder, or, it may be a *braced arch*, made up of tension and compression members like a simple truss. We shall consider the methods of determining the stress in the solid type only.

The solid arch may be treated as a curved bar, subjected to the action of external forces acting in the plane of curvature and, when the external forces are known, the bending moment and the normal force, or *thrust*, acting through the center of gravity, at any cross section can be found by the method in Art. (176). In ordinary cases, the radius of curvature is so large, compared with the dimensions of the cross section, that the stress intensity at any point can be calculated with sufficient accuracy by use of the formula for the straight bar,

$$f = \frac{P}{A} \pm \frac{My}{I} \text{ (Art. 125)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

rather than the more complex formulas in Art. (176).

Similarly, the displacements at any point can be accurately determined by use of the approximate formulas (19–21) (Art. 178) and in many cases the more approximate equations (8–10), of the same article, are sufficiently accurate.

Three cases will be considered, involving three different ways of supporting the arch. The determination of the stresses and displacements in two of these cases is somewhat difficult, owing to the fact that the reactions at the points of support cannot be determined from the statical conditions of equilibrium alone.

CASE I. *Three Hinged Arch.* — In this case, the arch is made up of two parts, or *ribs*, supported on hinges at O and E and connected with a third hinge at C (Fig. 252). When the external loads W_1, W_2, etc., are known, the horizontal and vertical components H_0, V_0, etc., of the reactions at the hinges can be easily computed by applying the statical conditions of equilibrium. The equation for the bending moment at any point D, whose coördinates with respect to horizontal and vertical axes through O are (x, y), may then be written

$$M = H_0 y + \Sigma Wa - V_0 x, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

where ΣWa = the sum of the moments about D of the forces acting between D and O, moments tending to increase the curvature being taken as positive.

Resolving the loads W_1, W_2, etc., into H and V components, H_1, V_1, H_2, V_2, etc., the equation for the thrust at the center of gravity of the cross section at D may be written

$$P = \Sigma H (\cos\alpha) + \Sigma V (\sin\alpha), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

where $\Sigma H = H_0 + H_1 +$ etc., $\Sigma V = V_0 - V_1 -$ etc., the summation being taken between O and D, and α = the angle between the tangent at D and the horizontal axis OX.

Similarly, the magnitude of the shearing force at D will be equal to

$$S = \Sigma H (\sin\alpha) - \Sigma V (\cos\alpha). \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

Having the values of M and P, the stress intensity at any point in the cross section D may be calculated by use of equation (1).

The foregoing solution would evidently apply equally as well if the hinges O and E were not on the same horizontal level as shown (Fig. 252), the H and V components in such a case being taken to represent the components respectively parallel and perpendicular to the axis OX, through the hinges O and E.

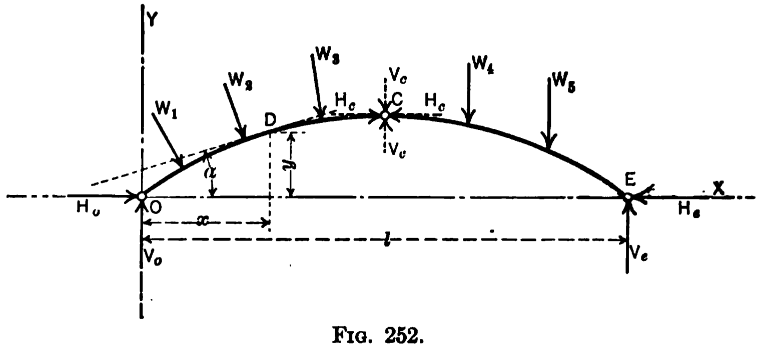


FIG. 252.

If the arch were subjected to a *distributed load*, a solution could evidently be made by dividing the load into small parts ΔW, resolving each increment into H and V components and making the summations indicated in (2) and (3) as before.

Case II. *Two Hinged Arch. First Solution.* — In this case the arch consists of a single rib held by hinges at the supports O and C (Fig. 253). When the loads W_1, W_2, etc., are known, the components V_0 and V_c at the hinges O and C, can be calculated by use of the statical conditions of equilibrium; but these conditions will fail to give a solution for the values of H_0 and H_c. The component H_0 can be determined, however, on the assumption that the supports are rigidly fixed and hence the displacement of the hinge O, relative to C, is equal to zero. Equation (2) may be written in the form

$$M = H_0 y + K, \quad \ldots \ldots \ldots \ldots \quad (5)$$

where

$$K = \sum_0^x Wa - V_0 x, \quad \ldots \ldots \ldots \ldots \quad (6)$$

which is the part of M which can be determined from the statical conditions of equilibrium alone.

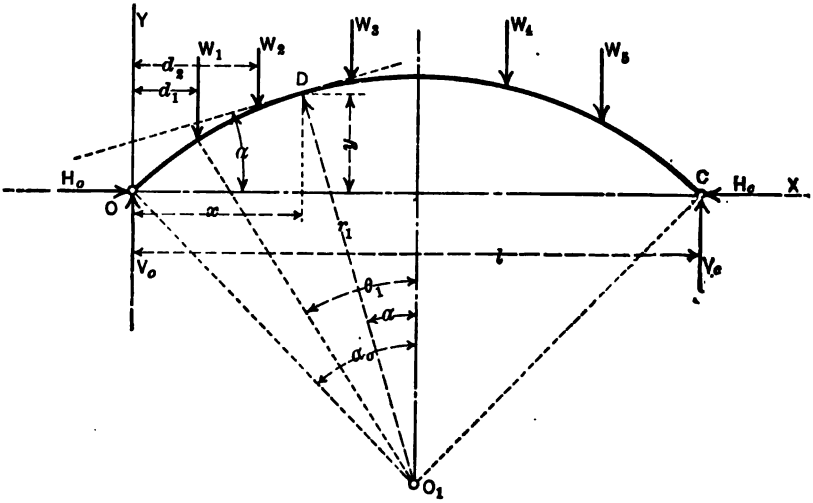


Fig. 253.

Substituting this value of M in equation (8), (Art. 178), we have, on the basis of the above assumption,

$$\delta x = \int_{x=0}^{x=l} \frac{H_0 y + K}{EI} y\, ds = H_0 \int_{x=0}^{x=l} \frac{y^2}{EI} ds + \int_{x=0}^{x=l} \frac{K}{EI} y\, ds = 0. \quad (7)$$

By solving (7) the value of H_0 can be obtained. Except in comparatively simple cases, however, the solution is complicated and hence the following illustrations are restricted to the arch of uniform cross section and material, in which the central axis is the arc of a circle, the end hinges are on the same level and the loads are vertical (Fig. 253). In such a case

$$H_0 = -\frac{\int Ky\, ds}{\int y^2\, ds} \quad \ldots \ldots \ldots \ldots \quad (8)$$

Having the values of H_0 and V_0, the bending moment, the thrust, and the maximum normal stress intensity, at any cross section, can be found from equations (2), (3) and (1) in the same manner as in the three hinged arch.

Concentrated Loads. — We will let l = the span OC, r_1 = the radius of curvature of the central axis, α_0 = the angle between OX and the tangent to the central axis at O, α = the angle between OX and the tangent to the central axis at any point D, whose coördinates are (x, y), and θ_1, θ_2, etc., equal the angles between OX and the tangents to the central axis at the points of intersection with the loads W_1, W_2, etc., which are located at distances d_1, d_2, etc., from the axis OY. Then $l = 2r_1 \sin\alpha_0$, $x = r_1(\sin\alpha_0 - \sin\alpha)$, $y = r_1(\cos\alpha - \cos\alpha_0)$, $d_1 = r_1(\sin\alpha_0 - \sin\theta_1)$, $(x - d_1) = r_1(\sin\theta_1 - \sin\alpha)$, etc., $(l - d_1) = r_1(\sin\alpha_0 + \sin\theta_1)$, $ds = -r_1\,d\alpha$, $dx = -r_1\cos\alpha\,d\alpha$ and $dy = -r_1\sin\alpha\,d\alpha$.

If the load W_1 were the only load on the arch, equation (6), for values of x from 0 to d_1, would take the form

$$K' = -V_0'x = -V_0'r_1(\sin\alpha_0 - \sin\alpha), \quad \ldots\ldots \quad (9)$$

and for values of x from d_1 to l,

$$K' = W_1(x - d_1) - V_0'x = W_1r_1(\sin\theta_1 - \sin\alpha) - V_0'r_1(\sin\alpha_0 - \sin\alpha), \quad (10)$$

the value of V_0' being

$$V_0' = \frac{W_1(l - d_1)}{l} = \frac{W_1(\sin\alpha_0 + \sin\theta_1)}{2\sin\alpha_0}. \quad \ldots\ldots \quad (11)$$

Substituting the values of K' in (7), observing that when $x = l$, $\alpha = -\alpha_0$, when $x = 0$, $\alpha = \alpha_0$, and when $x = d_1$, $\alpha = \theta_1$, and reducing and integrating we have

$$\begin{aligned}
\delta x = &-\frac{H_0'r_1^3}{EI}\int_{\alpha_0}^{-\alpha_0}(\cos\alpha - \cos\alpha_0)^2\,d\alpha \\
&-\frac{W_1r_1^3}{EI}\int_{\theta_1}^{-\alpha_0}(\sin\theta_1 - \sin\alpha)(\cos\alpha - \cos\alpha_0)\,d\alpha \\
&+\frac{V_0'r_1^3}{EI}\int_{\alpha_0}^{-\alpha_0}(\sin\alpha_0 - \sin\alpha)(\cos\alpha - \cos\alpha_0)\,d\alpha \\
= &\frac{r_1^3}{EI}\big[H_0'(\alpha_0 + 2\,\alpha_0\cos^2\alpha_0 - 3\sin\alpha_0\cos\alpha_0) \\
&- W_1\{(\theta_1 + \alpha_0)\sin\theta_1\cos\alpha_0 + (\cos\theta_1 - \cos\alpha_0)\cos\alpha_0 \\
&- \sin\theta_1\sin\alpha_0 - \tfrac{1}{2}(\sin^2\theta_1 + \sin^2\alpha_0)\} \\
&- 2\,V_0'(\sin^2\alpha_0 - \alpha_0\sin\alpha_0\cos\alpha_0)\big] = 0. \quad \ldots\ldots\ldots \quad (12)
\end{aligned}$$

Substituting the value of V_0' (equation 11) and solving for H_0', the horizontal component of the reaction at O due to the load W_1, we obtain

$$H_0' = \frac{W_1[(\theta_1\sin\theta_1 - \alpha_0\sin\alpha_0 + \cos\theta_1 - \cos\alpha_0)\cos\alpha_0 + \frac{1}{2}(\sin^2\alpha_0 - \sin^2\theta_1)]}{\alpha_0 + 2\,\alpha_0\cos^2\alpha_0 - 3\sin\alpha_0\cos\alpha_0} \quad .. \quad (13)$$

The expression for the value of H_0'', the horizontal component at O, due to the load W_2 acting alone, would evidently be an equation of the same form as (13), with the angle θ_2 replacing θ_1.

Hence the value of $H_0 = H_0' + H_0'' +$ etc., the horizontal component at O due to the entire system of vertical loads, will be represented by the expression

$$H_0 = \frac{\sum_0^l W\left[(\theta \sin\theta - \alpha_0 \sin\alpha_0 + \cos\theta - \cos\alpha_0)\cos\alpha_0 + \frac{1}{2}(\sin^2\alpha_0 - \sin^2\theta)\right]}{\alpha_0 + 2\,\alpha_0\cos^2\alpha_0 - 3\sin\alpha_0\cos\alpha_0}, \qquad (14)$$

where W represents any vertical load and θ = the angle between OX and the tangent to the central axis at its point of application and the summation includes all the loads on the arch. Since the loads are vertical, H_0 will evidently represent the horizontal component of the thrust at any cross section.

Uniform Load. — If the arch were subjected to a *uniform load* w per unit length of the span, the value of H_0 could be determined by substituting $w\,dx$ for W in equation (14) and integrating, or, it may be determined directly by substituting the value of K for this case in equation (7) and integrating, as follows: The expression for K (equation 6) will take the form

$$K = \frac{wx^2}{2} - \frac{wlx}{2} = \frac{wr_1^2}{2}(\sin^2\alpha - \sin^2\alpha_0) \quad \ldots\ldots \quad (15)$$

and substituting in (7) and integrating

$$\begin{aligned}\delta x = &-\frac{H_0 r_1^3}{EI}\int_{\alpha_0}^{-\alpha_0}(\cos\alpha - \cos\alpha_0)^2\,d\alpha \\ &-\frac{wr_1^4}{2\,EI}\int_{\alpha_0}^{-\alpha_0}(\sin^2\alpha - \sin^2\alpha_0)(\cos\alpha - \cos\alpha_0)\,d\alpha \\ = &\frac{r_1^3}{EI}\left[H_0(\alpha_0 + 2\,\alpha_0\cos^2\alpha_0 - 3\sin\alpha_0\cos\alpha_0)\right. \\ &- wr_1\left\{\sin^2\alpha_0\left(\tfrac{2}{3}\sin\alpha_0 - \alpha_0\cos\alpha_0\right)\right. \\ &\left.\left.+ \tfrac{1}{2}\cos\alpha_0(\alpha_0 - \sin\alpha_0\cos\alpha_0)\right\}\right] = 0, \quad \ldots\ldots\ldots \quad (16)\end{aligned}$$

the first term in the integral being the same as in (12).

Solving (16), we obtain

$$H_0 = \frac{wr_1\left[\sin^2\alpha_0\left(\frac{2}{3}\sin\alpha_0 - \alpha_0\cos\alpha_0\right) + \frac{1}{2}\cos\alpha_0(\alpha_0 - \sin\alpha_0\cos\alpha_0)\right]}{\alpha_0 + 2\,\alpha_0\cos^2\alpha_0 - 3\sin\alpha_0\cos\alpha_0} \,.\,. \quad (17)$$

Two Hinged Arch. Second Solution. — A more accurate solution for the value of H_0 can be made by allowing for the normal thrust P, as indicated in equation (19) (Art. 178), when applying the condition that the displacement δx_1 of O, relative to C, is equal to zero. The effect of a temperature change on the reactions at the hinges may also be included in the solution.

Concentrated Loads. — When the arch is subjected to vertical concentrated loads the expression for the thrust at any section D (equation 3) may be written

$$P = H_0\cos\alpha + \left(V_0 - \sum_0^x W\right)\sin\alpha \quad \ldots\ldots \quad (18)$$

and the expression for K at the section D (equation 6) may be written

$$K = \sum_0^x W(x - d) - V_0 x, \quad \ldots\ldots\ldots \quad (19)$$

where

$$V_0 = \frac{\sum_0^l W(l - d)}{l} = \frac{\sum_0^l W(\sin\alpha_0 + \sin\theta)}{2\sin\alpha_0} \quad \ldots\ldots \quad (20)$$

Substituting the values of P and $M = H_0 y + K$ in (19) (Art. 178), putting $\delta x_1 = 0$, and observing that the strain due to the thrust P will be compression, we have

$$\delta x_1 = \frac{H_0}{EI}\int y^2\,ds + \frac{1}{EI}\int \sum_0^l W\,(x-d)\,y\,ds - \frac{V_0}{EI}\int xy\,ds$$
$$+ \frac{H_0}{AE}\int \cos\alpha\,dx + \frac{1}{AE}\int \Big(V_0 - \sum_0^l W\Big)\sin\alpha\,dx - \epsilon T\int dx = 0, \qquad (21)$$

the integration being taken over the entire span.

The values of the integrals may be expressed as follows, the values of the first three being in the same form as in equation (12),

$$\frac{H_0}{EI}\int_{x=0}^{x=l} y^2\,ds = \frac{r_1^3}{EI}H_0(\alpha_0 + 2\,\alpha_0\cos^2\alpha_0 - 3\sin\alpha_0\cos\alpha_0), \quad . \quad (22)$$

$$\frac{1}{EI}\int_{x=d}^{x=l}\sum_0^l W\,(x-d)\,y\,ds = -\frac{r_1^3}{EI}\Big[\sum_0^l W\,\{(\theta+\alpha_0)\sin\theta\cos\alpha_0$$
$$+ (\cos\theta - \cos\alpha_0)\cos\alpha_0 - \sin\theta\sin\alpha_0 - \tfrac{1}{2}(\sin^2\theta + \sin^2\alpha_0)\}\Big], \quad (23)$$

where W = any load and θ = the angle between OX and the tangent to the central axis at the point of application of W,

$$-\frac{V_0}{EI}\int_{x=0}^{x=l} xy\,ds = -\frac{r_1^3}{EI}\,2\,V_0\,(\sin^2\alpha_0 - \alpha_0\sin\alpha_0\cos\alpha_0)$$
$$= -\frac{r_1^3}{EI}\Big[\sum_0^l W\,(\sin^2\alpha_0 - \alpha_0\sin\alpha_0\cos\alpha_0 + \sin\alpha_0\sin\theta - \alpha_0\sin\theta\cos\alpha_0)\Big], \quad (24)$$

$$\frac{H_0}{AE}\int_{x=0}^{x=l}\cos\alpha\,dx = -\frac{H_0 r_1}{AE}\int_{\alpha}^{-\alpha_0}\cos^2\alpha\,d\alpha = \frac{r_1}{AE}H_0\,(\alpha_0 + \sin\alpha_0\cos\alpha_0), \quad (25)$$

$$\frac{V_0}{AE}\int_{x=0}^{x=l}\sin\alpha\,dx = -\frac{V_0 r_1}{AE}\int_{\alpha_0}^{-\alpha_0}\sin\alpha\cos\alpha\,d\alpha = 0, \quad . \ . \quad (26)$$

$$-\frac{1}{AE}\int_{x=d}^{x=l}\sum^l W\sin\alpha\,dx = \frac{r_1}{AE}\int_{\theta}^{-\alpha_0}\sum_0^l W\sin\alpha\cos\alpha\,d\alpha$$
$$= \frac{r_1}{2\,AE}\sum_0^l W\,(\sin^2\alpha_0 - \sin^2\theta), \quad . \quad (27)$$

$$-\epsilon T\int_{x=0}^{x=l} dx = -\epsilon Tl. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (28)$$

Substituting these values in (21) and solving for H_0, we obtain

$$H_0 = \frac{\begin{array}{l}\sum_0^l W\,[(\theta\sin\theta - \alpha_0\sin\alpha_0 + \cos\theta - \cos\alpha_0)\cos\alpha_0 \\ \quad + \tfrac{1}{2}(\sin^2\alpha_0 - \sin^2\theta) - \dfrac{I}{2\,Ar_1^2}(\sin^2\alpha_0 - \sin^2\theta)] + \dfrac{EI}{r_1^3}\epsilon Tl\end{array}}{(\alpha_0 + 2\,\alpha_0\cos^2\alpha_0 - 3\sin\alpha_0\cos\alpha_0) + \dfrac{I}{Ar_1^2}(\alpha_0 + \sin\alpha_0\cos\alpha_0)}. \quad (29)$$

It will be observed that the first terms in the numerator and denominator of (29) are the same as in equation (14).

Uniform Load. — In the case of the arch subjected to a *uniform load* w per unit of span, if we allow for the effect of the thrust and change of temperature, equation (17) will be modified in a similar manner.

The expression for the thrust at any cross section will be

$$P = H_0 \cos\alpha + (V_0 - wx) \sin\alpha, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

which differs in the last term only from equation (18), and the expression for K will be the same as (15).

Substituting in equation (19) (Art. 178), and reducing, observing that the strain due to the thrust P will be compression, we have

$$\begin{aligned}\delta x_1 = &- \frac{H_0 r_1^3}{EI} \int_{\alpha_0}^{-\alpha_0} (\cos\alpha - \cos\alpha_0)^2\, d\alpha - \frac{wr_1^4}{2\,EI} \int_{\alpha_0}^{-\alpha_0} (\sin^2\alpha - \sin^2\alpha_0) \\ &(\cos\alpha - \cos\alpha_0)\, d\alpha - \frac{H_0 r_1}{AE} \int_{\alpha_0}^{-\alpha_0} \cos^2\alpha\, d\alpha - \frac{V_0 r_1}{AE} \int_{\alpha_0}^{-\alpha_0} \sin\alpha \cos\alpha\, d\alpha \\ &+ \frac{wr_1^2}{AE} \int_{\alpha_0}^{-\alpha_0} (\sin\alpha_0 - \sin\alpha) \sin\alpha \cos\alpha\, d\alpha - \epsilon T \int_0^l dx = 0. \quad . \; . \quad (31)\end{aligned}$$

The values of all the integrals are given in equations (16), (25), (26) and (28), with the exception of

$$\frac{wr_1^2}{AE} \int_{\alpha_0}^{-\alpha_0} (\sin\alpha_0 - \sin\alpha) \sin\alpha \cos\alpha\, d\alpha = \frac{wr_1^2}{AE}\, \frac{2}{3} \sin^3\alpha_0, \quad . \; . \quad (32)$$

and substituting these values in (31) and solving for H_0, we obtain

$$H_0 = \frac{\begin{array}{l} wr_1\, [\sin^2\alpha_0\, (\frac{2}{3} \sin\alpha_0 - \alpha_0 \cos\alpha_0) \\ \quad + \frac{1}{2} \cos\alpha_0\, (\alpha_0 - \sin\alpha_0 \cos\alpha_0)] - \dfrac{I}{Ar_1^2}\, \dfrac{2}{3} \sin^3\alpha_0 + \dfrac{EI}{r_1^3}\, \epsilon Tl \end{array}}{(\alpha_0 + 2\,\alpha_0 \cos^2\alpha_0 - 3 \sin\alpha_0 \cos\alpha_0) + \dfrac{I}{Ar_1^2}\, (\alpha_0 + \sin\alpha_0 \cos\alpha_0)} \quad . \; . \; . \quad (33)$$

Braced Arch. — In the braced arch, or, the solid arch of varying cross section, subjected to vertical loads, the value of H_0 may be estimated approximately by substituting the average values of I and A in the foregoing equations.

Case III. *Arch with Fixed Ends.* — In this case the ends of the arch are fixed in direction, similar to the beam with fixed ends (Art. 101), and a bending moment exists at each support. If we let M_0 and M_c equal the bending moments at the supports O and C, respectively (Fig. 253), the equation for the bending moment at any section D will take the form,

$$M = M_0 + H_0 y + \sum_0^x Wa - V_0 x, \quad . \; . \; . \; . \; . \; . \quad (34)$$

and the value of the thrust P will be represented by (3) as in the two preceding cases.

When the loads are vertical and the supports are on the same level (34) may be written in the form

$$M = M_0 + H_0 y + K, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (35)$$

the value of K being represented by (19). The value of P will be given by equation (18). Since the ends of the arch are fixed in direction, the change in the angle between the tangents at O and C, represented by equation (10), or (18) (Art. 178), will be equal to zero and this condition, in combination with the conditions that the horizontal and vertical displacements of the supports are equal to zero, will furnish a solution for the unknown quantities M_0, V_0 and H_0, the details being worked out in the same manner as in Case II.

186. Inverted Arch. — If the arch in any of the preceding cases were inverted and subjected to loads acting vertically upwards, the equations for H_0 would evidently take exactly the same form as those of Art. (185).

If the inverted arch in any case were subjected to loads acting *vertically downwards*, the signs of both the bending moment and the normal force at any cross section would be reversed and hence, by reversing the sign of the temperature term only, the equations for the corresponding case in Art. (185) would give the magnitude of H_0 for such a case.

187. Flexible Cords. — The determination of the forms taken by a flexible cord, or chain, fastened at the ends, when subjected to certain types of loading is of considerable importance. Evidently in such a cord there can be no bending, the resultant stress at every cross section being uniform tension.

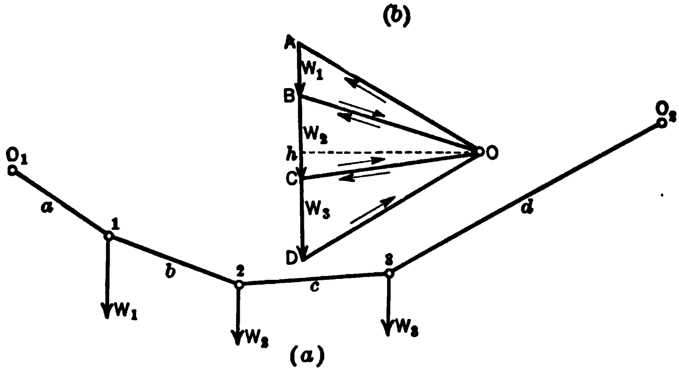


FIG. 254.

Concentrated Loads. — If the loads are concentrated and the weight of the cord is negligible, the form taken will be a broken line, or part of a polygon (Fig. 254 *a*), the shape of which must be such that the stresses in the sections, or strings, on either side of any load will balance that load.

If triangles are constructed, representing the forces acting at the point of application of each load, and the triangles for two consecutive points, such as 1 and 2, are placed so that the sides which represent the force in the section of the cord between them coincide, the force triangles for the entire load system will form a diagram (Fig. 254 *b*), in which the loads are represented by the vectors AB, BC, etc., and the stresses in the sections a, b, etc., of the cord, are represented by the vectors OA, OB, etc., diverging from the point O.

The diagram (Fig. 254 *b*) is called the ***stress diagram*** and, if the loads are vertical, the vectors W_1, W_2, etc., will form a straight line and the horizontal component of the stress in every section of the cord will be equal to Oh.

The broken line formed by the cord (Fig. 254 *a*) may be called the *equilibrium polygon.*

Distributed Loads. — If a flexible cord is subjected to a distributed load it will take the form of a curve which may be called a *catenary.* When the distributed load is vertical the differential equation of the curve may be obtained as follows: Let O be the lowest point in the cord (Fig. 255 *a*) which is suspended from the points A and C, the equilibrium polygon in this case being the curve AOC.

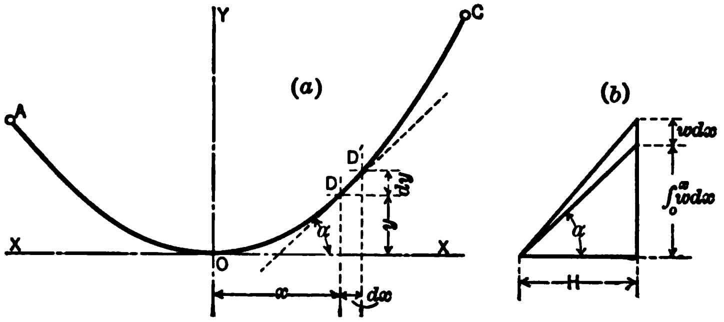


FIG. 255.

Let w = the load per unit of horizontal distance, T = the tension at the point D, whose coördinates with respect to the horizontal and vertical axes through O are (x, y), and let H = the tension at O, which is evidently equal to the horizontal component of T. The total load on the cord between the points O and D will be equal to $\int_0^x w\,dx$ and, if α = the angle between OX and the tangent at D,

$$\tan\alpha = \frac{dy}{dx} = \frac{\int_0^x w\,dx}{H} \quad \text{(Fig. 255b)}. \quad . \quad . \quad . \quad . \qquad (1)$$

Hence if D' is a point on the curve, whose coördinates are

$(x + dx,\ y + dy)$, the increment in $\tan\alpha$ between D and D' will be equal to

$$d\left(\frac{dy}{dx}\right) = \frac{w\,dx}{H},$$

or,

$$\frac{d^2y}{dx^2} = \frac{w}{H}, \quad \ldots \ldots \ldots \quad (2)$$

which is the equation of any vertically loaded flexible cord, referred to horizontal and vertical axes through its lowest point.

Uniform Load. — When the cord is subjected to a uniformly distributed load w per unit of distance horizontally, we obtain by integrating (2)

$$\frac{dy}{dx} = \frac{w}{H}x, \quad \ldots \ldots \ldots \quad (3)$$

the constant of integration being zero, since

$$\frac{dy}{dx} = 0, \text{ when } x = 0,$$

and by integrating again

$$y = \frac{w}{2H}x^2, \quad \ldots \ldots \ldots \quad (4)$$

the constant being zero, since $y = 0$, when $x = 0$.

Therefore, the equilibrium curve in this case is a parabola. The tension at any point D will be equal to

$$T = \sqrt{H^2 + (wx)^2} \quad \ldots \ldots \ldots \quad (5)$$

and this will evidently be a maximum at a point of support, the difference between the maximum tension and H depending on the *sag*, or *dip*, of the lowest point of the curve.

If we let (x_1, y_1) and (x_2, y_2) equal the coördinates of the supports C and A respectively (Fig. 255), and l = the horizontal length of the span, we shall have

$$y_1 = \frac{w}{2H}x_1^2 \quad \ldots \ldots \ldots \quad (6)$$

and

$$y_2 = \frac{w}{2H}x_2^2 = \frac{w}{2H}(l - x_1)^2. \quad \ldots \ldots \ldots \quad (7)$$

Therefore, when the span and the coördinates y_1 and y_2 of the supports are known, the coördinate x_1 and the value of H, for any load intensity w, can be obtained by solving (6) and (7) simultaneously. When the supports are on the same level, the lowest point is in the center of the span and the solution can be made by the use of (6) alone.

Variable Load. — When w is variable the equation of the curve and values of H and T can be obtained in a similar manner, provided w can be expressed as an integrable function of x.

Linear Arch. — It is evident that if a member designed to resist compression and having the form of the flexible cord were inverted and subjected to the same load the stress on every cross section would be uniform compression.

Hence if the central axis of an arch, designed to support a uniform load per unit length of span, were a curve represented by equation (4), the resultant stress on every cross section would be uniform compression and there would be no bending in the arch. Such a curve is, therefore, sometimes called a *linear arch*, or more frequently, the *equilibrium curve* for the arch.

188. The Common Catenary. — If a flexible cord of uniform section and material is suspended at the ends and hangs freely under its own weight (Fig. 256), the curve formed by the cord is known as the common catenary. The equation of the curve, referred to the axes OX and OY through the lowest point, may be obtained as follows:

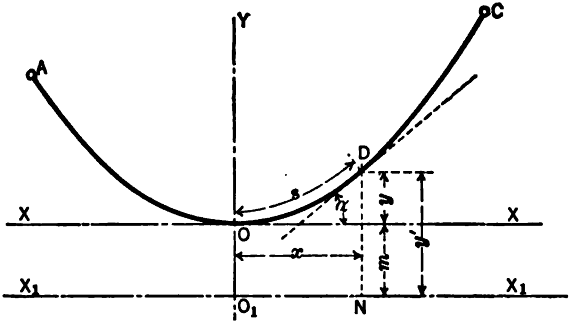


FIG. 256.

Let w_1 = the weight per unit length of the cord. The value of w (Art. 187) will then be equal to

$$w = w_1 \frac{ds}{dx} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Substituting this value in (2) (Art. 187) we have,

$$\frac{d^2y}{dx^2} = \frac{w_1}{H}\frac{ds}{dx} = \frac{1}{m}\sqrt{1+\left(\frac{dy}{dx}\right)^2} \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $m = \frac{H}{w_1}$ = a constant. Equation (2) may be written in the form

$$\frac{\frac{d^2y}{dx^2}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{1}{m}$$

and integrating, we obtain

$$\log_e \left[\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}\right] = \frac{x}{m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the constant of integration being zero. Hence,

$$\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} = e^{\frac{x}{m}}.$$

Transposing and squaring both sides of the equation,

$$1+\left(\frac{dy}{dx}\right)^2=\left(\frac{dy}{dx}\right)^2-2\,e^{\frac{x}{m}}\frac{dy}{dx}+e^{\frac{2x}{m}},$$

and therefore,

$$\frac{dy}{dx}=\frac{1}{2}\left(e^{\frac{x}{m}}-e^{-\frac{x}{m}}\right)=\sinh\frac{x}{m}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Integrating again

$$y=\frac{m}{2}\left(e^{\frac{x}{m}}+e^{-\frac{x}{m}}\right)-m=m\left(\cosh\frac{x}{m}-1\right), \quad . \quad . \quad . \quad . \qquad (5)$$

where $-m$ = the constant of integration. We also have

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{1}{2}\left(e^{\frac{x}{m}}+e^{-\frac{x}{m}}\right), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

and by integrating we obtain for the length of the portion of the curve between O and any point D,

$$s=\frac{m}{2}\left(e^{\frac{x}{m}}-e^{-\frac{x}{m}}\right)=m\sinh\frac{x}{m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

the constant of integration being zero.

If we transfer the equations to the axes O_1X_1 and O_1Y, where $OO_1=m$, (4) and (6) will remain in the same form and (5) will become

$$y'=y+m=\frac{m}{2}\left(e^{\frac{x}{m}}+e^{-\frac{x}{m}}\right)=m\cosh\frac{x}{m}. \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

By expanding the values of $\sinh\frac{x}{m}$ and $\cosh\frac{x}{m}$ in series and reducing, equation (5) will become

$$y=\frac{x^2}{2m}+\frac{x^4}{24m^3}+\frac{x^6}{720m^5}+\cdots \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Similarly, equation (7) will become

$$s=x+\frac{x^3}{6m^2}+\frac{x^5}{120m^4}+\cdots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The tension at the point D will evidently be equal to

$$T=\sqrt{H^2+(w_1s)^2}=w_1\sqrt{m^2+s^2}=w_1y'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

By transposing (6) and multiplying by w_1, we obtain

$$w_1\,ds=\frac{w_1}{2}\left(e^{\frac{x}{m}}+e^{-\frac{x}{m}}\right)dx=\frac{w_1}{m}\,y'\,dx=\frac{w_1^2}{H}\,y'\,dx. \quad . \quad . \quad . \qquad (12)$$

Hence the total weight of the section of the cord between O and D will be

$$w_1s=\frac{w_1^2}{H}\int_0^x y'\,dx=\frac{w_1^2}{H}\,(\text{area } O_1ODN); \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

that is, the load on any section of the catenary is equal to the area between the curve and the axis O_1X_1 multiplied by a constant $\frac{w_1^2}{H}=\frac{w_1}{m}$.

When the dip of the curve is small the higher powers of x in (9) and (10) may be neglected and

$$y = \frac{x^2}{2m} = \frac{w_1 x^2}{2H} \text{ (very nearly)}, \quad \ldots \ldots \ldots \ldots \quad (14)$$

$$s = x + \frac{x^3}{6m^2} = x + \frac{w_1^2 x^3}{6H^2} \text{ (very nearly)}, \quad \ldots \ldots \quad (15)$$

and the tension at any point in the cord (equation 11) will be equal to

$$T = H \text{ (very nearly)}. \quad \ldots \ldots \ldots \ldots \quad (16)$$

Therefore, in such a case the catenary very nearly coincides with the parabola.

Transformed Catenary. — It is evident that an inverted catenary would be the equilibrium curve for an arch designed to carry a vertical load, the intensity of which varies directly as the ordinates between the curve and a horizontal line at a definite distance $m = \frac{H}{w_1}$ above the highest point in the curve.

A more general type of an equilibrium curve for an arch would be that for a distributed load, whose intensity varies directly as the ordinate between the curve and a horizontal line at any distance d above the highest point. The equation of such a curve may be derived by a method similar to that employed in the case of the common catenary. If we let OO_1 (Fig. 256) represent the distance d, the load intensity at any point D will be equal to wy', where w = the load represented by each unit of the area between the curve AOC and the horizontal line $X_1O_1X_1$. The differential equation of the curve will be

$$\frac{d^2y'}{dx^2} = \frac{w}{H} y', \quad \ldots \ldots \ldots \ldots \ldots \quad (17)$$

which may be written, substituting y for y',

$$\frac{dy}{dx} d\left(\frac{dy}{dx}\right) = \frac{w}{H} y\, dy.$$

Integrating, observing that when $\frac{dy}{dx} = 0$, $y = d$,

$$\frac{dy}{dx} = \sqrt{\frac{w}{H}(y^2 - d^2)} = \frac{1}{m}\sqrt{y^2 - d^2}, \quad \ldots \ldots \ldots \quad (18)$$

where $$m^2 = \frac{H}{w}.$$

Equation (18) may be written

$$\frac{dy}{\sqrt{y^2 \quad d^2}} \quad \frac{1}{m} d$$

and integrating, observing that when $x = 0$, $y = d$,

$$\log_e\left[\frac{y + \sqrt{y^2 - d^2}}{d}\right] = \frac{x}{m}, \quad \ldots \ldots \ldots \quad (19)$$

which reduces to

$$y = \frac{d}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right), \quad \ldots \ldots \ldots \ldots \quad (20)$$

which becomes the equation of the common catenary when $d = m$.

189. Effect of Temperature and Load Changes. — When the dip of a suspended cord, or wire, is small compared with the span, the expansion, or contraction, due to a change in temperature, will result in a considerable change in the dip. In calculating this change it must be observed that the expansion due to an increase in temperature will be partly counteracted by the elastic contraction due to the change in stress accompanying the change in dip.

Let t = the temperature change, which is positive for an increase and negative for a decrease, and ϵ = the coefficient of linear expansion of the cord. Let (x_1, y_1) be the coördinates of a point of support C (Fig. 257), with reference to the horizontal and vertical axes through O_1, the lowest point in the cord, s_1 = the length of the curve O_1C, H_1 = the tension at the lowest point and w_1 = the weight per unit length in the initial condition.

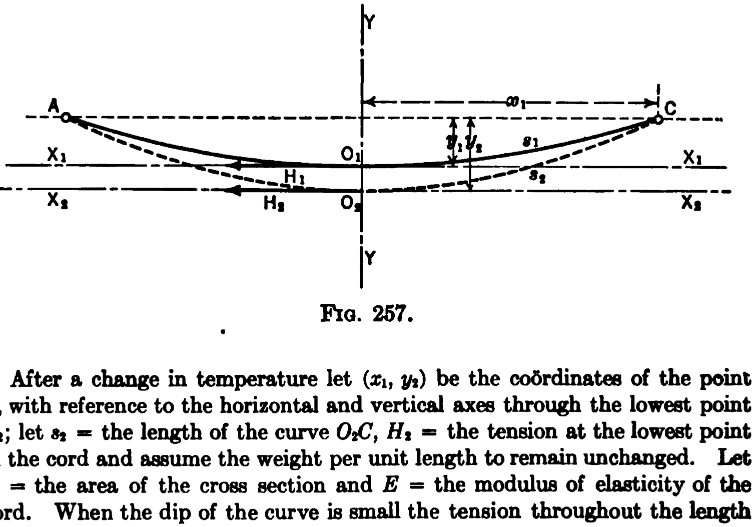


FIG. 257.

After a change in temperature let (x_1, y_2) be the coördinates of the point C, with reference to the horizontal and vertical axes through the lowest point O_2; let s_2 = the length of the curve O_2C, H_2 = the tension at the lowest point in the cord and assume the weight per unit length to remain unchanged. Let A = the area of the cross section and E = the modulus of elasticity of the cord. When the dip of the curve is small the tension throughout the length of the cord may be assumed to be uniform and equal to the tension at the lowest point.

Then, from (15) (Art. 188),

$$s_1 = x_1 + \frac{w_1^2 x_1^3}{6H_1^2} \quad \ldots\ldots\ldots\ldots \quad (1)$$

and

$$s_2 = x_1 + \frac{w_1^2 x_1^3}{6H_2^2}; \quad \ldots\ldots\ldots\ldots \quad (2)$$

and, subtracting (1) from (2),

$$s_2 - s_1 = \frac{w_1^2 x_1^3}{6}\left(\frac{1}{H_2^2} - \frac{1}{H_1^2}\right). \quad \ldots\ldots\ldots \quad (3)$$

But

$$s_2 - s_1 = \left(\frac{H_2 - H_1}{AE} + \epsilon t\right) s_1 \quad \ldots \ldots \ldots \quad (4)$$

and hence

$$\frac{1}{H_2^2} - \frac{1}{H_1^2} = \frac{6\,s_1}{w_1^2 x_1^3}\left(\frac{H_2 - H_1}{AE} + \epsilon t\right), \quad \ldots \ldots \ldots \quad (5)$$

where

$$H_1 = \frac{w_1 x_1^2}{2\,y_1} \quad \ldots \ldots \ldots \quad (6)$$

and

$$s_1 = x_1 + \frac{2\,y_1^2}{3\,x_1}, \quad \ldots \ldots \ldots \quad (7)$$

the value of s_1 being obtained by eliminating m between (14) and (15) (Art. 188). The solution of (5) will give the value of H_2 and the dip of the curve, after the temperature change, will be equal to

$$y_2 = \frac{w_1 x_1^2}{2\,H_2}. \quad \ldots \ldots \ldots \quad (8)$$

If the load on the cord changes, coincident with the change in temperature, and we let w_2 = the weight per unit length after the temperature change, equation (3) will become

$$s_2 - s_1 = \frac{x_1^3}{6}\left(\frac{w_2^2}{H_2^2} - \frac{w_1^2}{H_1^2}\right) \quad \ldots \ldots \ldots \quad (9)$$

and (5) will become

$$\frac{w_2^2}{H_2^2} - \frac{w_1^2}{H_1^2} = \frac{6\,s_1}{x_1^3}\left(\frac{H_2 - H_1}{AE} + \epsilon t\right). \quad \ldots \ldots \ldots \quad (10)$$

Having the values of w_1, H_1, x_1 and s_1, for the cord in the initial condition, and the value of w_2, after the change in temperature t, the value of H_2 may be obtained by the solution of (10) and the value of y_2 from (8), as before.

The change in dip, due to a uniform load change only, may evidently be obtained by putting $t = 0$ in (10) and solving for H_2 and y_2, as before.

190. Tramway Cable.

— The deflection of the track cable of a tramway for any position of the traveling load may be determined with a sufficient degree of accuracy by use of the approximate formulas for the common catenary (14) and (15) (Art. 188).

Let the curve AOB (Fig. 258) represent the form of the cable under a load W at the point O. Let L = the horizontal distance and y_0 = the difference in level between the points of support A and B and let (x_1, y_1) and (x_2, y_2) be the coördinates of A and B with respect to horizontal and vertical axes through O.

If we assume the cable to be perfectly flexible and of uniform weight, the curves OA and OB will be parts of two catenaries, intersecting at O, and the lowest points on the two curves may be represented by O_1 and O_2, respectively.

Let s_1 = the length of the curve OA, s_2 = the length of the curve OB and $s = s_1 + s_2$ = the total length of the cable. Let w = the weight of the cable per unit length, H = the horizontal component of the tension at any point, V_a = the vertical component of the tension at A, V_b = the vertical component of the tension at B and V_0 = the vertical component of the reaction between the parts OA and OB at the point O.

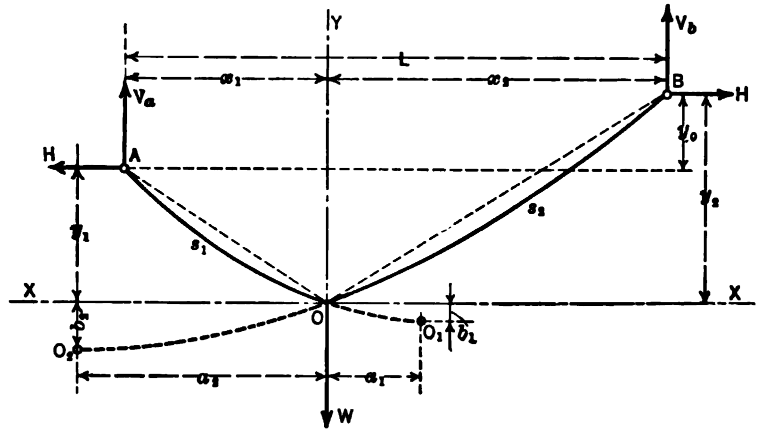


Fig. 258.

If we assume the load W to act on the part OA alone and apply the condition $\Sigma M = 0$ to the forces acting on OA, with the axis of moments at A, we shall have

$$ws_1\frac{x_1}{2} + Wx_1 - V_0x_1 - Hy_1 = 0 \text{ (very nearly)}, \quad . \quad . \quad . \quad (1)$$

and, similarly for the part OB,

$$-ws_2\frac{x_2}{2} - V_0x_2 + Hy_2 = 0 \text{ (very nearly)}. \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating V_0 between (1) and (2),

$$\frac{ws_1}{2} + \frac{ws_2}{2} + W - H\left(\frac{y_1}{x_1} + \frac{y_2}{x_2}\right) = \frac{ws}{2} + W - H\left(\frac{y_1}{x_1} + \frac{y_2}{x_2}\right) = 0;$$

and hence

$$H = \frac{W + \frac{ws}{2}}{\frac{y_1}{x_1} + \frac{y_2}{x_2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Taking the sum of the moments of all the forces acting on the cable about an axis through B and solving for V_a, we have

$$V_a = \frac{Wx_2 - Hy_0}{L} + \frac{ws}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

similarly,

$$V_b = \frac{Wx_1 + Hy_0}{L} + \frac{ws}{2}. \quad \dots \dots \quad (5)$$

The tension at *A* will be equal to

$$T_a = \sqrt{V_a^2 + H^2}; \quad \dots \dots \quad (6)$$

and the tension at *B*,

$$T_b = \sqrt{V_b^2 + H^2}; \quad \dots \dots \quad (7)$$

the greater of the last two values being the greatest tension in the cable for the given position of *W*. As the postition of *W* changes the greatest tension in the cable will evidently vary. In general, an exact solution for the position of *W* for which the greatest tension is a maximum, is somewhat difficult; but the maximum value may be obtained with sufficient accuracy by assuming that it occurs when *W* is at the middle of the span.

For most cases sufficiently accurate values of s_1 and s_2 may be obtained by assuming

$$s_1 = \sqrt{x_1^2 + y_1^2} \quad \dots \dots \quad (8)$$

and

$$s_2 = \sqrt{x_2^2 + y_2^2}. \quad \dots \dots \quad (9)$$

More exact values of s_1 and s_2 may be obtained by using equations (15) and (14) (Art. 188) as follows: Taking the origin at O_1 (Fig. 258),

$$s_1 = O_1A - O_1O = x_1 + a_1 + \frac{(x_1 + a_1)^3}{6\,m^2} - a_1 - \frac{a_1^3}{6\,m^2}$$

$$= x_1\left[1 + \frac{x_1^2 + 3\,x_1a_1 + 3\,a_1^2}{6\,m^2}\right]. \quad \dots \dots \quad (10)$$

and

$$y_1 = \frac{(x_1 + a_1)^2}{2\,m} - \frac{a_1^2}{2\,m} = \frac{x_1\,(2\,a_1 + x_1)}{2\,m}; \quad \dots \dots \quad (11)$$

hence

$$a_1 = \frac{my_1}{x_1} - \frac{x_1}{2}. \quad \dots \dots \quad (12)$$

Substituting the value of a_1 in (10), the value of s_1 is obtained in terms of y_1 and x_1. In a similar manner an expression for s_2 in terms of y_2 and x_2 can be easily obtained.

When *A* and *B* are on the same level the greatest tension occurs when *W* is at the middle of the span, in which case $y_1 = y_2$, $x_1 = x_2 = \frac{L}{2}$ and

$$H = \frac{L}{4\,y_1}\,(W + ws_1), \quad \dots \dots \quad (13)$$

$$V_a = V_b = \frac{W}{2} + ws_1, \quad \dots \dots \quad (14)$$

$$s_1 = s_2 = \sqrt{\frac{L^2}{4} + y_1^2}, \quad \dots \dots \quad (15)$$

or, a more exact value of s_1 may be obtained from equation (10).

In ordinary cases the deflection y_1 is so small, compared with the span, that

$$T_a = H \text{ (very nearly)};$$

and sufficiently accurate results can be obtained by assuming the cable to be under a uniform tension H. In such a case the dip of the curve, formed by the cable when the load is off the span, can be readily found by calculating s from (8) and (9) and determining the new value of m from (15) (Art. 188) and the dip y_1 from (14) (Art. 188). If allowance is to be made for the change in the length of the cable, due the change in tension when W is removed, the value of $m = \frac{H_1}{w}$ should be calculated first from (15) (Art. 188) and the correct value of H_2 obtained by the use of (5) (Art. 189), after which the dip y_1 may be found by substituting $m = \frac{H_2}{w}$ in (14) (Art. 188).

In the foregoing analysis the cable has been treated as a perfectly flexible cord, whereas in an ordinary cable the rigidity will be sufficient to produce a considerable bending stress where it bends around the carriage sheaves, supporting the load W, and the form of the curve taken by the cable will be somewhat different from that taken by the flexible cord.

The effect of the rigidity on the H components at the supports may be shown by introducing in equations (1) and (2) the term M, representing the bending couple at the section at O (Fig. 258), and solving for H, which will give

$$H = \frac{W + \frac{ws}{2} - 2M}{\frac{y_1}{x_1} + \frac{y_2}{x_2}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

showing that the value of H would be less than if the cable were perfectly flexible. The values of V_a and V_b (equations 4 and 5) would be the same as for a flexible cord.

In an ordinary case the value of M would not be large enough to change to any considerable extent the value of H from that given by (3); and the form of the curve can be determined with sufficient accuracy by treating the cable as a flexible cord.

In calculating the maximum stress intensity, however, allowance for the stress due to bending should be made. Rankine proposed that the bending stress be calculated on the assumption that all the wires in the cable take the same curvature when bending around a sheave. On this assumption, if d = the diameter of a single wire, E = the modulus of elasticity of the material and r = the radius of curvature of the axis of the bent cable, the maximum stress intensity due to bending would be equal to

$$\frac{Ed}{2r}.$$

Hence, if T = the tension and A = the total area of the cross section, the maximum stress intensity in the cable, where it bends around a sheave, would be equal to

$$f_t = \frac{T}{A} + \frac{Ed}{2r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

In applying (17) in practice allowance should be made for the fact that the "lay" of the strands in an ordinary cable is such that the curvature of the different wires in the bent cable is not the same; and also that the friction between the strands will prevent them from moving freely on each other, which would tend to produce a higher stress in the wires on the outside of a bend than in those on the inside.

Mr. F. C. Carstarphen has shown, by an extensive series of tests on steel cable, that as the tension on a cable increases the properties of the cable under bending approach those of a solid bar of the same material. To meet such a condition (17) should be modified to

$$f_t = \frac{T}{A} + \frac{E_1 d_1}{2\,r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where d_1 = the diameter of the cable and E_1 = a modified value of the modulus of elasticity, depending on the value of T.

Moreover the value of E_1 for a tramway cable under its working tension may be so large that the resistance of the cable to bending will be sufficient to prevent it from taking a radius of curvature as small as the radius of the carriage sheaves at O (Fig. 258), resulting in a smaller value for the bending stress than that given by (18).

191. Flexible Trough. — The form of the cross section of a perfectly flexible trough filled with any homogeneous fluid will be the same as that of a flexible cord subjected to a load which is everywhere normal to the curve and whose intensity at any point is proportional to the depth below a horizontal axis coinciding with the surface of the fluid.

Let AO_1B (Fig. 259) represent the cross section of a flexible trough of width $2\,b$ and depth a, supported at A and B and filled with a homogeneous liquid to the level AB. Take the axes of coördinates through O, the middle point in AB, and let w = the weight of the liquid per unit of volume, H = the tension per unit length of the trough at the lowest point O_1, T = the tension per unit length at any point C, whose coördinates are (x, y), p = the intensity of pressure at C, θ = the angle between the tangent at C and the horizontal and r = the radius of curvature at C.

Applying the condition of equilibrium, $\Sigma X = 0$, to the forces acting on the portion of the trough O_1C we have

$$\Sigma X = -H + \int_{y=a}^{y=y} p \sin\theta\, ds + T\cos\theta = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $$p = -wy = \frac{T}{r}, \quad \sin\theta = \frac{dy}{ds} \quad \text{and} \quad \cos\theta = \frac{dx}{ds}.$$

Hence

$$-H - w\int_a^y y\,dy + T\frac{dx}{ds} = -H - \frac{w}{2}(y^2 - a^2) + T\frac{dx}{ds} = 0;$$

and

$$T = \left[H - \frac{w}{2}(a^2 - y^2)\right]\frac{ds}{dx} = \left[H - \frac{w}{2}(a^2 - y^2)\right]\sec\theta. \quad . \quad . \quad (2)$$

But $T = -wyr = -wy\dfrac{ds}{d\theta}$; and hence $-wy = \left[H - \dfrac{w}{2}\left(a^2 - y^2\right)\right]\dfrac{d\theta}{dx}$

and

$$d\theta = \frac{-wy}{H - \frac{w}{2}(a^2 - y^2)}dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Multiplying by $\tan\theta = \frac{dy}{dx}$, we have

$$\tan\theta\, d\theta = \frac{-wy}{H - \frac{w}{2}(a^2 - y^2)}dy = \frac{-2y\,dy}{\frac{2H}{w} - a^2 + y^2} = \frac{-2y\,dy}{K + y^2}, \quad . \quad . \qquad (4)$$

where
$$K = \frac{2H}{w} - a^2.$$

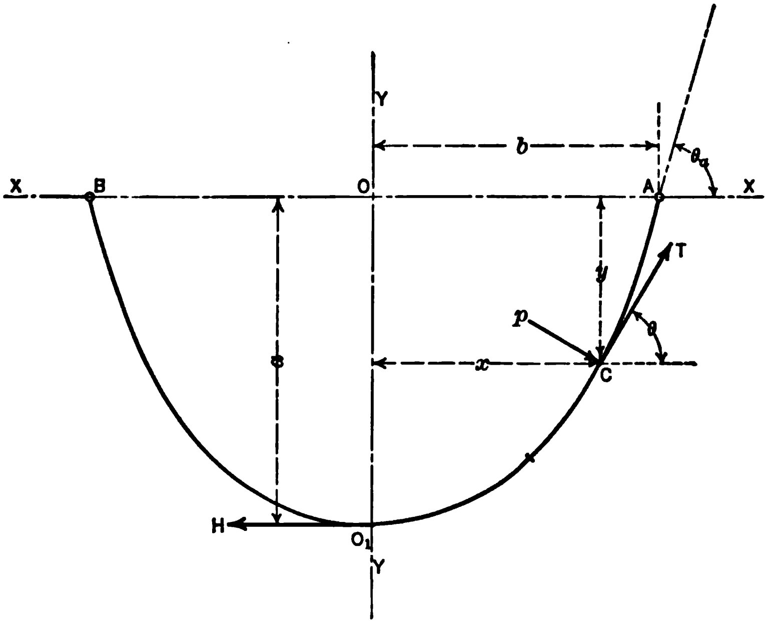


Fig. 259.

Integrating (4), we obtain

$$-\log\cos\theta = -\log(y^2 + K) + c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Observing that $\theta = 0$, when $y = a$,

$$c = \log(a^2 + K);$$

and hence

$$\log\cos\theta = \log\frac{y^2 + K}{a^2 + K},$$

or,

$$\cos\theta = \frac{y^2 + K}{a^2 + K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Therefore,

$$\tan\theta = \frac{dy}{dx} = \frac{\sqrt{1-\left(\frac{y^2+K}{a^2+K}\right)^2}}{\frac{y^2+K}{a^2+K}}$$

and

$$dx = \frac{y^2+K}{\sqrt{(a^2+K)^2-(y^2+K)^2}}\,dy = \frac{y^2+K}{\sqrt{(a^2+K+y^2+K)\,(a^2+K-y^2-K)}}\,dy$$

$$= \frac{(a^2+2K+y^2)-(a^2+K)}{\sqrt{(a^2+2K+y^2)\,(a^2-y^2)}}\,dy = \frac{\left(\frac{a^2+2K+y^2}{a^2}\right)-\left(\frac{a^2+K}{a^2}\right)}{\sqrt{\left(\frac{a^2+2K+y^2}{a^2}\right)\left(1-\frac{y^2}{a^2}\right)}}\,dy. \quad . \quad . \qquad (7)$$

Let $z^2 = 1-\frac{y^2}{a^2}$; then $\frac{y}{a} = \sqrt{1-z^2}$ and $dy = \frac{-az\,dz}{\sqrt{1-z^2}}$; and, substituting in (7) and reducing,

$$dx = \frac{-a\left[\left(\frac{2(a^2+K)}{a^2}-z^2\right)-\frac{a^2+K}{a^2}\right]}{\sqrt{\left(\frac{2(a^2+K)}{c^2}-z^2\right)\left(1-z^2\right)}}\,dz$$

$$= \frac{\sqrt{\frac{a^2+K}{2}}-\sqrt{2(a^2+K)}\left(1-\frac{a^2}{2(a^2+K)}z^2\right)}{\sqrt{(1-z^2)\left(1-\frac{a^2}{2(a^2+K)}z^2\right)}}\,dz \quad .$$

$$= \frac{\sqrt{\frac{a^2+K}{2}}-\sqrt{2(a^2+K)}\left(1-k^2z^2\right)}{\sqrt{(1-z^2)\,(1-k^2z^2)}}\,dz, \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

where $$k^2 = \frac{a^2}{2(a^2+K)} = \frac{wa^2}{4H}.$$

By substituting $z = \sin\phi$, $dz = \cos\phi\,d\phi = \sqrt{1-z^2}\,d\phi$,

$a^2+K = \frac{2H}{w}$ and integrating, equation (8) reduces to

$$x = \int \frac{\sqrt{\frac{H}{w}}-2\sqrt{\frac{H}{w}}(1-k^2\sin^2\phi)}{\sqrt{1-k^2\sin^2\phi}}\,d\phi$$

$$= \sqrt{\frac{H}{w}}\left[\int\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}-2\int\sqrt{1-k^2\sin^2\phi}\,d\phi\right], \quad . \quad . \qquad (9)$$

where $$\phi = \sin^{-1}z = \cos^{-1}\sqrt{1-z^2} = \cos^{-1}\frac{y}{a}, \text{ or,}$$

$$y = a\cos\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

and

$$k = \frac{a}{2}\sqrt{\frac{w}{H}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

The values of the integrals in (9), for any value of k and different values of ϕ can be readily determined from a table of elliptic integrals and, by substituting these values and the value of $\frac{H}{w}$ in the equation, values of x corresponding to the values of ϕ can be found. In this manner, as many values of x and y as may be required for plotting the curve can be obtained.

Substituting the value of $\cos\theta$ (equation 6) in (2) we have

$$T = \left[H - \frac{w}{2}(a^2 - y^2)\right]\frac{a^2 + K}{y^2 + K};$$

and reducing,

$$T = H. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Hence the tension throughout the length of the trough is uniform and equal to the tension at the lowest point.

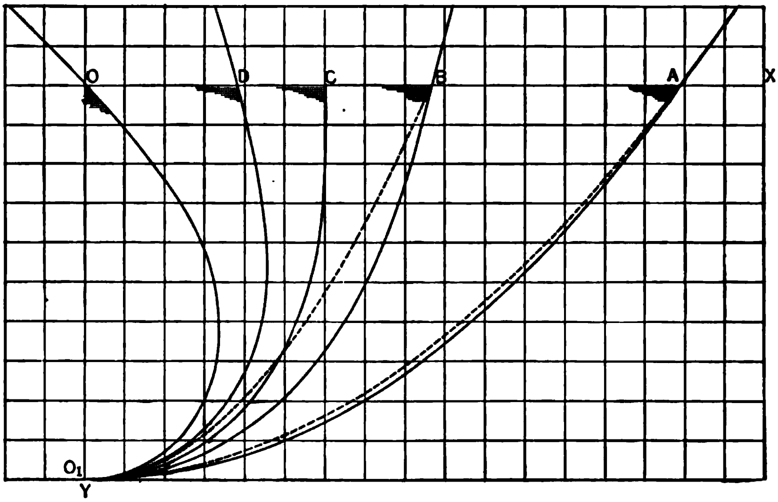


FIG. 260.

The form of the curve will depend upon the value of the ratio $\frac{b}{a}$; and the value of H will be determined by the values of w and a, in addition to the value of $\frac{b}{a}$. Different forms are represented by the curves O_1A, O_1B, O_1C, O_1D and O_1O (Fig. 260). As the ratio $\frac{b}{a}$ increases, the form of the curve approaches that of the parabola with the vertex at O_1. The dotted curves represent the parabolas through O_1 and the points A and B.

If we let θ_a = the slope of the tangent at a point of support, we have by putting $y = 0$ in (6)

$$\theta_a = \cos^{-1}\frac{K}{a^2 + K} = \cos^{-1}\left(1 - \frac{wa^2}{2H}\right). \quad . \quad . \quad . \quad . \quad (13)$$

It may be noted that when $wa^2 = 2H$ the tangent is vertical as shown by the curve O_1C (Fig. 260). In this case it will be found that the ratio $\frac{b}{a} = 0.599$ (very nearly).

The value of H for any given values of w, a and b can be found by trial; substituting $x = a$ in (9) and $y = b$ in (10) and interpolating in a table of elliptic integrals until the value of the ratio $\frac{H}{w}$ required to satisfy the equations is found.

When $\frac{b}{a} > 1$ the value of $\frac{H}{w}$ can be found, very nearly, by treating the curve as a parabola. For example, if we let $OA = b$ and $OO_1 = a$ (Fig. 260), the area of the half segment of the parabola OO_1A will be equal to $\frac{2}{3}\,ab$ and the distance of its center of gravity from the vertical through A will be equal to $\frac{5}{8}\,b$. Taking moments, about an axis through A, of the forces acting on the fluid in the half segment OO_1A, we have

$$-Ha + \frac{wa^2}{2} \times \frac{2}{3}a - w \times \frac{2}{3}ab \times \frac{5}{8}b = 0, \quad \ldots \quad (14)$$

and hence

$$\frac{H}{w} = \frac{1}{3}\left(a^2 + \frac{5}{4}\,b^2\right). \quad \ldots \quad (15)$$

192. Problems. — Arches and Catenaries.

Problem 1.

A three-hinged circular arch is made up of two equal ribs AB and BC, formed by bending 8″ steel I-beams to circular arcs, the radius of the central axis of each rib being 25 ft. The arch is subjected to four concentrated loads

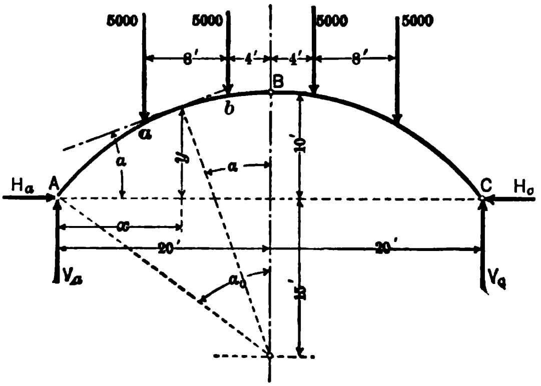


FIG. 261.

of 5000 lbs. each, as indicated (Fig. 261). Calculate the horizontal and vertical components of the supporting forces at A and C and determine the greatest fiber stress in the arch, assuming the area of the cross section = 6 sq. in., I = 60 (ins.)^4 and neglecting the weight of the arch ribs.

Solution. — Since the arch is symmetrically loaded, the distribution of the stresses in the two ribs will be the same and, from the conditions of equilibrium, the values of the H and V components at A,

$$H_a = 12{,}000 \text{ lbs.}, \quad V_a = 10{,}000 \text{ lbs.},$$

are readily obtained.

Taking the origin at A and substituting in equations (2) and (3) (Art. 185), we have, for any cross section of the arch between A and a,

$$M = 12{,}000\, y - 10{,}000\, x, \quad \ldots \ldots \quad (1)$$

$$P = 12{,}000 \cos\alpha + 10{,}000 \sin\alpha; \quad \ldots \ldots \quad (2)$$

for any cross section between a and b

$$M = 12{,}000\, y - 10{,}000\, x + 5000\,(x - 8), \quad \ldots \ldots \quad (3)$$

$$P = 12{,}000 \cos\alpha + 5000 \sin\alpha; \quad \ldots \ldots \quad (4)$$

and for any cross section between b and B,

$$M = 12{,}000\, y - 10{,}000\, x + 5000\,(x - 8) + 5000\,(x - 16), \quad \ldots \quad (5)$$

$$P = 12{,}000 \cos\alpha. \quad \ldots \ldots \quad (6)$$

For the slope of the tangent at A, we have

$$\alpha_0 = \sin^{-1}\tfrac{20}{25} = \cos^{-1}\tfrac{15}{25} = 53.1^\circ,$$

for the slope of the tangent at a,

$$\theta_1 = \sin^{-1}\tfrac{12}{25} = 28.7^\circ,$$

and for the slope of the tangent at b,

$$\theta_2 = \sin^{-1}\tfrac{4}{25} = 9.2^\circ.$$

Expressing x and y in terms of α, we have

$$x = r \sin\alpha_0 - r \sin\alpha = 20 - 25 \sin\alpha,$$

$$y = r \cos\alpha - r \cos\alpha_0 = 25 \cos\alpha - 15;$$

and, substituting in (1) and reducing,

$$M = 300{,}000 \cos\alpha + 250{,}000 \sin\alpha - 380{,}000. \quad \ldots \quad (7)$$

Differentiating, placing the derivative equal to zero and solving for α, we have

$$\frac{dM}{d\alpha} = -300{,}000 \sin\alpha + 250{,}000 \cos\alpha = 0,$$

$$\tan\alpha = \tfrac{25}{30}, \quad \alpha = \tan^{-1} 0.8333 = 39.8^\circ,$$

which lies between α_0 and θ_1.

Substituting the values $\sin 39.8^\circ = 0.640$ and $\cos 39.8^\circ = 0.768$ in (7) and (2) we have for the greatest bending moment between A and a,

$$M' = 300{,}000 \times 0.768 + 250{,}000 \times 0.640 - 380{,}000 = 10{,}400 \text{ ft. lbs.}$$

and for the normal thrust at the section of greatest bending moment,

$$P' = 12{,}000 \times 0.768 + 10{,}000 \times 0.640 = 15{,}600 \text{ lbs.}$$

Proceeding in a similar manner for any section between a and b, we obtain from (3),

$$M = 12{,}000\, y - 5000\, x - 40{,}000 = 300{,}000 \cos\alpha + 125{,}000 \sin\alpha - 320{,}000, \quad (8)$$

$$\frac{dM}{d\alpha} = -300{,}000 \sin\alpha + 125{,}000 \cos\alpha = 0,$$

$$\tan\alpha = \tfrac{125}{300}, \quad \alpha = \tan^{-1} 0.4167 = 22.6^\circ.$$

Since this value of α is less than θ_1, the greatest bending moment for any section from a to b will be at the section a. But, from the preceding part of the solution, M' is evidently greater than the bending moment M_a; and a comparison of (4) with (2) will show that P' is greater than the thrust on any section between a and b. Therefore the greatest fiber stress will be found by substituting the values of M' and P' in (1) (Art. 185), from which we obtain

$$f' = \frac{15{,}600}{6} + \frac{10{,}400 \times 12 \times 6}{60} = 2600 + 12{,}480 = 15{,}100 \text{ lbs. per sq. in.}$$

It is evident that, since the maximum fiber stress at a is greater than that on any section from a to b, it must also be greater than the fiber stress on any section between b and B.

Problem 2.

Solve Problem (1) omitting the load on each rib nearest the hinge B.

Problem 3.

Solve Problem (1) by constructing the equilibrium polygon for the forces acting on the arch, with the strings of the polygon passing through the hinges A, B and C; and determining the section of greatest bending moment by inspection and the normal thrust on this section by a graphical resolution of forces.

Problem 4.

Determine the greatest fiber stress in a two-hinged arch of the same dimensions and subjected to the same loads as the arch in Problem (1), the hinge B (Fig. 261) being omitted.

Note. — Calculate the value of H_a, the horizontal component of the supporting force at A, by using equation (14) (Art. 185) and proceed with the rest of the solution in the same manner as in Problem (1).

Problem 5.

Determine the deflection at the crown of the arch in Problem (4);

(a) Making an approximate solution by use of equation (9) (Art. 178);

(b) Making a more exact solution by use of equation (20) (Art. 178).

Problem 6.

Determine the greatest fiber stress in the arch given in Problem (4) if the concentrated loads were replaced with a uniformly distributed load of 600 lbs. per ft., measured horizontally.

Note. — Calculate H_a, the horizontal component of the supporting force at A, by using equation (17) (Art. 185) and write the general expression for M (equation 5, Art. 185), observing that the value of K is given by (15) (Art. 185). Differentiate and determine the greatest bending moment as in Problem (1).

Problem 7.

A flexible cable is suspended between the tops of two towers, 200 ft. apart, one of the towers being 20 ft. higher than the other. The cable is subjected to a uniform load of 100 lbs. per ft., measured horizontally, and the lowest point in the cable is 20 ft. below the level of the lower of the two supports. Find the greatest tension in the cable.

Solution. — Assume an origin of coördinates at O, the lowest point in the curve formed by the cable, and let (x_a, y_a) be the coördinates of A, the lower of the two supports, and (x_b, y_b) be the coördinates of B, the higher support, with reference to horizontal and vertical axes through O. Then, from (4) (Art. 187), we have

$$y_a = \frac{100}{2H} x_a^2, \quad \dots \dots \dots \dots \quad (1)$$

$$y_b = \frac{100}{2H} x_b^2 = y_a + 20 = \frac{100}{2H}\left(200 - x_a\right)^2. \quad \dots \dots \quad (2)$$

Substituting $y_a = 20$ and solving (1) and (2) simultaneously, we obtain

$$x_a = 82.8 \text{ ft.}, \ x_b = 117.2 \text{ ft.}$$

and

$$H = 17{,}140 \text{ lbs.} \quad \dots \dots \dots \dots \quad (3)$$

The greatest tension will occur at the support B and will be equal to

$$T_b = \sqrt{(H)^2 + (wx_b)^2} = 20{,}760 \text{ lbs.} \quad \dots \dots \quad (4)$$

Problem 8.

A steel guy rope, weighing 0.50 lb. per ft., is attached to the top of the mast of a derrick, 50 ft. high, and anchored at a distance of 150 ft. from and at the level of the foot of the mast. Find the deflection of the middle point of the guy rope, from a straight line between the anchor and the top of the mast, when the tension in the guy is 4000 lbs.

Note. — Use the approximate equation for the catenary (14) (Art. 188) and observe that the origin of coördinates is beyond the anchor for the guy. By letting (x_a, y_a) and (x_b, y_b) represent the coördinates of the point of anchorage and the top of the mast, respectively, referred to the horizontal and vertical axes through the origin, the problem may be solved in the same manner as Problem (7)

Problem 9.

A wire rope drive consisting of a steel cable, $\frac{3}{4}''$ diam., weighing 0.8 lb. per ft., running over two sheaves, 4 ft. diam., with centers on the same level and 150 ft. apart, is transmitting 100 h.p. If the speed of the sheaves is 150 r.p.m. and the tension in the tight portion of the drive is 5000 lbs., calculate the dip at the middle point in the tight portion of the cable and also at the middle point of the slack portion.

Note. — The tension will be very nearly uniform throughout each span of the cable and the approximate formula (14) (Art. 188) can be used, assuming H = tension throughout the span.

Problem 10.

A steel towing hawser, 1″ diam., weighing 1.4 lbs. per ft., and 800 ft. long is subjected to a horizontal pull of 8000 lbs. Find the dip at the middle point and the length of the span, assuming that the supports are on the same level.

Problem 11.

A steel wire, 0.12" diam., weighing 0.0382 lb. per ft., is suspended between two supports 200 ft. apart and on the same level. (a) Find the dip at the center of the span when the tensile stress in the wire is 15,000 lbs. per sq. in. (b) Find the increase in the dip due to an increase in temperature of 100° F.

$E = 30{,}000{,}000$ lbs. per sq. in., $\epsilon = 0.0000067$ per degree Fahr.

Solution. — (a) The area of the cross section of the wire $A = 0.01131$ sq. in. The tension throughout the span will be practically uniform and therefore

$$H = 15{,}000 \times 0.01131 = 170 \text{ lbs.}$$

Substituting in (14) (Art. 188),

$$y_1 = \frac{0.0382 \times (100)^2}{2 \times 170} = 1.12 \text{ ft.}$$

(b) Substituting in (7) (Art. 189),

$$s_1 = 100 + \frac{2 \times (1.12)^2}{3 \times 100} = 100.0084 = x_1 \text{ (very nearly)}$$

and, substituting $s_1 = x_1$ in (5) (Art. 189),

$$\frac{1}{H_2^2} - \frac{1}{(170)^2} = \frac{6 \times 100}{(0.0382)^2 \times (100)^3}\left(\frac{H_2 - 170}{0.01131 \times 30{,}000{,}000} + 0.0000067 \times 100\right),$$

which reduces to

$$H_2^3 + 85.8\, H_2^2 - 825{,}000 = 0.$$

Solving for H_2, we obtain

$$H_2 = 72 \text{ lbs.};$$

and hence,

$$y_2 = \frac{0.0382 \times (100)^2}{2 \times 72} = 2.65 \text{ ft.}$$

and the increase in dip

$$y_2 - y_1 = 1\ 53 \text{ ft.}$$

Problem 12.

Calculate the increase in the dip of the wire in Problem (11) due to a uniform ice load of 0.2 lb. per ft., the temperature remaining unchanged. Determine the tension in the wire under the additional load.

Problem 13.

Calculate the maximum deflection of a copper wire, 0.204" diam., weighing 0.126 lb. per ft., in a span of 120 ft. when the tension in the wire is 12,000 lbs. per sq. in.

Problem 14.

Calculate the change in the maximum deflection of the wire in Problem (13) due to an ice load of 0.25 lb. per ft., combined with a decrease in temperature of 80° F.

$E = 16{,}000{,}000$ lbs. per sq. in., $\epsilon = 0.000009$ per degree Fahr.

Problem 15.

A steel tramway cable, 1″ diam., weighing 2.20 lbs. per ft., has a span of 800 ft. Calculate the deflection under a load of 1500 lbs. at the middle of the span, assuming the greatest tension in the cable to be 10,000 lbs. and the supports on the same level. If the cable is made up of 19 wires, 0.2″ diam., and the value of *E* is assumed to be 25,000,000 lbs. per sq. in., calculate the maximum deflection of the cable under its own weight only.

CHAPTER XIII.

CYLINDERS AND PLATES.

193. Thin Oval Cylinder. — The expressions for the stress intensities at any point in a thin circular cylinder, subjected to uniform internal pressure, have already been deduced (Art. 54). If a thin cylinder of oval section is subjected to a uniform internal pressure it will tend to become circular in shape and bending stresses will be set up in the wall, in addition to direct tensile stresses.

Let a and b equal the semi-axes of a thin oval cylinder (Fig. 262), symmetrical with respect to the axes OX and OY, where $a > b$. Let t = the thickness and p = the intensity of the uniform internal pressure. Consider a portion of the cylinder between two transverse sections at a unit distance apart and let T_1 = the direct tension and M_1 = the bending moment at the section of this strip cut by the axis XX, T_2 = the tension and M_2 = the bending moment at the section cut by the axis YY, and T = the tension and M = the bending moment at any section C, through the point whose coördinates are (x, y).

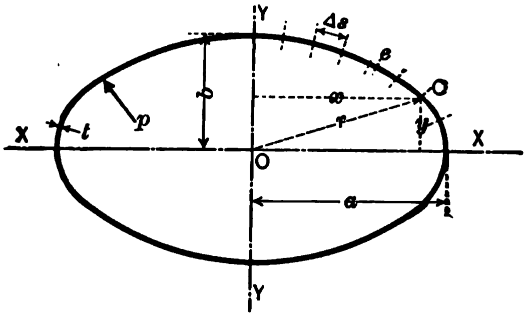


Fig. 262.

Then

$$T_1 = pa, \quad \ldots \ldots \quad (1)$$

$$T_2 = pb \quad \ldots \ldots \quad (2)$$

and

$$T = py\frac{dx}{ds} + \left[T_1 - p(a - x)\right]\frac{dy}{ds}$$
$$= py\frac{dx}{ds} + px\frac{dy}{ds}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

the greatest tension in the cylinder evidently being equal to T_1.

If we call bending moments which tend to increase the curvature positive, the expression for the bending moment at C will be

$$M = M_1 + T_1(a - x) - \frac{p}{2}(a - x)^2 - \frac{p}{2}y^2. \quad . \quad . \quad (4)$$

Substituting the value of T_1 from (1) and reducing,

$$M = M_1 + \frac{p}{2}(a^2 - x^2 - y^2) = M_1 + \frac{p}{2}(a^2 - r^2), \quad . \quad . \quad (5)$$

where $r^2 = x^2 + y^2$.

When $x = 0$, (5) becomes

$$M_2 = M_1 + \frac{p}{2}(a^2 - b^2); \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and hence,

$$M_2 - M_1 = \frac{p}{2}(a^2 - b^2). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From an inspection of (5) and (6) it is evident that M_1 is the greatest negative bending moment and M_2 is the greatest positive bending moment in the cylinder; and that the quantity $\frac{p}{2}(a^2 - b^2)$ represents the algebraic difference, or the numerical sum, of the two bending moments.

For an exact solution for the value of M_1, or M_2, the equation of the curve formed by the perimeter would be required and then, by applying the condition that the slopes of the curve at the intersections with the axes OX and OY would remain unchanged by the bending (Art. 178), an equation containing M_1 and M_2 might be obtained, which with (7) would give a solution.

An approximate solution, accurate enough for general purposes, however, can be made much more easily, as follows:

Draw the cross section of the cylinder to scale and divide any quadrant into a number of sections of length Δs and measure the value of r, the length of the radius vector from O to each of these sections. Calculate the value of the quantity $\frac{p}{2}(a^2 - r^2)$ for each value of r and plot each value as an ordinate on the development of the quadrant XY as a base line (Fig. 263). The value of M_1 will then be nearly equal to the mean of the ordinates of the area XYY_1; for, the angle between the tangents at the points X and Y (Fig. 262) will remain unchanged and, if we neglect the effect of the tension T on the change in curvature, we have from (10) (Art. 178),

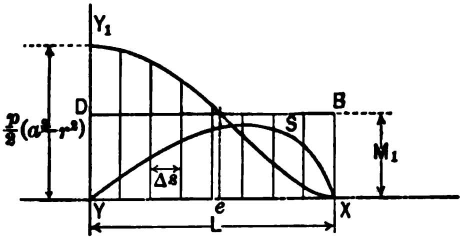


FIG. 263.

$$EI\delta\alpha = \Sigma M\Delta s = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and, substituting the value of M from (5),

$$M_1\Sigma\Delta s + \Sigma\frac{p}{2}(a^2 - r^2)\,\Delta s = 0; \quad . \quad . \quad . \quad . \quad (9)$$

and hence,

$$M_1 = -\frac{\Sigma\frac{p}{2}(a^2 - r^2)\,\Delta s}{L}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where L = the length of the base line XY and $\Sigma\frac{p}{2}(a^2 - r^2)\,\Delta s$ = the area XYY_1.

This relation is similar to that between the bending moment at the support of a symmetrically loaded beam, fixed at the ends, and the diagram of bending moments for a simple beam, similarly loaded (Art. 107).

If a line BD is drawn at the height of the mean ordinate M_1 and parallel to the base XY (Fig. 263), the bending moment at any section of the cylinder will be represented by the ordinate between BD as a base and the curve XY_1, ordinates above BD representing positive bending moments and those below BD, negative bending moments. The point of inflexion e can be located on the cylinder by laying off the distance Xe, measured from the bending moment diagram, along the perimeter (Fig. 262).

If A = the area and $\frac{I}{c}$ = the section modulus of any cross section C, the greatest stress intensity on the section will be equal to

$$f = \frac{T}{A} + \frac{Mc}{I}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where f is a tensile stress. The greatest stress intensity in the cylinder will evidently be equal to

$$f_1 = \frac{T_1}{A} + \frac{M_1 c}{I} = \frac{T_1}{t} + \frac{6\,M_1}{t^2}, \quad . \quad . \quad . \quad . \quad (12)$$

at the inner edge of a cross section at X.

The shearing force at any section C will evidently be equal to

$$S = \left[T_1 - p\,(a - x)\right]\frac{dx}{ds} - py\frac{dy}{ds}$$
$$= px\frac{dx}{ds} - py\frac{dy}{ds}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and its value can be found by measuring the angle between the tangent to the perimeter at C and the axis OX and calculating the products of its cosine and x and of its sine and y and substituting in (13). The shearing force is evidently zero at the sections at X and Y and its value at intermediate sections is represented by the ordinates of the curve XSY (Fig. 263).

If the cylinder is closed at the ends the intensity of the *end tension* will be very nearly equal to

$$f_2 = \frac{pA_1}{4\,Lt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where A_1 = the area of the end of the cylinder and $4\,L$ = the length of the perimeter.

194. Tube of Rectangular Cross Section. — If a tube, or pipe, of rectangular cross section and uniform thickness, is subjected to a uniform internal pressure, the sides will be subjected to combined bending and tensile stresses.

Let $2\,a$ and $2\,b$ represent the lengths of the longer and shorter sides, respectively, and t = the thickness of a rectangular tube (Fig. 264). Consider a portion of the tube between two transverse sections, at a unit distance apart, and let p = the intensity of internal pressure, T_1 = the direct tension and M_1 = the bending moment at the section through this portion, cut by the axis of symmetry OX; and T_2 = the tension and M_2 = the bending moment at the section, cut by the axis of symmetry OY. Let M_0 = the bending moment at a right section at either side of a corner C; and M = the bending moment at any section between X and C, or at any section between Y and C.

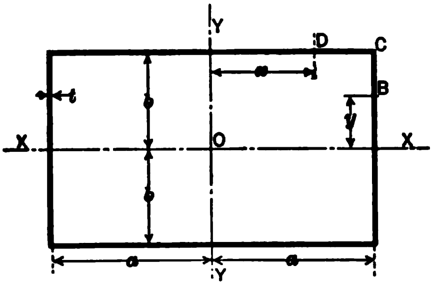


Fig. 264.

The direct tension, throughout the length of the short side, will then be equal to

$$T_1 = pa; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the direct tension, throughout the length of the long side,

$$T_2 = pb. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Assuming bending moments positive where the curvature is convex outwards and negative where the curvature is convex inwards, we shall have for the sections at X and Y,

$$M_1 = M_0 + \frac{pb^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$M_2 = M_0 + \frac{pa^2}{2} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For any cross section B, between X and C, the bending moment

$$M = M_1 - \frac{py^2}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and hence,

$$EIi = \int M\,dy = M_1 y - \frac{py^3}{6}, \quad . \quad . \quad . \quad . \quad (6)$$

the constant of integration being equal to zero.

Similarly, for any section D, between Y and C,

$$M = M_2 - \frac{px^2}{2} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

and

$$EIi = \int M\,dx = M_2 x - \frac{px^3}{6}. \quad . \; . \; . \; . \quad (8)$$

Substituting $y = b$ in (6) we have, for the product of EI and the slope at C,

$$EIi_c = M_1 b - \frac{pb^3}{6}; \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

and, substituting $x = a$ in (8), we have

$$EIi_c' = M_2 a - \frac{pa^3}{6}. \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

The values of i_c and i_c' will be equal in magnitude and have opposite signs and hence, by adding (9) and (10),

$$M_1 b + M_2 a - \frac{p}{6}(a^3 + b^3) = 0; \quad . \; . \; . \; . \quad (11)$$

and, by substituting the values of M_1 and M_2, from (3) and (4), and solving for M_0, we obtain

$$M_0 = -\frac{p}{3}\left(\frac{a^3 + b^3}{a + b}\right) = -\frac{p}{3}(a^2 - ab + b^2). \; . \; . \quad (12)$$

Putting the value of M_0 back in (3) and (4) and reducing,

$$M_1 = \frac{p}{6}(b^2 + 2\,ab - 2\,a^2) \; . \; . \; . \; . \; . \quad (13)$$

and

$$M_2 = \frac{p}{6}(a^2 + 2\,ab - 2\,b^2). \quad . \; . \; . \; . \quad (14)$$

It is evident, from an inspection of (12), (13) and (14), that the greatest bending moment in the tube is M_0; and hence the greatest stress intensity will be located at the inside edge of a cross section through the side XC, next to the corner C. This stress intensity will be tension and, if A = the area and $\frac{I}{c}$ = the section modulus of the cross section, it is evident that

$$f = \frac{T_1}{A_1} + \frac{M_0 c}{I} = \frac{T_1}{t} + \frac{6\,M_0}{t^2}. \; . \; . \; . \; . \quad (15)$$

By placing (5) and (7) equal to zero we have, for the points of inflexion,

$$y = \sqrt{\frac{2M_1}{p}} \quad \ldots \ldots \quad (16)$$

and

$$x = \sqrt{\frac{2M_2}{p}}. \quad \ldots \ldots \quad (17)$$

If desired, the bending moment and shearing force diagrams for the long and short sides may be readily plotted.

For a square tube, putting $b = a$, the values of the bending moments become

$$M_0 = -\frac{pa^2}{3}, \quad \ldots \ldots \quad (18)$$

$$M_1 = M_2 = \frac{pa^2}{6}. \quad \ldots \ldots \quad (19)$$

If the long and short sides of the tube are not of the same thickness, the values of the bending moments can be expressed in terms of I_1 and I_2, the moments of inertia of the cross sections of the short and long sides, respectively, about the axes through the centers of gravity. By substituting the values of I_1 and I_2 in (9) and (10) and solving for M_0, as before, we would obtain

$$M_0 = -\frac{p}{3}\left(\frac{a^3I_1 + b^3I_2}{aI_1 + bI_2}\right); \quad \ldots \ldots \quad (20)$$

and, by substituting this value in (3) and (4), values for M_1 and M_2 can be easily obtained.

195. Thick Hollow Cylinder.—A method of calculating the stress intensity in the wall of a circular cylinder subjected to uniform internal pressure, applicable when the thickness of the wall is small, has been given in Art. (54).

In general, when a circular cylinder is subjected to a uniform internal or a uniform external radial pressure, or to both pressures simultaneously, and the material is of uniform elasticity, the stress at any point in the wall may be determined from the fundamental relations between stresses and strains in an elastic body (Art. 46), on the assumption that the cylinder is under plane stress, with the plane of stress at right angles to its axis.

Let r_1 = the internal radius and r_2 = the external radius of a cylinder, which is subjected to a uniform internal pressure of intensity P_1, combined with a uniform external pressure of intensity P_2. (Fig. 265.)

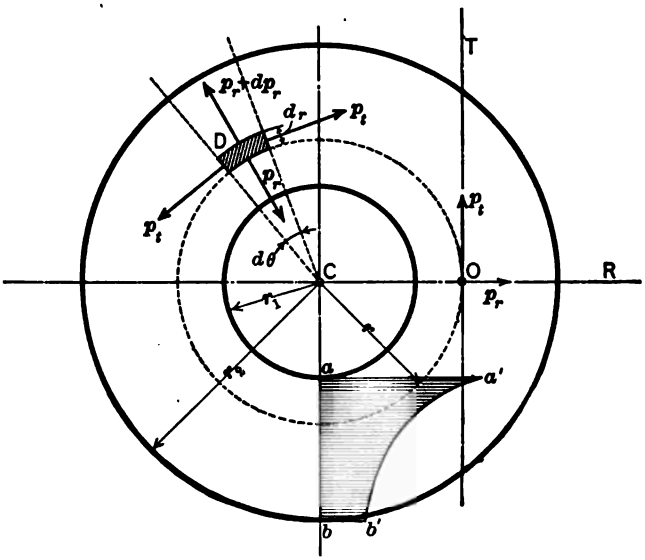


Fig. 265.

Through any point O in the wall, at a distance r from the center C, the principal planes of stress will be the radial plane OR and the plane OT, tangent to a cylinder of radius r. At the point O let p_t = the stress intensity on the radial plane OR, p_r = the stress intensity on the tangential plane OT, e_t = the strain in the direction OT, and e_r = the strain in the direction OR; and assume tensile stresses and strains positive and compressive stresses and strains negative. Let u = the change in the length of the radius r, due to the distortion of the cylinder under pressure.

Then

$$e_r = \frac{du}{dr}, \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$e_t = \frac{2\pi(u+r) - 2\pi r}{2\pi r} = \frac{u}{r}; \quad \ldots \ldots \quad (2)$$

and, from (5) and (6) (Art. 46),

$$p_r = \frac{mE}{m^2-1}(me_r + e_t) = \frac{mE}{m^2-1}\left(m\frac{du}{dr} + \frac{u}{r}\right) \quad . \; . \quad (3)$$

and

$$p_t = \frac{mE}{m^2 - 1}(me_t + e_r) = \frac{mE}{m^2 - 1}\left(m\frac{u}{r} + \frac{du}{dr}\right). \quad . \quad . \quad (4)$$

If we consider any particle D, at the distance r from the center of the cylinder and bounded by two transverse planes, at a unit distance apart, two cylindrical surfaces, of radii r and $r + dr$, and two radial planes subtending an angle $d\theta$ (Fig. 265), the stress intensities on the two transverse planes will be equal to zero; and p_r = the stress intensity on the inner cylindrical surface, $p_r + dp_r$ = the stress intensity on the outer cylindrical surface and p_t = the stress intensity on each of the radial planes.

Since the stresses acting on the particle are in equilibrium, the sum of the radial components of the stresses acting on its faces will be equal to zero and hence

$$(p_r + dp_r)(r + dr)\,d\theta - p_r r\,d\theta - 2\,p_t\,dr \sin \tfrac{1}{2}\,d\theta = 0,$$

which reduces to

$$\frac{p_r - p_t}{r} + \frac{dp_r}{dr} = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

giving a relation existing between the radial and tangential stress intensities throughout the wall of the cylinder. Differentiating (3),

$$\frac{dp_r}{dr} = \frac{mE}{m^2 - 1}\left(m\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right);$$

and, substituting this value, together with the values of p_r and p_t from (3) and (4), in equation (5) and reducing, we obtain the differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation may be written in the form

$$\frac{d}{dr}\left(\frac{du}{dr}\right) = -\frac{d}{dr}\left(\frac{u}{r}\right);$$

and hence, by integrating,

$$\frac{du}{dr} = -\frac{u}{r} + 2A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $2A$ = the constant of integration.

By transforming (7) we obtain

$$u + r\frac{du}{dr} = \frac{d}{dr}(ur) = 2Ar;$$

and integrating,

$$ur = Ar^2 + B,$$

or,

$$u = Ar + \frac{B}{r}, \tag{8}$$

where B = the constant of integration.
Hence

$$e_r = \frac{du}{dr} = A - \frac{B}{r^2} \quad \ldots\ldots\ldots \tag{9}$$

and

$$e_t = \frac{u}{r} = A + \frac{B}{r^2}; \quad \ldots\ldots\ldots \tag{10}$$

and, by substituting the values of the strains in (3) and (4) and reducing,

$$p_r = \frac{mE}{m-1}A - \frac{mE}{m+1}\left(\frac{B}{r^2}\right) \quad \ldots\ldots \tag{11}$$

and

$$p_t = \frac{mE}{m-1}A + \frac{mE}{m+1}\left(\frac{B}{r^2}\right). \quad \ldots\ldots \tag{12}$$

To determine A and B we may substitute in (11) $p_r = P_1$, when $r = r_1$, and $p_r = P_2$, when $r = r_2$, and solve the two equations simultaneously, obtaining

$$A = \frac{m-1}{mE}\left(\frac{P_1 r_1^2 - P_2 r_2^2}{r_1^2 - r_2^2}\right) \quad \ldots\ldots \tag{13}$$

and

$$B = \frac{m+1}{mE}\left(\frac{(P_1 - P_2)\, r_1^2 r_2^2}{r_1^2 - r_2^2}\right). \quad \ldots\ldots \tag{14}$$

By substituting the values of A and B in equations (8) to (12) inclusive, the change in the radius and the principal strains and stress intensities at any point in the cylinder, due to any combination of internal and external pressures, can be found.

When the external pressure $P_2 = 0$, the above-named equations reduce to

$$u = \frac{P_1 r_1^2\,[(m-1)\,r^2 + (m+1)\,r_2^2]}{mEr\,(r_1^2 - r_2^2)}, \quad \ldots\ldots \tag{15}$$

$$e_r = \frac{P_1 r_1^2\,[(m-1)\,r^2 - (m+1)\,r_2^2]}{mEr^2\,(r_1^2 - r_2^2)}, \quad \ldots\ldots \tag{16}$$

$$e_t = \frac{P_1 r_1^2\,[(m-1)\,r^2 + (m+1)\,r_2^2]}{mEr^2\,(r_1^2 - r_2^2)}, \quad \ldots\ldots \tag{17}$$

$$p_r = \frac{P_1 r_1^2\,(r^2 - r_2^2)}{r^2\,(r_1^2 - r_2^2)} \quad \ldots\ldots\ldots\ldots \tag{18}$$

and

$$p_t = \frac{P_1 r_1^2 (r^2 + r_2^2)}{r^2 (r_1^2 - r_2^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

It will be evident from an inspection of the foregoing equations that the greatest stresses and strains in a cylinder under internal pressure only will occur at points on the inner surface of the wall and, by putting $r = r_1$, in (18) and (19), we obtain for the maximum stress intensities,

$$\text{max. } p_r = P_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$\text{max. } p_t = \frac{P_1 (r_1^2 + r_2^2)}{(r_1^2 - r_2^2)}. \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The variation in the tangential stress intensity p_t, on a radial section of the cylinder subjected to internal pressure only, is indicated by the diagram *aa'b'b*, constructed by erecting ordinates representing values of p_t at different points on the base *ab*.

If we assume $m = 3$, the values of the products of E and the principal strains at any point in the inner surface, when the cylinder is subjected to internal pressure only, become equal to

$$\text{max. } Ee_r = \frac{P_1 (2 r_1^2 - 4 r_2^2)}{(r_1^2 - r_2^2)} \quad . \quad . \quad . \quad . \quad (22)$$

and

$$\text{max. } Ee_t = \frac{P_1 (2 r_1^2 + 4 r_2^2)}{3 (r_1^2 - r_2^2)}. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If we assume $m = 4$ these products become equal to

$$\text{max. } Ee_r = \frac{P_1 (3 r_1^2 - 5 r_2^2)}{4 (r_1^2 - r_2^2)} \quad . \quad . \quad . \quad . \quad (24)$$

and

$$\text{max. } Ee_t = \frac{P_1 (3 r_1^2 + 5 r_2^2)}{4 (r_1^2 - r_2^2)}. \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If e_z = the extension in the direction of the axis of the cylinder it is evident that

$$Ee_z = -\frac{p_r}{m} - \frac{p_t}{m} = -\frac{2E}{m-1} A = \text{a constant.} \quad . \quad . \quad (26)$$

If a uniform longitudinal tension P_3 is applied to the cylinder in combination with the radial pressure, the stress intensities p_r and p_t will not be affected; but the constant $-\frac{p'}{m}$ will be added to the values of Ee_r and Ee_t, determined from the radial pressure alone.

The longitudinal stress intensity will be given by the expression:

$$p' = \frac{P_2}{\pi\,(r_2^2 - r_1^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

196. Thick Hollow Sphere. — The principal stress intensities and strains at any point in a thick hollow sphere, subjected to uniform internal or external pressure, may be determined by a method similar to that employed in the case of the cylinder.

Let r_1 = the internal radius and r_2 = the external radius of a hollow sphere which is subjected to a uniform internal pressure, of intensity P_1, and a uniform external pressure, of intensity P_2. Through any point O, at a distance r from the center, the principal planes of stress will be any two meridian planes at right angles to each other, intersecting in the radius CR, and the tangent plane through O, intersecting either meridian plane in a line OT, as indicated in the sketch of a section of the sphere (Fig. 266).

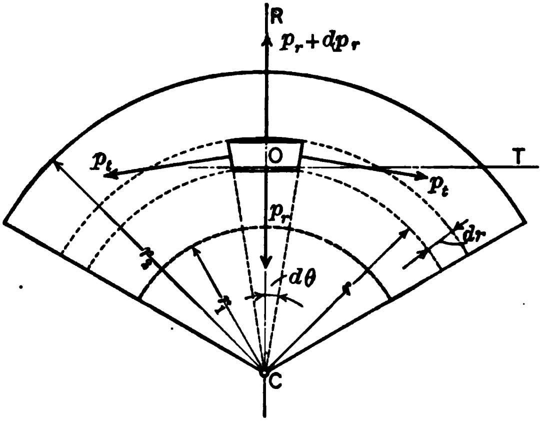


Fig. 266.

Let p_r = the intensity of the stress on the tangent plane and p_t = the intensity of stress on a meridian plane through O, e_r = the strain in the direction OR and e_t = the strain in the direction of the tangent OT. Assume tensile stresses and strains positive and compressive stresses and strains negative; and let u = the change in length of the radius r, due to the distortion of the sphere under pressure.

Then

$$e_r = \frac{du}{dr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$e_t = \frac{2\pi(u+r) - 2\pi r}{2\pi r} = \frac{u}{r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and, from (6) and (4), (Art. 49),

$$p_r = \frac{mE}{(m+1)(m-2)}\left[(m-1)\frac{du}{dr} + 2\frac{u}{r}\right] \quad . \quad . \quad (3)$$

and

$$p_t = \frac{mE}{(m+1)(m-2)}\left[m\frac{u}{r} + \frac{du}{dr}\right]. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we consider a small particle at O, bounded by the surfaces of two concentric spheres of radii r and $r + dr$ and a circular cone, having $2\,d\theta$ for the angle at the vertex C, as indicated in Fig. (266), the stress intensities on the inner and outer spherical surfaces will be equal to p_r and $p_r + dp_r$, respectively, and at any point on the conical surface the stress intensity will be equal to p_t. For equilibrium the sum of the radial components of the stresses on the particle must be equal to zero and hence

$$(p_r + dp_r)\,\pi\,[(r + dr)\,d\theta]^2 - p_r\pi\,(r d\theta)^2 - p_t \sin d\theta\; 2\pi r\, d\theta\, dr = 0,$$

which reduces to

$$\frac{2(p_r - p_t)}{r} + \frac{dp_r}{dr} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

giving a relation existing between the radial and tangential stress intensities throughout the wall of the sphere.

Differentiating (3),

$$\frac{dp_r}{dr} = \frac{mE}{(m+1)(m-2)}\left[(m-1)\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - \frac{2u}{r^2}\right]; \quad . \quad (6)$$

and, substituting this value, together with the values of p_r and p_t, in (5) and reducing, we obtain

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - \frac{2u}{r^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Writing (7) in the form

$$\frac{d}{dr}\left(\frac{du}{dr}\right) = -2\frac{d}{dr}\left(\frac{u}{r}\right),$$

and integrating, we obtain

$$\frac{du}{dr} = -2\frac{u}{r} + 3A, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where $3A$ = an unknown constant. Multiplying by r^2 and transposing, we have

$$r^2\frac{du}{dr} + 2ur = \frac{d}{dr}(r^2u) = 3Ar^2; \quad . \quad . \quad . \quad . \quad (9)$$

and integrating,

$$r^2u = Ar^3 + B,$$

or,

$$u = Ar + \frac{B}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where B = the constant of integration.
Hence,

$$e_r = \frac{du}{dr} = A - \frac{2B}{r^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$e_t = \frac{u}{r} = A + \frac{B}{r^3}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and, by substituting the values of the strains in (3) and (4) and reducing,

$$p_r = \frac{mE}{m-2}A - \frac{2mE}{m+1}\left(\frac{B}{r^3}\right) \quad . \quad . \quad . \quad . \quad (13)$$

and

$$p_t = \frac{mE}{m-2}A + \frac{mE}{m+1}\left(\frac{B}{r^3}\right). \quad . \quad . \quad . \quad . \quad (14)$$

To determine A and B we may substitute in (13) $p_r = P_1$, when $r = r_1$, and $p_r = P_2$, when $r = r_2$, and, by solving the two equations simultaneously, obtain

$$A = \frac{m-2}{mE}\left(\frac{P_1r_1^3 - P_2r_2^3}{r_1^3 - r_2^3}\right), \quad . \quad . \quad . \quad . \quad . \quad (15)$$

$$B = \frac{m+1}{2mE}\left(\frac{(P_1 - P_2)\,r_1^3r_2^3}{r_1^3 - r_2^3}\right). \quad . \quad . \quad . \quad . \quad (16)$$

By substituting the values of A and B in equations (10) to (14), inclusive, the change in the radius and the principal strains and stresses at any point, due to any combination of internal and external pressures, can be found.

When the external pressure $P_2 = 0$, these equations reduce to

$$u = \frac{P_1 r_1^3 \left[(m-2) r^3 + (m+1) r_2^3\right]}{mEr^2 (r_1^3 - r_2^3)}, \quad . \quad . \quad . \quad (17)$$

$$e_r = \frac{P_1 r_1^3 \left[(m-2) r^3 - (m+1) r_2^3\right]}{mEr^3 (r_1^3 - r_2^3)}, \quad . \quad . \quad . \quad (18)$$

$$e_t = \frac{P_1 r_1^3 \left[(m-2) r^3 + \left(\frac{m+1}{2}\right) r_2^3\right]}{mEr^3 (r_1^3 - r_2^3)}, \quad . \quad . \quad . \quad (19)$$

$$p_r = \frac{P_1 r_1^3 (r^2 - r_2^3)}{r^3 (r_1^3 - r_2^3)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and

$$p_t = \frac{P_1 r_1^3 (2 r^3 + r_2^3)}{2 r^3 (r_1^3 - r_2^3)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The greatest values of the strains and stress intensities, when the sphere is subjected to internal pressure only, will occur at points on the inside surface; and, by putting $r = r_1$ in (20) and (21), we obtain for the maximum stress intensities

$$\text{max. } p_r = P_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and

$$\text{max. } p_t = \frac{P_1 (2 r_1^3 + r_2^3)}{2 (r_1^3 - r_2^3)}. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

If we assume $m = 3$, the values of the products of E and the principal strains at any point in the inner surface, when the sphere is subjected to internal pressure only, become equal to

$$\text{max. } Ee_r = \frac{P_1 (r_1^3 - 4 r_2^3)}{3 (r_1^3 - r_2^3)} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and

$$\text{max. } Ee_t = \frac{P_1 (r_1^3 + 2 r_2^3)}{3 (r_1^3 - r_2^3)}. \quad . \quad . \quad . \quad . \quad (25)$$

If we assume $m = 4$, these products become equal to

$$\text{max. } Ee_r = \frac{P_1 (2 r_1^3 - 5 r_2^3)}{4 (r_1^3 - r_2^3)} \quad . \quad . \quad . \quad . \quad (26)$$

and

$$\text{max. } Ee_t = \frac{P_1 (4 r_1^3 + 5 r_2^3)}{8 (r_1^3 - r_2^3)}. \quad . \quad . \quad . \quad . \quad (27)$$

197. Circular Flat Plate Uniformly Loaded. — The expressions for the stress intensities and the strains due to a uniform normal pressure on the surface of a circular flat plate will be derived for two special cases, namely: (a) when the plate is fixed in direction at the circumference and (b) when the plate is freely supported at the circumference. For convenience the plate will be taken to be horizontal and the load to act vertically downwards in each case. The material is assumed to be homogeneous and of uniform elasticity.

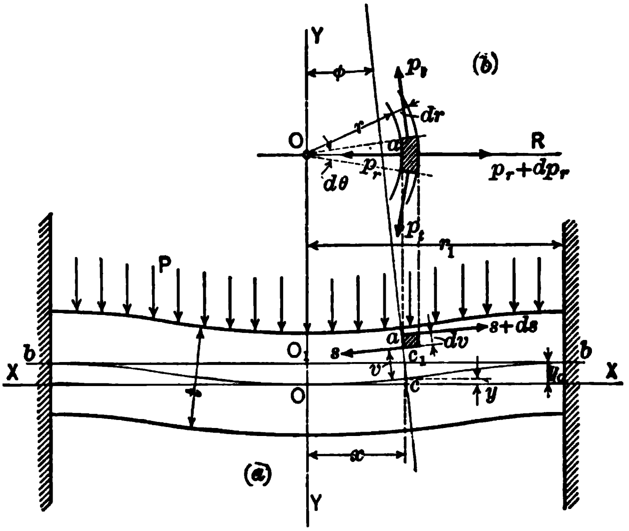


FIG. 267.

(*a*) *Plate Fixed at the Circumference.* — Let the sketch (Fig. 267a) represent a vertical cross section through the center, or a meridian section, of the plate. Let P = the load intensity, r_1 = the outside radius and t = the thickness of the plate. Let W = the total load on the plate. Then

$$W = P\pi r_1^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let bO_1b represent a section through the middle layer of the plate in the unstrained state and bOb the section through the middle layer after the load is applied, the middle layer of the strained plate being, evidently, the surface of revolution formed by revolving the curve Ob about the axis OY. Assume the axis OX through the point O and let c be any point in the meridian section Ob, having the coördinates (x, y), the original position of the point being c_1, having the coördinates $(x, 0)$. Evidently all points on a circle of radius x, through c_1, will be displaced the same vertical distance under the load.

If the thickness t is considerably smaller than the radius r_1, it may be assumed: (1) *That all straight lines perpendicular to the plane of the middle layer, before flexure, remain straight and normal to the middle layer after bending takes place;* and, since the material is homogeneous and elastic, it may be assumed; (2) *That the stress intensity is proportional to the strain in all directions through the plate.* Strains and stresses in the vertical direction, due directly to the pressure on the surface, will be so small as to be negligible.

It will follow from these assumptions that the middle layer will be the neutral layer, in which there is no tensile or compressive strain or stress. A normal to the neutral layer at c_1, before bending, will be perpendicular to the neutral layer at c after bending, making the angle

$$\phi = \tan^{-1}\frac{dy}{dx}$$

with OY. Let a be any point on the normal at c and let the distance $ca = v$. The distance of a from OY will evidently be equal to

$$r = x - v\sin\phi = x - v\,\frac{dy}{dx}\ \text{(very nearly)},$$

where v is positive, when measured above the neutral layer, and negative, when measured below the neutral layer.

The reciprocal of the radius of curvature of the meridian section Ob at c will be equal to

$$\frac{1}{R} = \frac{d^2y}{dx^2}\ \text{(very nearly)};$$

and, if we assume tensile stresses and strains plus and compressive stresses and strains minus, the radial strain at a, in the direction parallel to the tangent to Ob at c, will be equal to

$$e_r = -\frac{v}{R} = -v\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

The tangential strain at a, in the direction of the perpendicular to the plane XOY, will be equal to

$$e_t = \frac{2\pi\left(x - v\,\dfrac{dy}{dx}\right) - 2\pi x}{2\pi x} = -\frac{v}{x}\left(\frac{dy}{dx}\right). \quad . \quad . \quad . \quad . \qquad (3)$$

Then, if we let p_r = the radial stress intensity at the point a, that is, the normal stress intensity on the plane, perpendicular to the plane XOY and containing the element ca, and p_t = the tangential stress intensity at a, that is, the normal stress intensity on the plane XOY, we shall have, from (5) and (6) (Art. 46),

$$p_r = \frac{mE}{m^2-1}(me_r + e_t) = -\frac{mEv}{m^2-1}\left(m\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx}\right) \quad . \quad . \qquad (4)$$

and

$$p_t = \frac{mE}{m^2-1}\left(me_t + e_r\right) = -\frac{mEv}{m^2-1}\left(\frac{m}{x}\frac{dy}{dx} + \frac{d^2y}{dx^2}\right). \quad . \quad . \qquad (5)$$

If we consider a small particle at a (Fig. 267b), bounded by two planes, parallel to the tangent plane to the neutral surface at c and at distances v and $v + dv$ from c, two radial planes, intersecting at the axis OY and subtending the angle $d\theta$, and two curved surfaces, normal to the neutral surface, which, without appreciable error, may be considered as cylindrical surfaces having radii r and $r + dr$; the intensities of the stress components, acting on the faces of the particle in directions parallel to the tangent plane, will be the shearing components s and $s + ds$, the normal components p_t and the normal components p_r and $p_r + dp_r$, acting as indicated.

Since the stresses on the faces of the particle must be in equilibrium, the sum of the components in the radial direction OR will equal zero; and hence

$$(p_r + dp_r)\, dv\, (r + dr)\, d\theta - p_r\, dv\, r\, d\theta - 2\, p_t\, dv\, dr \sin \tfrac{1}{2}\, d\theta + (s + ds)\, r\, d\theta\, dr - s\, r\, d\theta\, dr = 0,$$

which reduces to

$$\frac{p_r - p_t}{r} + \frac{dp_r}{dr} + \frac{ds}{dv} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Since the deflection of the plate is small, we may write $r = x$ and $dr = dx$, in which case (6) may be written

$$-\frac{ds}{dv} = \frac{p_r - p_t}{x} + \frac{dp_r}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

Differentiating (4) with respect to x, we obtain, for any layer at a distance v from the neutral layer,

$$\frac{dp_r}{dx} = -\frac{mEv}{m^2 - 1}\left(m\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx}\right);$$

and, substituting the value of the derivative, together with the values of p_r and p_t in (7) and reducing, we obtain

$$\frac{ds}{dv} = \frac{m^2Ev}{m^2 - 1}\left(\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx}\right) = vZ, \quad . \quad . \quad . \quad . \qquad (8)$$

where

$$Z = \frac{m^2E}{m^2 - 1}\left(\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx}\right). \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

Integrating (8), observing that Z is constant for any one value of x, we obtain

$$s = \frac{v^2Z}{2} - \frac{t^2Z}{8} = \frac{Z}{8}(4v^2 - t^2), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

where $-\frac{t^2Z}{8}$ is the constant of integration, determined from the condition that $s = 0$, when $v = \pm\frac{t}{2}$

Equation (10) evidently gives the radial, or horizontal, shearing stress intensity in terms of Z, at any point a at a distance x from OY, on a layer of the plate parallel to and at the distance v from the neutral layer; and hence (Art. 24) it gives the vertical shearing stress intensity at the point a in the conical surface, formed by revolving the normal ac about the axis OY. Since the deflection of the plate is small, this conical surface will be very nearly the same as that of a right cylinder, of radius x and length t; and the total shearing stress around its circumference will be equal to the resultant pressure on the portion of the surface of the plate within a circle of radius x.

Hence

$$P\pi x^2 = 2\pi x\int_{-\frac{t}{2}}^{\frac{t}{2}} s\, dv = \frac{\pi x Z}{4}\int_{-\frac{t}{2}}^{\frac{t}{2}} (4v^2 - t^2)\, dv = -\pi x Z\frac{t^3}{6}; \quad . \qquad (11)$$

and, substituting the value of Z and transposing terms,

$$\frac{d^3y}{dx^3}+\frac{1}{x}\frac{d^2y}{dx^2}-\frac{1}{x^2}\frac{dy}{dx}=-\frac{6\,(m^2-1)\,P}{m^2Et^3}\,x=Ax, \quad . \quad . \quad . \quad (12)$$

where
$$A=-\frac{6\,(m^2-1)\,P}{m^2Et^3}=-\frac{6\,(m^2-1)\,W}{\pi r_1^2m^2Et^3}$$

Equation (12) may be written in the form

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)+\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dx}\right)=Ax;$$

and hence, by integrating,

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=A\left(\frac{x^2}{2}+B\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Equation (13) may be written in the form

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d}{dx}\left(x\,\frac{dy}{dx}\right)=A\left(\frac{x^3}{2}+Bx\right)$$

and, by integrating,

$$x\frac{dy}{dx}=A\left(\frac{x^4}{8}+\frac{Bx^2}{2}+C\right),$$

where $C=0$, since $\frac{dy}{dx}=0$ when $x=0$; and hence

$$\frac{dy}{dx}=A\left(\frac{x^3}{8}+\frac{Bx}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Integrating (14),

$$y=A\left(\frac{x^4}{32}+\frac{Bx^2}{4}+D\right),$$

where $D=0$, since $y=0$ when $x=0$; and hence

$$y=\frac{A}{32}\left(x^4+8\,Bx^2\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

To determine the remaining unknown constant B, we have

$\frac{dy}{dx}=0$ when $x=r_1$; and hence, from (14),

$$B=-\frac{r_1^2}{4}.$$

Substituting this value in (15), we have, for the equation of the neutral section Ob,

$$y=\frac{A}{32}\,(x^4-2\,r_1^2x^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

By differentiation, we obtain

$$\frac{dy}{dx}=\frac{A}{8}\,(x^3-r_1^2x), \quad \frac{d^2y}{dx^2}=\frac{A}{8}\,(3\,x^2-r_1^2), \quad \frac{d^3y}{dx^3}=\frac{3\,Ax}{4};$$

and, substituting the values of the derivatives in (2) and (3), we obtain the expressions for the radial and tangential strains at any point in the plate,

$$e_r=-\frac{Av}{8}\,(3\,x^2-r_1^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and

$$e_t=-\frac{Av}{8}\,(x^2-r_1^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

By substituting the values of e_r and e_t in (4) and (5), we obtain the expressions for the radial and tangential stress intensities at any point,

$$p_r = \frac{A_1 v}{8}[(3m+1)x^2 - (m+1)r_1^2] \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and

$$p_t = \frac{A_1 v}{8}[(m+3)x^2 - (m+1)r_1^2], \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where

$$A_1 = -\frac{mE}{m^2-1}A = \frac{6P}{mt^3} = \frac{6W}{\pi r_1^2 m t^3}.$$

If we determine the value of Z, by substituting the values of the derivatives in (9), and then substitute its value in (10), we obtain the value of the horizontal and vertical shearing stress intensities at any point,

$$s = \frac{3Px}{4t^3}(t^2 - 4v^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The greatest values of the radial and tangential stress intensities and strains will evidently occur at the surfaces of the plate, where $v = \pm\frac{t}{2}$; and it follows, from an inspection of (17) and (18), that the greatest strain in the plate is the radial strain at the surface of the plate at the circumference. Similarly, it is evident from an inspection of (19) and (20) that the greatest normal stress intensity is the radial stress intensity at the surface of the plate at the circumference.

Hence, if we assume $m = 3$ and substitute $x = r_1$, $v = \pm\frac{t}{2}$, together with the value of A in (17), we obtain, for the greatest value of the product of E and a principal strain,

$$\text{max. } Ee_r = \pm\frac{2Pr_1^2}{3t^2} = \pm\frac{2W}{3\pi t^2}; \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and, for the greatest value of a principal stress, we have, by putting $m = 3$, $x = r_1$, $v = \pm\frac{t}{2}$ and substituting the value of A_1 in (19),

$$\text{max. } p_r = \pm\frac{3Pr_1^2}{4t^2} = \frac{3W}{4\pi t^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

It is evident from an inspection of (21) that the greatest shearing stress intensity will be located at the neutral layer at the circumference of the plate; and, putting $x = r_1$ and $v = 0$, we obtain

$$\text{max. } s = \frac{3Pr_1}{4t} = \frac{3W}{4\pi r_1 t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The principal stress intensities $\pm p_1$, at a point of maximum shear, will also be represented by (24), the planes of principal stress making angles of 45° with the neutral layer (Art. 30); and hence, letting $m = 3$, the product of E and a principal strain at these points will be equal to

$$Ee_1 = p_1 + \frac{p_1}{m} = \frac{Pr_1}{t} = \frac{W}{\pi r_1 t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

By putting $x = r_1$ and substituting the value of A in (16), we obtain for the value of the greatest deflection of the plate,

$$y_0 = \frac{3}{16} \frac{m^2 - 1}{m^2 E t^3} P r_1^4; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and, if we assume $m = 3$,

$$y_0 = \frac{P r_1^4}{6 E t^3} = \frac{W r_1^2}{6 \pi E t^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

By putting

$$\frac{d^2 y}{dx^2} = \frac{A}{8} (3 x^2 - r_1^2) = 0,$$

we obtain

$$x = \frac{r_1}{\sqrt{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

for the distance of the point of inflexion of the curve Ob from the center of the plate, and a circle of radius $\frac{r_1}{\sqrt{3}}$ will evidently be the line of inflexion in the neutral layer of the plate.

If we consider a vertical section *one unit in width*, perpendicular to any radius at a distance x from the center of the plate, the normal stress on this section will be uniformly varying and the moment of resistance of the stress on the section will be equal to

$$M_r = -\frac{p_r' t^2}{6}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where the moment is considered positive when the curvature is convex downwards and p_r' is the normal stress intensity at the top surface of the plate. Letting $m = 3$ and substituting the value of A_1 and the value of p_r', obtained by putting $v = \frac{t}{2}$ in (19), we obtain

$$M_r = -\frac{P}{24} (5 x^2 - 2 r_1^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Similarly, if we consider a vertical strip *one unit in width* through any radial section, at a distance x from the center of the plate, the moment of resistance will be equal to

$$M_t = -\frac{p_t' t^2}{6}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where p_t' is the normal stress intensity at the top surface of the plate; and, substituting the value of A_1 and the value of p_t', obtained by putting $v = \frac{t}{2}$ and $m = 3$ in (20), we have

$$M_t = -\frac{P}{24} (3 x^2 - 2 r_1^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

The quantities M_r and M_t are frequently called the *bending moments* in the plate; and for the maximum values we have, when $x = r_1$,

$$\max. M_r = -\frac{P r_1^2}{8} = -\frac{W}{8 \pi}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

and, when $x = 0$,

$$\text{max.}\, M_t = \frac{Pr_1^2}{12} = \frac{W}{12\pi}, \quad \dots\dots\dots \quad (34)$$

which is evidently equal to the value of M_r at the center of the plate.

If m is assumed to be equal to 4, or any other quantity, the strains, stress intensities, deflections and bending moments can easily be determined by substituting the value of m in the fundamental equations (17), (18), (19), (20) and (26).

(b) *Plate Freely Supported at the Circumference.* — The assumptions and conditions for this case and the derivation of the formulas, as far as equation (15), will be identical with Case (a).

In this case the constant B can be determined from the condition that when $x = r_1$, $p_r = 0$; and by differentiating (15), we obtain

$$\frac{dy}{dx} = \frac{A}{8}(x^3 + 4\,Bx), \quad \frac{d^2y}{dx^2} = \frac{A}{8}(3\,x^2 + 4\,B), \quad \frac{d^3y}{dx^3} = \frac{3\,Ax}{4};$$

and, by substituting the values of the derivatives in (4),

$$p_r = -\frac{mEvA}{8\,(m^2 - 1)}[(3\,m + 1)\,x^2 + 4\,(m + 1)\,B]. \quad \dots\dots \quad (35)$$

Then, by substituting $x = r_1$, putting (35) equal to zero, and solving for B, we obtain

$$B = -\frac{3\,m + 1}{4\,(m + 1)}r_1^2.$$

Substituting the value of B in the above derivatives and then substituting the values of the derivatives, together with the value of A, in equations (2) and (3), we obtain

$$e_r = -\frac{Av}{8}\left[3\,x^2 - \frac{3\,m + 1}{m + 1}r_1^2\right] \quad \dots\dots\dots \quad (36)$$

and

$$e_t = -\frac{Av}{8}\left[x^2 - \frac{3\,m + 1}{m + 1}r_1^2\right]; \quad \dots\dots\dots \quad (37)$$

and, by substituting in (4) and (5),

$$p_r = \frac{A_1v}{8}[(3\,m + 1)\,(x^2 - r_1^2)] \quad \dots\dots\dots \quad (38)$$

and

$$p_t = \frac{A_1v}{8}[(m + 3)\,x^2 - (3\,m + 1)\,r_1^2]. \quad \dots\dots \quad (39)$$

By substituting the values of the derivatives in (9) and then calculating the value of s from (10), we would obtain the same expression for s as in Case (a) (equation 21).

It is evident from an inspection of the above equations that the greatest stress intensities and strains occur when $x = 0$ and $v = \pm\frac{t}{2}$; and, putting $m = 3$, and substituting the value of A in (36), or (37), we have

$$\text{max.}\, Ee_r = \text{max.}\, Ee_t = \mp\frac{5}{6}\frac{Pr_1^2}{t^2} = \mp\frac{5\,W}{6\,\pi t^2}. \quad \dots\dots \quad (40)$$

Substituting the value of A_1, together with the above values of x, v and m, in (38), or (39), we have

$$\text{max.}\, p_r = \text{max.}\, p_t = \mp \frac{5}{4}\frac{Pr_1^2}{t^2} = \mp \frac{5W}{4\pi t^2}. \quad . \quad . \quad . \quad . \quad (41)$$

The maximum value of s is given by (24), as in the preceding case, with the accompanying product of E and the principal strain, given by (25).

The equation of the neutral section, obtained by substituting the value of B in (15), will be

$$y = \frac{A}{32}\left[x^4 - 2\left(\frac{3m+1}{m+1}\right)r_1^2x^2\right]; \quad . \quad . \quad . \quad . \quad . \quad (42)$$

and, by substituting $x = r_1$ and the value of A, we obtain for the value of the greatest deflection, when $m = 3$,

$$y_0 = \frac{2\,Pr_1^4}{3\,Et^3} = \frac{2\,Wr_1^2}{3\pi Et^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

For the values of the bending moments we shall have, putting $v = \frac{t}{2}$ in (38) and (39), and substituting the values of p_r' and p_t' in (29) and (31), together with $m = 3$,

$$M_r = -\frac{P}{24}(5x^2 - 5r_1^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

$$M_t = -\frac{P}{24}(3x^2 - 5r_1^2); \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and for the maximum values, evidently,

$$\text{max.}\, M_r = \text{max.}\, M_t = \frac{5\,Pr_1^2}{24} = \frac{5W}{24\pi}. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

If m is assumed to have a value other than 3, the values of the stress intensities, strains, etc., may be easily derived in the same manner as the above.

198. Circular Flat Plate Centrally Loaded. — The expressions for the stress intensities and strains in a circular flat plate, due to a uniform load over the central portion only, can be determined by the method employed in Art. (197). Two cases will be considered, similar to the two cases in the preceding article, the plate being considered horizontal and the load as acting vertically downward in each case.

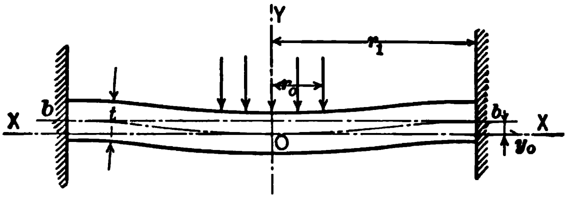


FIG. 268.

(a) *Plate Fixed at the Circumference.* — Let r_1 = the outside radius and t = the thickness of the plate, P = the load intensity and W = the total load, uniformly distributed over a circular area of radius r_0 (Fig. 268). Then

$$W = P\pi r_0^2, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

where the value of r_0, for any given value of W must be greater than a certain limiting value to be determined.

Making the same assumptions and following the same method as in the preceding article we obtain the same fundamental equations (2) to (10), inclusive (Art. 197).

Equating the total shear on a cylindrical section through the plate, of radius x, to the total load on the surface within the circle of radius x, we obtain, for values of x, from 0 to r_0,

$$P\pi x^2 = -\pi x Z \frac{t^2}{6} \text{ (Art. 197)}; \quad . \; . \; . \; . \; . \; . \quad (2)$$

and, for values of x from r_0 to r_1,

$$P\pi r_0^2 = -\pi x Z \frac{t^2}{6}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

Substituting the value of Z from (9) (Art. 197) and reducing, we have, for values of x from 0 to r_0,

$$\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx} = Ax; \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

and, for values of x from r_0 to r_1,

$$\frac{d^3y}{dx^3} + \frac{1}{x}\frac{d^2y}{dx^2} - \frac{1}{x^2}\frac{dy}{dx} = \frac{Ar_0^2}{x}, \quad . \; . \; . \; . \; . \; . \quad (5)$$

where

$$A = -\frac{6(m^2-1)P}{m^2Et^3} = -\frac{6(m^2-1)W}{\pi r_0^2 m^2 E t^3}.$$

Integrating (4), observing that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$, we obtain, as in Art. (197), for values of r from 0 to r_0,

$$y = A\left(\frac{x^4}{32} + \frac{Bx^2}{4}\right); \quad . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

and, for the values of the derivatives,

$$\frac{dy}{dx} = A\left(\frac{x^3}{8} + \frac{Bx}{2}\right), \quad . \; . \; . \; . \; . \; . \; . \quad (7)$$

$$\frac{d^2y}{dx^2} = A\left(\frac{3x^2}{8} + \frac{B}{2}\right). \quad . \; . \; . \; . \; . \; . \; . \quad (8)$$

$$\frac{d^3y}{dx^3} = A\frac{3x}{4}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

Similarly from (5), for values of r from r_0 to r_1,

$$y = Ar_0^2\left[\frac{x^2}{4}(\log x - 1) + \frac{Ex^2}{4} + F\log x + G\right]; \quad . \; . \; . \quad (10)$$

and

$$\frac{dy}{dx} = Ar_0^2\left[\frac{x}{2}\left(\log x - \frac{1}{2}\right) + \frac{Ex}{2} + \frac{F}{x}\right], \quad . \; . \; . \quad (11)$$

$$\frac{d^2y}{dx^2} = Ar_0^2\left[\frac{1}{2}\left(\log x + \frac{1}{2}\right) + \frac{E}{2} - \frac{F}{x^2}\right], \quad . \; . \; . \; . \quad (12)$$

$$\frac{d^3y}{dx^3} = Ar_0^2\left[\frac{1}{2x} + \frac{2F}{x^3}\right]. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

To determine the constants B, E, F and G, we have the conditions: when $x = r_0$, the values of y given by (6) and (10) are equal, the values of $\frac{dy}{dx}$ given by (7) and (11) are equal and the values of $\frac{d^2y}{dx^2}$ given by (8) and (12) are equal; and, when $x = r_1$, the value of $\frac{dy}{dx}$ given by (11) is equal to zero. The solution of these equations will give:

$$B = r_0^2\left(\log\frac{r_0}{r_1}\right) - \frac{r_0^4}{4r_1^2},$$

$$E = \frac{1}{2} - \log r_1 - \frac{r_0^2}{4r_1^2}$$

$$F = \frac{r_0^2}{8},$$

$$G = \frac{5r_0^2}{32} - \frac{r_0^2}{8}\log r_0.$$

Substituting these constants in the equations for the derivatives and then substituting in equations (2) and (3) (Art. 197), we obtain, for values of x from 0 to r_0,

$$e_r = -\frac{Av}{8}\left[3x^2 + 4r_0^2\log\frac{r_0}{r_1} - \frac{r_0^4}{r_1^2}\right], \quad . \; . \; . \; . \; . \; . \quad (14)$$

$$e_t = -\frac{Av}{8}\left[x^2 + 4r_0^2\log\frac{r_0}{r_1} - \frac{r_0^4}{r_1^2}\right]; \quad . \; . \; . \; . \; . \; . \quad (15)$$

and, for values of x from r_0 to r_1,

$$e_r = -\frac{Avr_0^2}{8}\left[4\log\frac{x}{r_1} - \frac{r_0^2}{x^2} - \frac{r_0^2}{r_1^2} + 4\right], \quad . \; . \; . \; . \quad (16)$$

$$e_t = -\frac{Avr_0^2}{8}\left[4\log\frac{x}{r_1} + \frac{r_0^2}{x^2} - \frac{r_0^2}{r_1^2}\right]. \quad . \; . \; . \; . \; . \; . \; . \quad (17)$$

Substituting the values of the strains in (4) and (5) (Art. 197), we obtain, for values of x from 0 to r_0,

$$p_r = \frac{A_1v}{8}\left[(3m+1)x^2 + (m+1)\left(4r_0^2\log\frac{r_0}{r_1} - \frac{r_0^4}{r_1^2}\right)\right], \quad . \; . \quad (18)$$

$$p_t = \frac{A_1v}{8}\left[(m+3)x^2 + (m+1)\left(4r_0^2\log\frac{r_0}{r_1} - \frac{r_0^4}{r_1^2}\right)\right]; \quad . \; . \quad (19)$$

and, for values of x from r_0 to r_1,

$$p_r = \frac{A_1vr_0^2}{8}\left[(m+1)\left(4\log\frac{x}{r_1} - \frac{r_0^2}{r_1^2}\right) - (m-1)\frac{r_0^2}{x^2} + 4m\right], \quad . \quad (20)$$

$$p_t = \frac{A_1vr_0^2}{8}\left[(m+1)\left(4\log\frac{x}{r_1} - \frac{r_0^2}{r_1^2}\right) + (m-1)\frac{r_0^2}{x^2} + 4\right], \quad . \; . \quad (21)$$

where

$$A_1 = -\frac{mE}{m^2-1}A = \frac{6P}{mt^3} = \frac{6W}{\pi r_0^2 m t^3}.$$

Substituting the values of the derivatives in (9) and (10) (Art. 197), we obtain for the shearing stress intensity, for values of x from 0 to r_0,

$$s = \frac{3Px}{4t^3}(t^2 - 4v^2), \quad \ldots \ldots \ldots \quad (22)$$

and, for values of x from r_0 to r_1,

$$s = \frac{3Pr_0^2}{4t^3x}(t^2 - 4v^2). \quad \ldots \ldots \ldots \quad (23)$$

At the center of the plate, when $v = \pm\frac{t}{2}$, we obtain, by substituting the values of A and A_1 in (14), (15), (18) and (19), assuming $m = 3$,

$$Ee_r = Ee_t = \pm\frac{W}{3\pi t^2}\left[4\log\frac{r_0}{r_1} - \left(\frac{r_0}{r_1}\right)^2\right] \quad \ldots \ldots \quad (24)$$

and

$$p_r = p_t = \pm\frac{W}{2\pi t^2}\left[4\log\frac{r_0}{r_1} - \left(\frac{r_0}{r_1}\right)^2\right]; \quad \ldots \ldots \quad (25)$$

and, similarly, at the circumference we obtain from (16) and (20),

$$Ee_r = \pm\frac{W}{3\pi t^2}\left[4 - 2\left(\frac{r_0}{r_1}\right)^2\right] \quad \ldots \ldots \quad (26)$$

and

$$p_r = \pm\frac{W}{2\pi t^2}\left[3 - \frac{3}{2}\left(\frac{r_0}{r_1}\right)^2\right]. \quad \ldots \ldots \quad (27)$$

On investigation it will be found that when $\frac{r_0}{r_1} > 0.42$ (nearly) the greatest strain will be located at the circumference of the plate and the maximum value of Ee_r will be given by (26). Similarly, when $\frac{r_0}{r_1} > 0.59$ (nearly) the greatest stress intensity will be given by (27). For ratios of $\frac{r_0}{r_1}$ less than the above values the maximum strains and stresses will be at the center.

From an inspection of (22) and (23) it will be evident that the greatest shearing stress intensity is located at the intersection of the neutral layer and the cylindrical surface of radius r_0 and will be equal to

$$\text{max. } s = \frac{3Pr_0}{4t} = \frac{3W}{4\pi r_0 t}; \quad \ldots \ldots \ldots \quad (28)$$

and, if $m = 3$, the product of E and the principal strain accompanying this stress will be equal to

$$Ee_1 = \frac{Pr_0}{t} = \frac{W}{\pi r_0 t}. \quad \ldots \ldots \ldots \quad (29)$$

If r_0 is very small the values of the stress intensity and the strain given by (28) and (29) for a total load W may exceed those at any other point in the plate, the values approaching infinity as r_0 approaches zero. The equations may be used therefore to determine the limiting radius of the circle over which the load may be distributed.

By putting the values of the constants in (6) and (10) we obtain the equations of the neutral section *Ocb*, for values of x from 0 to r_0,

$$y = \frac{A}{4}\left(\frac{x^4}{8} + r_0^2x^2\left(\log\frac{r_0}{r_1}\right) - \frac{r_0^4x^2}{4\,r_1^2}\right), \quad . \quad (30)$$

and, for values of x from r_0 to r_1,

$$y = \frac{Ar_0^2}{4}\left[x^2\log\frac{x}{r_1} + \frac{r_0^2}{2}\log\frac{x}{r_0} - \frac{x^2}{2} - \frac{r_0^2x^2}{4\,r_1^2} + \frac{5\,r_0^2}{8}\right]. \quad (31)$$

For the greatest deflection, putting the value of A and $x = r_1$ in (31), we obtain, assuming $m = 3$,

$$y_0 = \frac{2\,W}{3\,\pi Et^3}\left[r_1^2 - \frac{3\,r_0^2}{4} - r_0^2\log\frac{r_1}{r_0}\right]. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

When $r_1 > 2\,r_0$ (nearly) the point of inflexion in the neutral section *Ocb* can be found by putting (12) equal to zero and solving for x, the equation being

$$\log\frac{x}{r_1} - \frac{r_0^2}{4\,x^2} - \frac{r_0^2}{4\,r_1^2} + 1 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

(b) *Plate Freely Supported at the Circumference.* — For this case the fundamental equations will be in the same form as equations (1) to (13), inclusive, for Case (a). To determine the constants in the equations, we have the conditions: when $x = r_0$, the values of y given by (6) and (10) are equal, the values of $\frac{dy}{dx}$ given by (7) and (11) are equal and the values of $\frac{d^2y}{dx^2}$ given by (8) and (12) are equal; and, when $x = r_1$, the stress intensity p_r is equal to zero. The solution of these equations will give

$$B = r_0^2\left(\log\frac{r_0}{r_1}\right) + \frac{m-1}{m+1}\,\frac{r_0^4}{4\,r_1^2} - \frac{m}{m+1}\,r_0^2,$$

$$E = -\frac{1}{2}\left(\frac{m-1}{m+1}\right) - \log r_1 + \left(\frac{m-1}{m+1}\right)\frac{r_0^2}{4\,r_1^2},$$

$$F = \frac{r_0^2}{8},$$

$$G = \frac{5\,r_0^2}{32} - \frac{r_0^2}{8}\log r_0.$$

Proceeding as in Case (a), we obtain, for values of x from 0 to r_0,

$$e_r = -\frac{Av}{8}\left[3\,x^2 + 4\,r_0^2\log\frac{r_0}{r_1} + \left(\frac{m-1}{m+1}\right)\frac{r_0^4}{r_1^2} - \frac{4\,mr_0^2}{m+1}\right], \quad . \quad (34)$$

$$= -\frac{Av}{8}\left[x^2 + 4\,r_0^2\log\frac{r_0}{r_1} + \left(\frac{m-1}{m+1}\right)\frac{r_0^4}{r_1^2} - \frac{4\,mr_0^2}{m+1}\right]; \quad . \quad . \quad (35)$$

and, for values of x from r_0 to r_1,

$$e_r = -\frac{Avr_0^2}{8}\left[4\log\frac{x}{r_1} - \frac{r_0^2}{x^2} + \left(\frac{m-1}{m+1}\right)\frac{r_0^2}{r_1^2} + \frac{4}{m+1}\right], \quad . \quad . \quad (36)$$

$$e_t = -\frac{Avr_0^2}{8}\left[4\log\frac{x}{r_1} + \frac{r_0^2}{x^2} + \left(\frac{m-1}{m+1}\right)\frac{r_0^2}{r_1^2} - \frac{4\,m}{m+1}\right]. \quad . \quad . \quad (37)$$

For the stress intensities we have, for values of x from 0 to r_0,

$$p_r = \frac{A_1 v}{8}\left[(3m+1)x^2 + (m+1)\,4r_0^2\log\frac{r_0}{r_1} + (m-1)\frac{r_0^4}{r_1^2} - 4mr_0^2\right], \quad . \quad (38)$$

$$p_t = \frac{A_1 v}{8}\left[(m+3)x^2 + (m+1)\,4r_0^2\log\frac{r_0}{r_1} + (m-1)\frac{r_0^4}{r_1^2} - 4mr_0^2\right]; \quad . \quad (39)$$

and, for values of x from r_0 to r_1,

$$p_r = \frac{A_1 v r_0^2}{8}\left[(m+1)\,4\log\frac{x}{r_1} - (m-1)\left(\frac{r_0^2}{x^2} - \frac{r_0^2}{r_1^2}\right)\right], \quad . \quad (40)$$

$$p_t = \frac{A_1 v r_0^2}{8}\left[(m+1)\,4\log\frac{x}{r_1} + (m-1)\left(\frac{r_0^2}{x^2} + \frac{r_0^2}{r_1^2} - 4\right)\right]. \quad . \quad (41)$$

It will be apparent from an inspection of the foregoing equations that the greatest values of the strains and stress intensities occur at the center of the plate and that, if $m = 3$,

$$\text{max. } Ee_r = \text{max. } Ee_t = \pm\frac{W}{3\pi t^2}\left[4\log\frac{r_0}{r_1} + \frac{1}{2}\left(\frac{r_0}{r_1}\right)^2 - 3\right], \quad . \quad (42)$$

$$\text{max. } p_r = \text{max. } p_t = \pm\frac{W}{2\pi t^2}\left[4\log\frac{r_0}{r_1} + \frac{1}{2}\left(\frac{r_0}{r_1}\right)^2 - 3\right]. \quad . \quad . \quad (43)$$

The values of s will be given by (22) and (23), as in Case (a), the maximum value of s and the accompanying principal strain being given by (28) and (29).

The equations of the meridian section through the neutral surface, obtained by putting the values of the constants in (6) and (10), will be, for values of x from 0 to r_0,

$$y = \frac{A}{4}\left(\frac{x^4}{8} + r_0^2x^2\log\frac{r_0}{r_1} + \frac{m-1}{m+1}\right)\frac{r_0^2x^2}{4r_1^2} - \frac{mr_0^2x^2}{m+1}\Big). \quad . \quad . \quad (44)$$

and, for values of x from r_0 to r_1,

$$y = \frac{Ar_0^2}{4}\left(x^2\log\frac{x}{r_1} + \frac{r_0^2}{2}\log\frac{x}{r_0} - \frac{3m+1}{2(m+1)}x^2 + \left(\frac{m-1}{m+1}\right)\frac{r_0^2x^2}{4r_1^2} + \frac{5r_0^2}{8}\right). \quad . \quad (45)$$

By substituting the value of A and putting $x = r_1$ in (45) we obtain for the value of the greatest deflection, assuming $m = 3$,

$$y_0 = \frac{2W}{3\pi Et^3}\left[\frac{5}{2}r_1^2 - \frac{3}{2}r_0^2 - r_0^2\log\frac{r_1}{r_0}\right]. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

In either of the cases considered under this article the values of the bending moments M_r and M_t, at any point in the plate, can be readily determined by substituting the expressions for the stress intensities p_r' and p_t', at the surface of the plate, in equations (29) and (31) (Art. 197).

199. Circular Flat Plate Uniformly Loaded and Supported at the Center.

— If a circular flat plate is subjected to a uniformly distributed load combined with a center load, the strains, stress intensities and deflections, at any point in the plate, can be determined by adding the values obtained for each load separately, from the equations in Arts. (197) and (198). This method can be used for a plate which is fixed at the circumference, or when the plate is freely supported. The case under consideration is a special case of the latter type in which the supporting force, or center load, is equal in magnitude to the total uniform load and opposite in direction, the resultant reaction at the circumference being equal to zero.

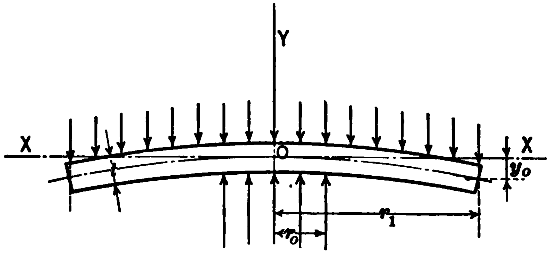


FIG. 269.

Let r_1 = the outside radius and t = the thickness of the plate, P_1 = the intensity of the load and P_0 = the intensity of the supporting force, which is uniformly distributed over a circle of radius r_0 (Fig. 269), the minimum value of r_0 being determined by the conditions stated in Art. (198). Then the total load

$$W = P_1\pi r_1^2 = -P_0\pi r_0^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The following expressions for the strains and stress intensities throughout the plate can be readily obtained by adding the values under Case (b) (Art. 197) and Case (b) (Art. 198), calling the upward pressure negative.

For values of x from 0 to r_0,

$$e_r = Av\left[\frac{3x^2}{r_1^2} - \frac{3x^2}{r_0^2} - 4\log\frac{r_0}{r_1} - \left(\frac{m-1}{m+1}\right)\frac{r_0^2}{r_1^2} + \frac{m-1}{m+1}\right], \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$e_t = Av\left[\frac{x^2}{r_1^2} - \frac{x^2}{r_0^2} - 4\log\frac{r_0}{r_1} - \left(\frac{m-1}{m+1}\right)\frac{r_0^2}{r_1^2} + \frac{m-1}{m+1}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$p_r = A_1v\left[(3m+1)\left(\frac{x^2}{r_1^2} - \frac{x^2}{r_0^2}\right) - 4(m+1)\log\frac{r_0}{r_1} - (m-1)\frac{r_0^2}{r_1^2} + (m-1)\right], \quad . \quad (4)$$

$$p_t = A_1v\left[(m+3)\left(\frac{x^2}{r_1^2} - \frac{x^2}{r_0^2}\right) - 4(m+1)\log\frac{r_0}{r_1} - (m-1)\frac{r_0^2}{r_1^2} + (m-1)\right]; \quad (5)$$

and, for values of x from r_0 to r_1,

$$e_r = Av\left[\frac{3x^2}{r_1^2} + \frac{r_0^2}{x^2} - 4\log\frac{x}{r_1} - \left(\frac{m-1}{m+1}\right)\frac{r_0^2}{r_1^2} - \frac{3m+5}{m+1}\right], \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$e_t = Av\left[\frac{x^2}{r_1^2} - \frac{r_0^2}{x^2} - 4\log\frac{x}{r_1} - \left(\frac{m-1}{m+1}\right)\frac{r_0^2}{r_1^2} + \frac{m-1}{m+1}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$p_r = A_1v\left[(3m+1)\frac{x^2}{r_1^2} + (m-1)\frac{r_0^2}{x^2} - 4(m+1)\log\frac{x}{r_1} - (m-1)\frac{r_0^2}{r_1^2} - (3m+1)\right], \quad (8)$$

$$p_t = A_1v\left[(m+3)\frac{x^2}{r_1^2} - (m-1)\frac{r_0^2}{x^2} - 4(m+1)\log\frac{x}{r_1} - (m-1)\frac{r_0^2}{r_1^2} + (m-5)\right]; \quad . \quad (9)$$

the values of the constants being

$$A = \frac{3\,(m^2 - 1)\,W}{4\,\pi m^2 E t^3} \quad \text{and} \quad A_1 = \frac{3\,W}{4\,\pi m t^3}.$$

The bending moments, so-called, will be given by (29) and (31) (Art. 197), the values of p_r' and p_t' being obtained by putting $v = \pm \frac{t}{2}$ in (4) and (8) and (5) and (9).

Values of s, for values of x from 0 to r_0, will be represented by the following expression, obtained by combining (21) (Art. 197) and (22) (Art. 198),

$$s = \frac{3\,Wx}{4\,\pi t^3}\left(\frac{1}{r_1^2} - \frac{1}{r_0^2}\right)(t^2 - 4\,v^2); \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and, by combining (21) (Art. 197) and (23) (Art. 198) we have, for values of x from r_0 to r_1,

$$s = \frac{3\,W}{4\,\pi t^3}\left(\frac{x}{r_1^2} - \frac{1}{x}\right)(t^2 - 4\,v^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The equations of the meridian section of the neutral layer can be obtained by combining (42) (Art. 197) with (44) and (45) (Art. 198).

For the maximum values of the strains and stress intensities we have the following, assuming $m = 3$;

$$\text{max.}\, Ee_r = \text{max.}\, Ee_t = \pm \frac{W}{3\,\pi t^2}\left[\frac{1}{2} - \frac{1}{2}\left(\frac{r_0}{r_1}\right)^2 - 4\log\frac{r_0}{r_1}\right], \quad . \quad . \quad (12)$$

$$\text{max.}\, p_r = \text{max.}\, p_t = \pm \frac{W}{2\,\pi t^2}\left[\frac{1}{2} - \frac{1}{2}\left(\frac{r_0}{r_1}\right)^2 - 4\log\frac{r_0}{r_1}\right], \quad . \quad . \quad (13)$$

located at the center of the plate;

$$\text{max.}\, s = \frac{3\,W}{4\,\pi r_0 t}\left[\left(\frac{r_0}{r_1}\right)^2 - 1\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

when $x = r_0$, with the accompanying value of the product of E and the principal strain

$$Ee_1 = \frac{W}{\pi r_0 t}\left[\left(\frac{r_0}{r_1}\right)^2 - 1\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The greatest deflection, obtained by adding (43) (Art. 197) and (46) (Art. 198), making due allowance for signs, will be equal to

$$y_0 = \frac{W}{\pi E t^3}\left[r_0^2 - r_1^2 + \frac{2}{3}\,r_0^2 \log\frac{r_1}{r_0}\right]. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

200. Square and Rectangular Flat Plates Uniformly Loaded. — If we attempt to deduce equations for the strains and stresses in a square, or rectangular, flat plate, uniformly loaded and fixed, or freely supported, at the edges, on the basis of the assumptions made in the case of the circular plate, an exact solution will be found to be impossible Approximate formulas, however, which must be regarded as largely empirical, can be obtained by making additional assumptions in regard to the distribution of the stresses in the plate. Four cases will be considered.

(a) *Rectangular Plate Fixed at the Edges.* — Let 2 a and 2 b be the dimensions of the surface, t = the thickness and P = the pressure intensity (Fig. 270). Neglecting stresses and strains in the direction normal to the surface, the same assumptions may be made as in the case of the circular plate, namely: (1) *That all straight lines, perpendicular to the surface of the plate before flexure, remain straight and normal to the middle, or neutral, layer of the plate after bending takes place,* and (2) *that stress intensities are proportional to strains throughout the plate.*

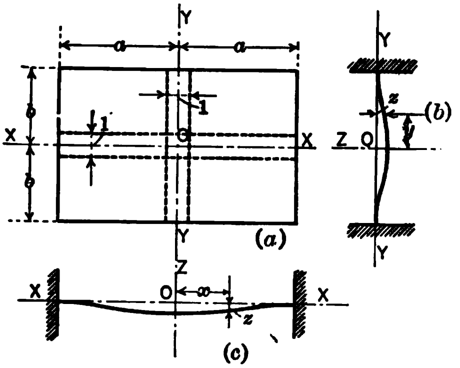


FIG. 270.

If we consider a strip through the center of the plate, one unit in width and parallel to the axis OY, it is evident that, if the dimension a is increased indefinitely, the strains and stresses in the strip will approach the values of the stresses and strains in a rectangular beam, fixed at the ends, of the same cross section as the strip and subjected to the same uniform load. Hence, at the limit when $a = \infty$, the bending moment at any cross section of the strip, at a distance y from the center (Fig. 270*b*), will be equal to

$$M = M_0 - \frac{Py^2}{2} = \frac{Pb^2}{6} - \frac{Py^2}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the deflection at the cross section,

$$z = \int\int \frac{M}{EI}\,dx\,dx = -\frac{P}{24\,EI}(b^2 - y^2)^2 = -\frac{P}{2\,Et^3}(b^2 - y^2)^2. \quad . \quad (2)$$

Similarly, if we consider a strip one unit wide along OX, and b is increased indefinitely; at the limit, when $b = \infty$, the value of the bending moment at a section, at any distance x from the center (Fig. 270c), will be equal to

$$M = \frac{Pa^2}{6} - \frac{Px^2}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the deflection at the cross section

$$z = -\frac{P}{2\,Et^3}(a^2 - x^2)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

It was assumed by Grashof that the following equation, which would satisfy each of the limiting cases mentioned, could be taken as the equation of the surface formed by the neutral layer after bending; namely,

$$z = -\frac{P}{2\,Et^3}\,\frac{(a^2 - x^2)^2\,(b^2 - y^2)^2}{a^4 + b^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Then, if we let r_x = the radius of curvature of a section of the neutral layer, parallel to OX, through any point in the layer, whose coördinates are (x, y, z), and r_y = the radius of curvature of the section parallel to OY, through the same point, we shall have, making the same approximation as in the common beam theory,

$$\frac{1}{r_x} = \frac{\partial^2 z}{\partial x^2} \quad \text{and} \quad \frac{1}{r_y} = \frac{\partial^2 z}{\partial y^2};$$

and hence, on the basis of the assumptions stated above, the strains in any layer of the plate, at a distance v from the neutral layer, in directions parallel to OX and OV, will be respectively equal to

$$e_x = -\frac{v}{r_x} = -v\,\frac{\partial^2 z}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

$$e_y = -\frac{v}{r_y} = -v\,\frac{\partial^2 z}{\partial y^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

tensile strains and stresses being plus and v being plus when measured upwards.

Obtaining the values of the derivatives from (5) and substituting in (6) and (7),

$$e_x = -\frac{2\,Pv\,(a^2 - 3\,x^2)\,(b^2 - y^2)^2}{Et^3\,(a^4 + b^4)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

and

$$e_y = -\frac{2\,Pv\,(b^2 - 3\,y^2)\,(a^2 - x^2)^2}{Et^3\,(a^4 + b^4)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

It will be evident from an inspection of (8) and (9), that the maximum strains will be located at the surface of the plate, at the ends of the axes OX and OY, and for these points,

$$\text{max.}\,Ee_x = \pm\frac{2\,Pa^2}{t^2}\left(\frac{b^4}{a^4 + b^4}\right), \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

$$\text{max.}\,Ee_y = \pm\frac{2\,Pb^2}{t^2}\left(\frac{a^4}{a^4 + b^4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

At the center of the plate, when $v = \pm\frac{t}{2}$,

$$Ee_x = \mp\frac{Pa^2}{t^2}\left(\frac{b^4}{a^4 + b^4}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

$$Ee_y = \mp\frac{Pb^2}{t^2}\left(\frac{a^4}{a^4 + b^4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

It will be evident that when $a > b$ the greatest value of the product of E and a strain for the entire plate will be that given by (11).

By use of equations (5) and (6) (Art. 46) we may obtain, for the sections at the ends of the axis OX, when $v = \pm \frac{t}{2}$,

$$\max.\, p_x = \pm \frac{m^2}{m^2-1} \frac{2\,Pa^2}{t^2} \left(\frac{b^4}{a^4+b^4}\right); \quad . \quad . \quad . \quad . \quad (14)$$

and, similarly for the sections at the ends of the axis OY,

$$\max.\, p_y = \pm \frac{m^2}{m^2-1} \frac{2\,Pb^2}{t^2} \left(\frac{a^4}{a^4+b^4}\right). \quad . \quad . \quad . \quad . \quad (15)$$

It will be evident from an inspection of equations (14) and (15) that, when $a > b$, the greatest stress intensity will be given by (15), where, if $m = 3$,

$$\max.\, p_y = \frac{9\,Pb^2}{4\,t^2} \left(\frac{a^4}{a^4+b^4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The quantity $\frac{2\,Pb^2}{t^2}$ (equation 11) is evidently equivalent to $\frac{M_1c}{I}$, where $M_1 = \frac{P\,4\,b^2}{12}$ = the greatest bending moment in a strip one unit wide along OY (Fig. 270*a*), considered as a beam under a uniform load of intensity P, and $\frac{I}{c} = \frac{t^2}{6}$ = the section modulus of the cross section of the strip. Similarly, the quantity $\frac{2\,Pa^2}{t^2}$ is equivalent to $\frac{M_2C}{I}$, where $M_2 = \frac{P\,4a^2}{12}$. Hence equations (10) and (11) may be interpreted as follows, assuming $a > b$:

To determine the greatest values of the products of E and the principal strains in a rectangular plate, fixed at the edges and subjected to a uniform load; calculate the maximum fiber stresses in the two strips, one unit wide, along the central axes of the plate, considered as uniformly loaded beams fixed at ends; the load on the shorter strip being equal to $\left(\frac{a^4}{a^4+b^4}\right)P$ *and that on the longer strip equal to* $\left(\frac{b^4}{a^4+b^4}\right)P$, *where* P = *the load intensity on the plate.*

Since the strains along the edges of the plate at the ends of the axes OX and OY are equal to zero the stress intensities p_x and p_y, from (14) and (15), are greater than the values of Ee_x and Ee_y, from (10) and (11), instead of being equal to them, as would be the case if the strips were separate beams.

The greatest deflection in the plate will evidently occur at the center and, by putting $x = 0$ and $y = 0$ in (5), we obtain for its value

$$z_0 = -\frac{Pa^4b^4}{2\,Et^3\,(a^4+b^4)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It will be evident, from an inspection of (2) and (4), that the division of the load intensity, required to produce the same maximum deflection in two strips of unit width, along the central axes of plate, when acting as independent beams, is the same as that called for in the foregoing rule for calculating the maximum strains, by considering the strips along the central axes as uniformly loaded beams.

The diagram (Fig. 271) represents the variation in the value of the quantity $\frac{a^4}{a^4 + b^4}$ with the variation in $\frac{a}{b}$; and it is evident that when $\frac{a}{b} > 3$ the value of $\frac{a^4}{a^4 + b^4} = 1$ (nearly); that is, the maximum stresses and strains are practically the same as if the plate were of indefinite length.

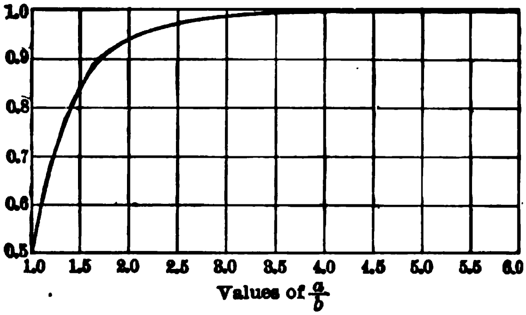


FIG. 271.

Expressions for the shearing stress intensities through the plate might be obtained by use of the general equations of equilibrium (Art. 49); but the magnitudes would be so small that the results would be of no considerable importance.

(b) *Square Plate Fixed at the Edges.* — For a square plate, putting $b = a$ in (10), (11), (14) and (15), we have

$$\text{max. } Ee_x = \text{max. } Ee_y = \pm \frac{Pa^2}{t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and, when $m = 3$,

$$\text{max. } p_x = \text{max. } p_y = \pm \frac{9\,Pa^2}{8\,t^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Putting $b = a$ in (17), we have for the maximum deflection of the plate

$$z_0 = -\frac{Pa^4}{4\,Et^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The numerical coefficients in (18) and (19) are somewhat higher than the coefficients in the empirical formulas, proposed by Bach and others, for calculating the strains and stress intensities in square plates.

The results of a series of measurements of the deflection of a square steel plate made under the supervision of one of the authors showed that the values given by (20) averaged 15 to 20 per cent higher than the measured deflections, which would also indicate that the values given (18) and (19) are too high and hence, if we write these equations in the forms

$$\text{max. } Ee_x = \pm \frac{8\,Pa^2}{9\,t^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and

$$\max . p_x = \pm \frac{Pa^2}{t^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

they will probably represent more nearly the true values of the greatest strain and stress intensity in the plate.

Grashof, by a solution of the differential equation of the neutral surface of the square plate found the coefficient in (21) to be $\frac{2}{3}$ when $m = 3$.

It would follow, perhaps, that if the numerical coefficients in (10) and (16) were modified, in accordance with these facts, to $\frac{16}{9}$ and 2, respectively, the equations would represent more nearly the true values of the maximum strain and stress intensities in the rectangular plate.

(c) *Square Plate Freely Supported at the Edges.* — A theoretical treatment will not lead to a satisfactory solution in this or the following cases. We will, therefore, write for these cases empirical formulas, simply, omitting signs and stating in each case the reasons for the proposed form of the equation.

If we compare equations (21) and (22) with equations (22) and (23) (Art. 197) we find that the maximum values of the strains and stress intensities given by the formulas for the square plate, fixed at the edges and uniformly loaded, are one-third higher than the values given by the formulas for a circular plate, having a diameter equal to the side of the square, and loaded and supported in the same manner.

Hence, if we assume a similar relation to exist in the case of the square and circular plates freely supported at the edges we may, by increasing the numerical coefficient in (40) (Art. 197) by one-third, estimate the value of the product of E and the greatest strain to be

$$\max . Ee_x = \max . Ee_y = \frac{10\, Pa^2}{9\, t^2}, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

the strain evidently being located at the center of the plate.

By substituting in (5) (Art. 46) we shall have, for the greatest stress intensity, if $m = 3$,

$$\max . p_x = \max . p_y = \frac{5\, Pa^2}{3\, t^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

(d) *Rectangular Plate Freely Supported at the Edges.* — Since the load intensities on two uniformly loaded beams, of the same cross section and having the same maximum deflection, are inversely proportional to the fourth powers of the lengths, we may assume, from the analogy with Case (a), that the ratio of the maximum strains in the two unit strips, taken along the central axes OX and OY (Fig. 270), is the same as in that case.

Hence, using the same notation as before, we shall have for the maximum values of the products of E and the strains e_x and e_y, located at the center of the plate,

$$\max . Ee_x = \frac{20\, Pa^2}{9\, t^2}\left(\frac{b^4}{a^4 + b^4}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\max . Ee_y = \frac{20\, Pb^2}{9\, t^2}\left(\frac{a^4}{a^4 + b^4}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

both (25) and (26) reducing to the form of (23) when $b = a$.

When $a > b$ the greatest value of the product of E and a principal strain for the entire plate will evidently be given by (26).

By substituting in (6) (Art. 46) we obtain for the value of the greatest stress intensity in the plate, when $m = 3$,

$$\max.\, p_y = \frac{5\, Pa^2b^2\,(3\,a^2 + b^2)}{6\,t^2\,(a^4 + b^4)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

201. Square and Rectangular Flat Plates Centrally Loaded.

— In each of the following cases we will assume a load W to be uniformly distributed over a small circular area of radius r_0 at the center of the plate. We will use the same notation for the dimensions of the plate as in Art. (200). Four cases will be considered, empirical formulas, simply, being written for each case.

(a) *Square Plate Fixed at the Edges.* — If we assume the maximum strain in the square plate to be one-third higher than in the circular plate, having a diameter equal to the side of the square and similarly loaded, as in Case (c) (Art. 200), we shall have, from (24) (Art. 198), for the product of E and the greatest strain, located at the center of the plate,

$$\max.\, Ee_x = \frac{4\,W}{9\,\pi t^2}\left[4 \log \frac{r_0}{a} - \left(\frac{r_0}{a}\right)^2\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we assume $\frac{r_0}{a} = 0.1$, equation (1) reduces to

$$\max.\, Ee_x = \max.\, Ee_y = \frac{21\,W}{16\,t^2}\ \text{(nearly)}\,; \quad . \quad . \quad . \quad . \quad (2)$$

and, from (5) (Art. 46), if $m = 3$,

$$\max.\, p_x = \max.\, p_y = \frac{2\,W}{t^2}\ \text{(nearly)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(b) *Square Plate Freely Supported at the Edges.* — Making an assumption similar to that in Case (a) we shall have, from (42) (Art. 198),

$$\max.\, Ee_x = \frac{4\,W}{9\,\pi t^2}\left[4 \log \frac{r_0}{a} + \frac{1}{2}\left(\frac{r_0}{a}\right)^2 - 3\right] \quad . \quad . \quad . \quad . \quad . \quad (4)$$

If we assume $\frac{r_0}{a} = 0.1$ equation (4) reduces to

$$\max.\, Ee_x = \max.\, Ee_y = \frac{7\,W}{4\,t^2}\ \text{(nearly)}\,; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and from (5) (Art. 46), if $m = 3$,

$$\max.\, p_x = \max.\, p_y = \frac{21\,W}{8\,t^2}\ \text{(nearly)}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(c) *Rectangular Plate Fixed at the Edges.* — Since the greatest bending moments in two centrally loaded beams of the same cross section and having the same maximum deflection are inversely proportional to the squares of the lengths, if we assume the maximum moments of resistance of the two unit strips along the central axes of the plate to be proportional to the maximum values of Ee_x and Ee_y, we may estimate that, when $\frac{r_0}{b} = 0.1$, the maximum values of the products of E and the strains e_x *and* e_y will be equal to

$$\text{max. } Ee_x = \frac{21\,W}{8\,t^2}\left(\frac{b^2}{a^2+b^2}\right), \quad \ldots\ldots\ldots \quad (7)$$

$$\text{max. } Ee_y = \frac{21\,W}{8\,t^2}\left(\frac{a^2}{a^2+b^2}\right), \quad \ldots\ldots\ldots \quad (8)$$

located at the center of the plate.

When $a > b$ the value of the product of E and the greatest strain for the entire plate will evidently be that given by (8); and both (7) and (8) will evidently reduce to the form of (2) when $b = a$.

By substituting in (6) (Art. 46) we would have for the greatest stress intensity in the plate, when $m = 3$,

$$\text{max. } p_y = \frac{W\,(3\,a^2+b^2)}{t^2\,(a^2+b^2)}, \quad \ldots\ldots\ldots \quad (9)$$

which reduces to the form of (3) when $b = a$.

(d) *Rectangular Plate Freely Supported at the Edges.* — By reasoning similar to that in the preceding case we have for the maximum values of the products of E and the strains e_x and e_y, assuming $\frac{r_0}{b} = 0.1$,

$$\text{max. } Ee_x = \frac{7\,W}{2\,t^2}\left(\frac{b^2}{a^2+b^2}\right), \quad \ldots\ldots \quad (10)$$

$$\text{max. } Ee_y = \frac{7\,W}{2\,t^2}\left(\frac{a^2}{a^2+b^2}\right), \quad \ldots\ldots \quad (11)$$

located at the center of plate; the maximum value for the entire plate, when $a > b$, being that given by (11); both (10) and (11) reducing to the form of (5) when $b = a$.

The greatest stress intensity in the plate, obtained from (6) (Art. 46), would then be equal to, assuming $m = 3$,

$$\text{max. } p_y = \frac{21\,W}{16\,t^2}\left(\frac{3\,a^2+b^2}{a^2+b^2}\right), \quad \ldots\ldots \quad (12)$$

which reduces to the form of (6) when $b = a$.

202. Problems. — Cylinders and Plates.

Problem 1.

Determine the allowable internal pressure intensity in an elliptical cylinder ½″ thick, having maximum and minimum inside diameters 8″ and 5″, assuming a working stress intensity of 12,000 lbs. per sq. in.

Problem 2.

Determine the required thickness of a square tube, 2″ × 2″ inside measurement, to withstand an internal pressure of 40 lbs. per sq. in., assuming $f =$ 12,000 lbs. per sq. in.

Problem 3.

A circular cylinder, 10″ inside diameter with a wall 3″ thick, is subjected to an internal pressure of 5000 lbs. per sq. in.

(a) Calculate the maximum stress intensity in the wall;

(b) Calculate the maximum value of the product Ee_t, assuming $m = 4$;

(c) Determine the increase in the inside diameter due to the pressure, assuming $E = 30{,}000{,}000$ lbs. per sq. in.

Problem 4.

Plot the diagrams for the cylinder in Problem (3), showing the variation of the following quantities along any radial section, assuming $E = 30{,}000{,}000$ lbs. per sq. in. and $m = 4$:

(a) Tangential stress intensity;

(b) Radial stress intensity;

(c) Values of Ee_t;

(d) Values of Ee_r;

(e) Increase in radius.

Problem 5.

Solve Problem (3), assuming the cylinder to be subjected to an external pressure only of 8000 lbs. per sq. in.

Problem 6.

Solve Problem (4), assuming the cylinder to be subjected to an external pressure only of 8000 lbs. per sq. in.

Problem 7.

Determine the thickness of a cast-iron cylinder, 12″ inside diameter, required to withstand an internal pressure of 1200 lbs. per sq. in.:

(a) Assuming the working stress intensity = 4000 lbs. per sq. in.;

(b) Assuming the value of the product $Ee_t = 4000$ lbs. per sq. in.

Calculate the increase in the inside diameter of the cylinder, assuming $E = 14{,}000{,}000$ lbs. per sq. in.

Problem 8.

Solve Problem (7), substituting a cast-iron spherical shell, 12″ inside diameter, for the cylinder.

Problem 9.

A circular flat plate, 24″ diameter and 1″ thick, fixed at the circumference, is subjected to a uniform pressure of 120 lbs. per sq. in. Assuming $m = 3$;

(a) Determine the maximum stress intensity in the plate;

(b) Determine the maximum value of the product Ee_r;

(c) Determine the greatest deflection of the plate, assuming $E = 30{,}000{,}000$ lbs. per sq. in.

Problem 10.

Solve Problem (9), assuming the plate to be freely supported at the circumference.

Problem 11.

Plot diagrams representing the variation of the following quantities along any radial section of the plate in Problem (9), assuming $m = 3$:

(a) Radial stress intensity p_r;
(b) Tangential stress intensity p_t;
(c) Values of Ee_r;
(d) Values of Ee_t.

Determine the radius of the line of inflexion in the neutral layer of the plate.

Problem 12.

Calculate the maximum values of p_r and Ee_r for a plate, 20″ diameter and 1″ thick, supported at the center only on a post 2″ diameter, and subjected to a uniform load of 150 lbs. per sq. in. Calculate the greatest deflection of the plate. Assume $E = 30{,}000{,}000$ lbs. per sq. in., $m = 3$.

Problem 13.

A circular flat plate, 30″ diameter and ¾″ thick, fixed at the circumference and supported by a stay at the center, is subjected to a uniform pressure of intensity P. Determine the central supporting force in terms of P, assuming the support to be on the same level as the circumference and the supporting force to be distributed over a circle 2″ diameter. Calculate the allowable value of P, assuming $m = 3$:

(a) Assuming $p_r = 16{,}000$ lbs. per sq. in.;
(b) Assuming $Ee_r = 16{,}000$ lbs. per sq. in.

Problem 14.

Find the allowable pressure intensity on a rectangular plate 20″ × 12″ and ¾″ thick, assuming the working stress intensity = 16,000 lbs. per sq. in.:

(a) When the plate is fixed at the edges;
(b) When the plate is freely supported at the edges.

Find the allowable pressure on the assumption that the product of E and the greatest strain = 16,000 lbs. per sq. in.:

(a) When the plate is fixed at the edges;
(b) When the plate is freely supported at the edges.

Problem 15.

Find the allowable center load W on the plate in Problem (14), under each of the conditions stated, on the assumption that the load is distributed over a circle having a diameter equal to about 0.1 of the short dimension of the plate.

CHAPTER XIV.

REINFORCED CONCRETE BEAMS AND COLUMNS.

203. Conditions and Assumptions. — It is our purpose in this chapter to give the derivation of the formulas commonly used in making calculations of the stresses due to bending, shear and compression in beams and columns, or struts, made up of concrete and steel reinforcement.

In the development of the theory it will be necessary to assume the existence of certain ideal conditions which, at best, are only approximately met in practice. The formulas which are deduced, however, will be in proper form for use, provided the required constants are obtained from the results of experiments on the strength of members subjected to loading under conditions similar to those actually existing in concrete structures.

In all the cases considered the following conditions will be imposed:

(a) Every member will be considered to be made up of layers, or fibers, parallel to its central axis, and each of the component materials will be considered to be homogeneous.

(b) The theory will be subject to the limitations (c), (d) and (e) (Art. 63) of the ordinary theory of beams.

In addition to the foregoing conditions it will be necessary in every case to make the following assumptions, the first two of which are similar to the assumptions of the beam theory (Art. 66).

(1) That a plane cross section, perpendicular to the central axis of a member before loading, remains plane and normal to the central axis after loading.

(2) That each material follows Hooke's law, within the limits of stress due to the working load, the ratio of stress intensity to strain in the concrete bearing a fixed relation to the ratio of stress intensity to strain in the steel, in both tension and compression.

(3) It follows from the first assumption that at any surface of contact the concrete and the steel must elongate, or contract, the same amount in any given length; that is, there must be a perfect bond, or no slipping, between the two materials.

(4) The resistance of concrete to tension will be assumed to be negligible.

The last assumption is probably in close agreement with actual conditions in many cases, the concrete on the tension side of a member subjected to bending failing, on account of its low tensile strength, before the safe working load is reached, even when evidence of the failure is not visible to the eye. Under certain conditions this failure in tension may not take place and the resistance of the concrete to tension will have an appreciable effect on the moment of resistance at any given section.* In such cases, however, the error due to assuming the tension equal to zero will not be very great; and formulas based on this assumption will be adaptable to all cases.

In the development of the theory, in each of the following cases, the foregoing conditions and assumptions will be rigidly adhered to, although leading in some cases to refinements which the conditions of practice may not justify. It is the part of the Engineer to modify the theoretical equations into such forms as may simplify the necessary computations and still give results sufficiently accurate for use under actual working conditions.

For the sake of simplicity all beams will be considered as horizontal and the formulas for columns and struts will be derived by considering the central axes vertical.

204. Rectangular Beam Reinforced for Tension Only. — A section of such a beam is represented by the sketch (Fig. 272), one or more reinforcing bars being embedded in the concrete near the under side of the beam to resist tension, the compression in the upper portion of the beam being resisted entirely by the concrete. In accordance with the assumptions (Art. 203) the stress in the concrete at any cross section will be uniformly varying compression above the neutral axis *NN* and of zero intensity at all points below the neutral axis.

The steel reinforcing bars would be placed in a single horizontal plane, when possible; but, if necessary, would be arranged in two, or more, horizontal layers. Since the total area of the cross section of the steel bars is small in extent, compared with the area of the entire cross section, the stress in the steel may be considered to be uniform. without any considerable error.

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* See Experimental Researches on Reinforced Concrete; by A. Considère.

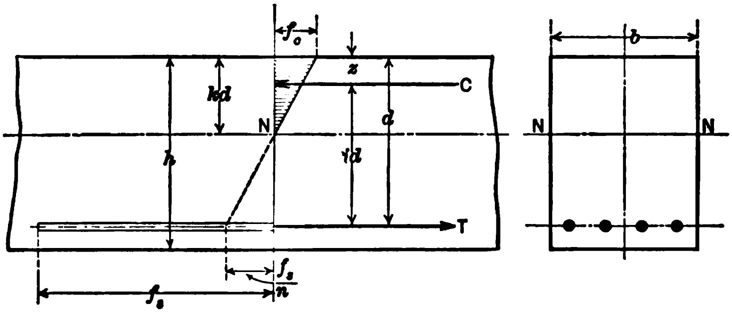

FIG. 272.

The following notation has been generally adopted:

b = the breadth of the beam,
h = the total depth of the beam,
d = the depth to the center of gravity of the total cross section of the steel reinforcing bars, called the *effective depth of the beam*,
A = the total area of the cross section of the steel reinforcement,
f_c = the maximum intensity of stress in the concrete,
f_s = the intensity of stress in the steel,
C = the resultant compressive stress in the concrete,
T = the resultant tensile stress in the steel reinforcement,
M = the moment of resistance of the stress on a cross section, equal and opposite to the bending moment due to the external forces,
k = the ratio of the depth of the neutral axis to the effective depth d,
z = the distance from the top of the beam to the resultant compressive stress C,
j = the ratio of the moment arm of the resisting couple to the effective depth d,
p = the ratio of the steel area A to the effective area of the concrete bd,
e_c = the maximum longitudinal strain in the concrete,
e_s = the longitudinal strain in the steel,
E_c = the ratio of stress intensity to strain, or the modulus of elasticity, for the concrete,
E_s = the modulus of elasticity of the steel,
n = the ratio of the value of E_s to the value of E_c.

It will follow from assumption (1) (Art. 203) that the longitudinal strain in the layers of the beam will be uniformly varying (see equation 1, Art. 95) from zero at the neutral layer to the values e_c, at the upper layer, and e_s, at the layer through the center of the steel, and therefore

$$\frac{e_s}{e_c} = \frac{d - kd}{kd} = \frac{1 - k}{k}. \quad \ldots \ldots \quad (1)$$

From assumption (2) (Art. 203) we have

$$\frac{e_s}{e_c} = \frac{f_s}{E_s} \times \frac{E_c}{f_c} = \frac{f_s}{nf_c}; \quad \ldots \ldots \quad (2)$$

and, combining (1) and (2),

$$\frac{f_s}{f_c} = n\left(\frac{1 - k}{k}\right). \quad \ldots \ldots \quad (3)$$

Since the loads on the beam are vertical, it is evident that $C = T$; and hence

$$\frac{f_c bkd}{2} = f_s A; \quad \ldots \ldots \quad (4)$$

from which we obtain

$$\frac{f_s}{f_c} = \frac{bkd}{2A}. \quad \ldots \ldots \quad (5)$$

But, from the foregoing definitions, $\frac{A}{bd} = p$ and, substituting this value in (5) and equating to (3), we have

$$\frac{k}{2p} = n\frac{1 - k}{k};$$

and solving for k,

$$k = \sqrt{2pn + (pn)^2} - pn, \quad \ldots \ldots \quad (6)$$

which gives the ratio k in terms of the steel ratio p and the ratio of the moduli of elasticity of the steel and the concrete n; showing that the position of the neutral axis depends solely on the proportion of the steel reinforcement and the ratio of the two moduli of elasticity.

It is evident that

$$z = \frac{kd}{3}$$

and that

$$jd = d - z, \quad \ldots \ldots \quad (7)$$

hence

$$j = 1 - \frac{k}{3}. \quad \ldots \ldots \quad (8)$$

The moment of resistance of the stress on the cross section will evidently be equal to

$$M = Tjd = Cjd; \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and, substituting the values of T and C, we have

$$M = f_s Ajd = \frac{f_c bkd}{2} jd, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

from which we may obtain the stress intensities in terms of the bending moment at the section,

$$f_s = \frac{M}{Ajd} = \frac{M}{pjbd^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$f_c = \frac{2M}{kjbd^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

From (3) we also have

$$f_c = \frac{f_s}{n}\left(\frac{k}{1-k}\right); \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

the magnitude of the quantity $\frac{f_s}{n}$ being represented in the sketch (Fig. 272).

From (11) and (12) we obtain

$$bd^2 = \frac{M}{f_s pj} = \frac{2M}{f_c kj}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

from which the size of the beam, required to resist a bending moment M, may be obtained when the constants in the equation are known.

The beam of minimum weight, required to resist a given bending moment, will evidently be that in which both the stress intensity in the steel and the maximum stress intensity in the concrete have the greatest allowable values. This condition will not exist unless the amount of the steel reinforcement bears a certain definite proportion to the amount of concrete. In order to determine the proportion of steel required we have, from (3),

$$\frac{1}{k} = 1 + \frac{f_s}{nf_c};$$

and from (5),

$$k\frac{f_c}{2f_s} = \frac{A}{bd} = p.$$

Eliminating k between these two equations and reducing,

$$p = \frac{1}{2\frac{f_s}{f_c}\left(1 + \frac{f_s}{nf_c}\right)}; \quad . \quad . \quad . \quad . \quad . \quad (15)$$

which is the value of p, in terms of the allowable working stress intensities and the ratio n, for the beam of minimum weight, required to resist any given bending moment.

The design of a beam under this condition, therefore, consists in calculating the value of bd^2 for the required bending moment, by substituting the values of the constants in (14), and then selecting values for b and d which will satisfy the equation, the total cross section of the steel reinforcement required being equal to $A = pbd$, the value of p being obtained from (15).

205. Shearing and Bond Stresses. — If the bending in the rectangular concrete beam is due to vertical loading, longitudinal shearing stresses will be developed, in the same manner as in the homogeneous beam (Art. 88). The longitudinal shearing stresses along the layers in contact with the reinforcing bars will tend to break the bond, or cause slipping, between the steel and the concrete. The bond stress intensity must therefore be considerably below the stress intensity at which the adhesion between the concrete and the steel rods will be overcome. It is customary to assume that the intensity of the bond stress is the same at all points in the perimeters of the reinforcing bars, which lie in the plane of any given cross section, but that the intensity varies in the longitudinal direction, in the same manner as the shearing stress.

At any point in the beam the intensity of the shearing stress on a vertical section will evidently be equal to that on a longitudinal section (Art. 24). The shearing stress intensity at any point on any longitudinal section may be found by the following method, similar to that employed in the common beam theory (Art. 88).

We will use the same notation as before and, in addition, let

V = the shearing force at any cross section of the beam.
v = the shearing stress intensity on a vertical, or longitudinal, section at any point below the neutral layer,
u = the bond stress per unit of area of the surface of a reinforcing bar,
o = the perimeter of a single reinforcing bar,
Σo = the sum of the perimeters of all the reinforcing bars.

Then from equation (1) (Art. 88) we have, for the total shearing stress on the portion of a longitudinal section between two cross sections at the distance x apart,

$$sbx = R_2 - R_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

where R_2 and R_1 represent the resultant normal stresses on the portions of the two cross sections, between the longitudinal layer and the top of the beam, and s represents the average longitudinal shearing stress intensity between the two cross sections. Under assumption (4) (Art. 203) it is evident that for all layers below the neutral layer,

$$R_2 - R_1 = C_2 - C_1 = \text{a constant},$$

where C_2 and C_1 represent the values of the total compression in the concrete at the two cross sections. Hence, by taking x sufficiently small, we may put $v = s$ and write (1) in the form

$$v = \frac{C_2 - C_1}{bx}. \qquad (2)$$

But from (9) (Art. 204) we have

$$C_2 - C_1 = \frac{M_2 - M_1}{jd}, \qquad (3)$$

where M_2 and M_1 represent the bending moments at the two cross sections. Hence,

$$v = \frac{M_2 - M_1}{xbjd} = \frac{V}{bjd}; \qquad (4)$$

since, by taking x sufficiently small, we may write

$$\frac{M_2 - M_1}{x} = V \text{ (Art. 73)}.$$

The longitudinal shearing stress intensity, on sections above the neutral layer, will evidently vary in the same manner as in a homogeneous beam of rectangular section (Art. 90). Hence the diagram, representing the variation in either the longitudinal or the vertical shearing stress intensities at any cross section, will be in the form shown (Fig. 273), the portion above the neutral layer being a parabola and the portion below, a vertical line.

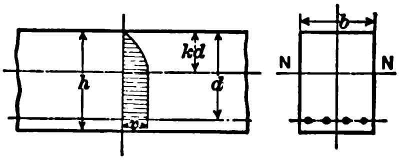

FIG. 273.

To obtain the intensity of the bond stress between the concrete and the steel, observe that the average longitudinal shearing stress per unit of length of the surface of contact with the steel will be equal to vb, the total surface of contact per unit of length of steel being equal to Σo. Hence the average bond stress intensity will be equal to

$$u = \frac{vb}{\Sigma o} = \frac{V}{jd\Sigma o}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is evident that the allowable value of u will depend on the nature of the surface of the reinforcing bars, whether rough or smooth; and it is important that the maximum value of u in any beam shall not exceed the allowable value for the type of reinforcing bar used.

It is also evident that sufficient space must be left between the bars if the full strength of the bond between the steel and the concrete is to be obtained. The distance between the centers of adjacent bars should be at least $2\frac{1}{2}$ to 3 times their diameters and the distance between the center of a bar and the side of a beam at least 2 diameters.

206. Floor Slabs. — *Slab supported on two sides.* The stresses in a reinforced concrete floor slab, supported on two sides only, may be determined by considering the slab as divided into transverse strips, one unit wide, and treating each strip as a rectangular beam. Slabs are usually made continuous over the supports; that is, the reinforcing bars are arranged so as to resist negative bending, or tension at the top of the slab, at every support.

When the continuity exists for several spans it is evident that each intermediate span approaches the condition of a beam fixed at the ends; and, for uniform loading, the greatest bending moment may be determined with sufficient accuracy by use of the formula

$$M_2 = \frac{wl^2}{12} \text{ (Art. 101)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where M_2 = the bending moment at the end of the span and l = the distance from center to center of the supports (Fig. 274). If the clear span l_1 is considerably less than the distance between centers a somewhat less value may be used in place of l in (1). For uniform loading, the bending moment M_1, at the middle of the span, would be one-half the value of M_2; but, when the slab is of uniform thickness, the maximum bending moment only need be considered.

Where the reinforcement is continuous the bars should evidently cross the central layer in the vicinity of the points of inflexion, which would be located at distances approximately 0.2 l from the ends of the span (Art. 101).

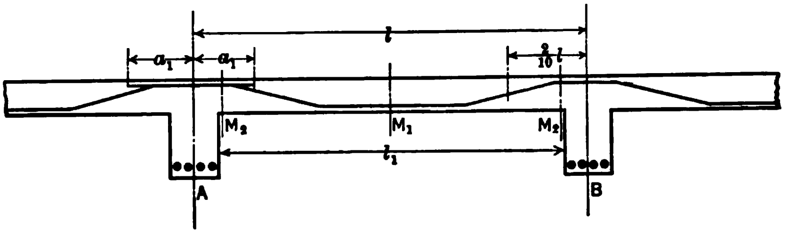

FIG. 274.

Where it becomes necessary to join the reinforcement, the rods may be clamped together, or lapped by each other in the concrete, a distance 2 a_1, as indicated at the support A, the bond in the concrete being relied upon to furnish the required continuity. In the latter case the overlap should be sufficient to provide a total bond stress in the length a_1 equal to the total tension in the rod at the section over the support.

Since the maximum bending moments in the end spans of a uniformly loaded continuous beam are greater than in the intermediate spans (Art. 121), the end spans should be designed for a greater moment of resistance. It is customary to use the equation

$$M_2 = \frac{wl^2}{10} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

in calculating the greatest bending moment in the end spans of a slab which is continuous over several supports. Owing to the uncertainty which usually exists in regard to the conditions at the different supports, the calculation of bending moments by more exact methods is of little value when the slab is continuous over several spans.

When the number of spans is less than four, however, or the intensity of loading varies in different spans, the bending moments should be calculated by use of the more exact formulas for continuous beams, with due allowance for the uncertainty of conditions at the supports.

Slab supported on four sides. — If a slab is supported on four sides, reinforcing rods are used in both the transverse and the lengthwise directions. When the length of the slab is over twice the width, the entire load should be assumed to be carried by the transverse reinforcement. When the ratio of length to width is less than two, the stresses in the slab may be estimated in the same manner as in the case of the homogeneous flat plate (Art. 200); the stress intensities on the cross sections at the ends of a transverse strip, one unit wide, through the center, being calculated on the assumption that the load intensity on the strip is equal to $\frac{a^4}{a^4 + b^4}$ times the total load intensity on the slab, where a = the length and b = the width of the slab; and the stress intensities on the cross sections at the ends of a longitudinal strip, one unit wide, being calculated for a load intensity $\frac{b^4}{a^4 + b^4}$ times the total load intensity.

For a square slab one-half of the load would evidently be assumed to be carried on each set of reinforcement.

Since the stress intensities in strips, parallel to and near the sides of the slab, will be less than in the strips through the center, less reinforcement is required near the sides than through the center; that is, the distance between the rods may be made larger as the sides are approached. The allowable increase in spacing can best be determined by experiment.

207. Rectangular Beam Reinforced for Tension and Compression. — In this case reinforcing bars are embedded in the concrete near the side of the beam under compression as well as in the tension side. The distribution of stress on the cross section of such a beam is indicated by the sketch (Fig. 275).

We will follow the same notation as in Art. (204) and, in addition, let.

A' = the total area of the cross section of the reinforcement on the compression side,
d' = the depth of the center of gravity of the cross section of the compression reinforcing bars,
f_s' = the intensity of stress in the compression reinforcement, assumed to be uniform,
C' = the resultant compressive stress in the steel,
p' = the ratio of the steel area A' to the effective area of the concrete bd.

From assumptions (1) and (2) (Art. 203) we have, as in the beam reinforced for tension only,

$$\frac{f_s}{f_c} = n\left(\frac{1-k}{k}\right); \quad \ldots \ldots \quad (1)$$

and also, since the tensile and compressive moduli of elasticity are assumed to be equal,

$$\frac{f_s'}{f_s} = \frac{kd - d'}{d - kd}. \quad \ldots \ldots \quad (2)$$

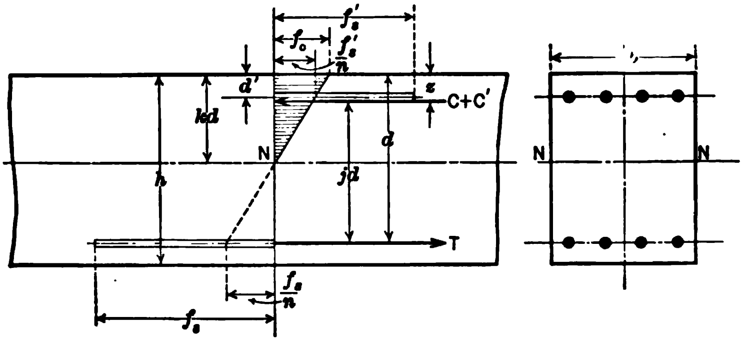

FIG. 275.

Since, for a vertically loaded beam, $C + C' = T$, we shall have

$$\frac{f_c bkd}{2} + f_s'A' = f_sA \text{ (very nearly)}, \quad \ldots \quad (3)$$

the resultant stress in the concrete being calculated without deducting from the area of the cross section the space occupied by the compression reinforcement.

Substituting the values $A = pbd$ and $A' = p'bd$, together with the value of f_s' from (2), in equation (3) and transposing terms, we obtain

$$\frac{f_s}{f_c} = \frac{k}{2\left[p - p'\left(\frac{kd - d'}{d - kd}\right)\right]}. \quad \ldots \ldots \quad (4)$$

Equating (1) and (4) and transposing terms,

$$\begin{aligned} k^2 &= 2n(1-k)\left[p - p'\left(\frac{kd-d'}{d-kd}\right)\right] \\ &= 2n\left[p(1-k) - p'\left(k - \frac{d'}{d}\right)\right] \\ &= 2n\left(p + p'\frac{d'}{d}\right) - 2n(p+p')k; \end{aligned}$$

and solving for k,

$$k = \sqrt{2n\left(p + p'\frac{d'}{d}\right) + n^2(p + p')^2} - n(p + p'). \quad . \qquad (5)$$

By adding the moments of the components of the compressive stress about an axis through the top of the cross section we obtain

$$(C + C')z = \frac{f_c bkd}{2} \times \frac{kd}{3} + f_s'A'd'. \quad . \quad . \quad . \quad . \qquad (6)$$

Substituting the values of C and C' and also the value of f_s' from (2), we have

$$z = \frac{\dfrac{f_c bk^2d^2}{6} + f_s A'd'\left(\dfrac{kd - d'}{d - kd}\right)}{\dfrac{f_c bkd}{2} + f_s A'\left(\dfrac{kd - d'}{d - kd}\right)};$$

and eliminating f_s, by substituting its value in terms of f_c from (1), substituting $A' = p'bd$ and reducing, we obtain

$$z = \frac{\dfrac{k^3 d}{3} + 2p'nd'\left(k - \dfrac{d'}{d}\right)}{k^2 + 2p'n\left(k - \dfrac{d'}{d}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

It follows that

$$jd = d - z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

and that the moment of resistance is equal to

$$M = Tjd = (C + C')jd, \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

or,

$$M = f_s Ajd = \left(\frac{f_c bkd}{2} + f_s'A'\right)jd. \quad . \quad . \quad . \quad . \qquad (10)$$

Hence,

$$f_s = \frac{M}{Ajd} = \frac{M}{pjbd^2}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

and, substituting f_s' in terms of f_c and $A' = p'bd$ and solving,

$$f_c = \frac{2M}{jbd^2\left[k + 2p'n\left(1 - \dfrac{d'}{kd}\right)\right]}. \quad . \quad . \quad . \quad . \qquad (12)$$

The value of f_c can evidently be obtained more easily from equation (1), from which the value

$$f_c = \frac{f_s}{n}\left(\frac{k}{1 - k}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

can be computed after the value of f_s has been found.

From (2) we have for the value of the stress intensity in the compression reinforcement.

$$f_s' = f_s \frac{kd - d'}{d - kd} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Shearing and Bond Stresses. — Since there is assumed to be no tension in the concrete below the neutral layer, the distribution of the shearing stress on any cross section will be similar to that for the beam reinforced for tension only (Art. 205), the shearing stress intensity being uniform below the neutral layer; and hence, by the same reasoning as before,

$$v = \frac{V}{bjd}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and, for the bond stress intensity on the *tension* reinforcement,

$$u = \frac{vb}{\Sigma o} = \frac{V}{jd\Sigma o}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

As the difference between the stress intensities in the compression reinforcing bars and the adjacent layers of concrete is less than the stress intensity in the tension reinforcing bars, it is unnecessary to develop a formula for the bond stress intensity on the compression steel.

208. Tee Beam. — When a reinforced concrete floor is supported on beams of the same material, as indicated in Fig. (274), the beams and the floor are bonded together so as to form a continuous structure. The bond is usually strengthened beyond that due to the adhesion in the concrete alone by the use of special reinforcement. The floor slab, therefore, forms the flange of a beam, tee-shaped in section, as shown in Fig. (276), which resists the compressive stress, while the tensile stress is carried by reinforcing bars placed near the bottom of the web, or stem, of the tee.

In the deduction of the formulas for the maximum stress intensity in the concrete and the stress intensity in the steel we will follow the notation previously adopted and, in addition, let

b = the width of the flange,
t = the thickness of the flange,
b' = the width of the web, or stem, of the tee.

In floors of ordinary proportions it would not be safe to assume that one-half of the entire slab on either side of any beam would act as a compression flange and fulfill the conditions stated in Art. (203); and no satisfactory theory can be developed to determine just what proportion of the floor slab can be considered to form a part of the beam in order that the formulas for stress intensity, deduced under the above-mentioned conditions, may give accurate results.

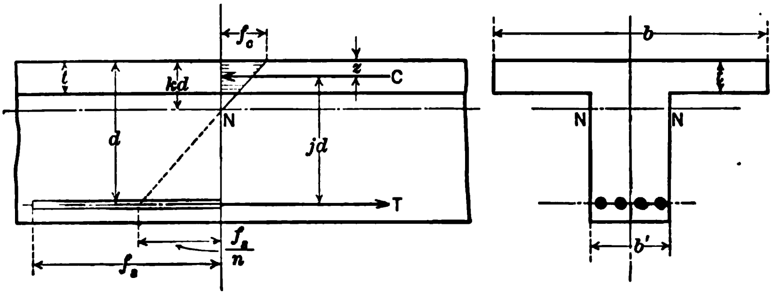

FIG. 276.

The rules which have been proposed for obtaining the value of b are, therefore, entirely empirical and as a result have varied to a large extent. Of these, the rule that

$$b \leqq 5t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

can be considered as conservative.

This rule should be interpreted to mean that when the flange of the tee is a part of a floor, b should ordinarily be taken equal to approximately five times the thickness of the slab, but in case the spans of the floor slabs were very short, or in the case of a single beam, having a tee section, the width b might evidently be less than $5t$. As less conservative, the rule that $b \leqq 6t$, or even the rule that $b \leqq 8t$, may be quoted.

After having determined the value of b the expressions for stress intensity may be deduced under the same assumptions as in the preceding cases. Two, or three, cases will arise, depending on the position of the neutral axis.

(a) *When the neutral axis lies in the flange of the tee.* — When the thickness of the flange t is large, in proportion to the depth d, the neutral axis may be located in the flange, the value of kd being less than t. It is evident that, since there is assumed to be no tensile stress in the concrete, the distribution of stress on any cross section would in this case be identical with that for a rectangular beam of width b; and, therefore, the maximum stress intensities in the concrete and the steel bars would be given by the equations in Art. (204).

(b) *When the neutral axis lies in the web near the flange of the tee.* — This is the condition which usually exists in the tee beam and, since the formulas for stress intensity are found to be simplified considerably if the compression stress in the web between the neutral axis and the under side of the flange is neglected, it is customary to deduce the equations on the assumption that the entire compression stress on a cross section is carried by the flange of the tee, as indicated (Fig. 276). The error introduced by making this assumption is evidently very small when NN is near the flange.

Then, in the same manner as in the rectangular beam (Art. 204), we have, from assumptions (1) and (2) (Art. 203),

$$\frac{f_s}{f_c} = n\left(\frac{1-k}{k}\right). \quad \ldots \ldots \quad (2)$$

The stress intensity in the concrete at the under side of the flange will be equal to

$$f_c' = f_c \frac{kd - t}{kd}; \quad \ldots \ldots \quad (3)$$

and, since $T = C$ for any vertically loaded beam,

$$f_s A = \frac{1}{2}(f_c + f_c')\,bt = f_c bt\left(\frac{2\,kd - t}{2\,kd}\right); \quad \ldots \quad (4)$$

and hence

$$\frac{f_s}{f_c} = bt\left(\frac{2\,kd - t}{2\,Akd}\right). \quad \ldots \ldots \quad (5)$$

Equating (2) and (5) and solving for kd, we have

$$kd = \frac{2\,ndA + bt^2}{2\,nA + 2\,bt}. \quad \ldots \ldots \quad (6)$$

Since the value of z will evidently be equal to the distance from the top of the beam to the center of gravity of the trapezoid, representing the variation in the stress intensity over the flange, we shall have

$$z = \frac{\frac{f_c t}{2} \times \frac{t}{3} + \frac{f_c' t}{2} \times \frac{2\,t}{3}}{(f_c + f_c')\,\frac{t}{2}} = \frac{(f_c + 2 f_c')\,t}{3\,(f_c + f_c')};$$

and, substituting the value of f_c' from (3) and reducing,

$$z = \frac{t}{3} \times \frac{3\,kd - 2\,t}{2\,kd - t}. \quad \quad (7)$$

Having the value of z, we obtain

$$jd = d - z; \quad \quad (8)$$

and, substituting the values of T and C in the equation

$$M = Tjd = Cjd, \quad \quad (9)$$

we have

$$M = f_sAjd = f_cbt\left(\frac{2\,kd - t}{2\,kd}\right)jd. \quad \quad (10)$$

Solving for f_s and f_c,

$$f_s = \frac{M}{Ajd} \quad \quad (11)$$

and

$$f_c = \frac{2\,Mkd}{(2\,kd - t)\,btjd}. \quad \quad (12)$$

From equation (2) we obtain, as in previous cases, the expression

$$f_c = \frac{f_s}{n}\left(\frac{k}{1 - k}\right), \quad \quad (13)$$

from which the value of f_c can be found after the value of f_s is known.

(c) *When the compression in the web is not negligible.* — If the flange of the tee is comparatively small, the neutral axis may lie so far below the under side of the flange that a considerable error will be introduced, if the compression stress in the web is neglected, and the following equations, derived by allowing for the stress in the web, will give more accurate results.

Equations (2) and (3) will hold true for this case, as before, but equation (4) will be modified to the form

$$f_sA = \frac{f_cb'kd}{2} + f_c\,(b - b')\,t\left(\frac{2\,kd - t}{2\,kd}\right). \quad \quad (14)$$

Solving for $\frac{f_s}{f_c}$ and equating to (2), we have

$$\frac{b'kd}{2\,A} + \frac{(b - b')\,t\,(2\,kd - t)}{2\,Akd} = n\left(\frac{1 - k}{k}\right),$$

which reduces to the form

$$b'\,(kd)^2 + 2\,[nA + (b - b')\,t]\,kd = 2\,n\,dA + (b - b')\,t^2;$$

and, solving for *kd*,

$$kd=\sqrt{\frac{2ndA+(b-b')t^2}{b'}+\left(\frac{nA+(b-b')t}{b'}\right)^2}-\frac{nA+(b-b')t}{b'}. \qquad (15)$$

By taking moments of the components of the compression stress, about an axis through the top of the flange, we obtain, for the distance of the center of the compressive stress from the top of the beam,

$$z=\frac{\dfrac{f_cb'kd}{2}\times\dfrac{kd}{3}+f_c(b-b')t\left(\dfrac{2kd-t}{2kd}\right)\times\dfrac{t}{3}\left(\dfrac{3kd-2t}{2kd-t}\right)}{\dfrac{f_cb'kd}{2}+f_c(b-b')t\left(\dfrac{2kd-t}{2kd}\right)},$$

which reduces to

$$z=\frac{b'(kd)^3+(b-b')(3kd-2t)t^2}{3[b'(kd)^2+(b-b')(2kd-t)t]}. \quad . \quad . \quad (16)$$

Having the values of k and z, the values of jd, f_s and f_c can be found from (8), (11) and (13).

Shearing and Bond Stresses. — Since there is assumed to be no tension in the concrete, the distribution of the shearing stress will be similar to that in the rectangular beam, the shearing stress intensity on any cross section being uniform between the neutral axis and the center of the tension reinforcement. Hence, by the same reasoning as in Art. (205), we shall have for the value of the maximum shearing stress intensity on any cross section

$$v=\frac{V}{b'jd}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and, for the bond stress intensity,

$$u=\frac{vb'}{\Sigma o}=\frac{V}{jd\Sigma o}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Bending Moments. — When a concrete floor beam, or girder, is supported at more than two points the reinforcement is usually arranged to resist negative bending over the intermediate supports; and, under uniform loading, the bending moments should be calculated in the same manner as for the continuous floor slab (Art. 206). When the beam is continuous over several spans and the loading is uniform, the greatest bending moment in any intermediate span may be taken equal to

$$M=\frac{wl^2}{12}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and in the end spans the greatest bending moment may be assumed to be equal to

$$M = \frac{wl^2}{10}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

For a continuous beam having three spans, or two only, or when the loading is not uniform, the bending moments should be estimated by use of the more exact equations for continuous beams. The values in (19) and (20) represent the bending moments at the ends of the spans; but it is customary to design the middle cross section for a moment of resistance equal to the bending moments given by these equations. The value of l is ordinarily taken equal to the distance from center to center of the supports, unless the clear span between the supports is considerably less than this distance, when a smaller value for l may be allowed.

Stresses at a Section Over a Support. — The reinforcement required to resist the negative bending moment over a support of a continuous floor beam may be provided by bending up one-half of the reinforcing bars in the spans on each side of the support, as indicated at the support B (Fig. 277), the remainder of the bars being run through the support on the lower side of the beam. If the bars are symmetrically arranged this will provide, at sections near a support, equal amounts of tension and compression reinforcement, as shown in the section A-A; the total area of each set of reinforcing bars being the same as that of the tension reinforcement at the section C–C, which would be similar to the cross section represented in Fig. (276).

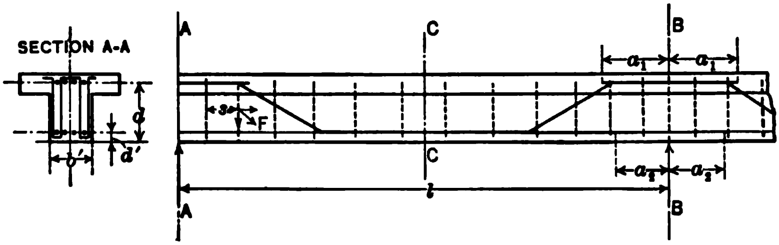

FIG. 277.

Since it is assumed that the concrete does not resist tension, the section A–A is equivalent to that of a rectangular beam, reinforced for both tension and compression (Fig. 275). The moment of resistance of this section must evidently be as great as, or greater than, the moment of resistance of the tee section at C–C. The stress intensities at the section A–A can be obtained by use of the equations in Art. (207), the width of the beam being equal to b', the effective depth equal to d, and the distance from the compression side of the beam to the center of the compression reinforcement equal to d'.

The bent-up rods should cross the neutral layer near the points of inflexion, at distances $0.2\,l$ to $0.25\,l$ from the supports. When the bond in the concrete is relied upon to develop the full strength of the steel at the section over the supports, the overlap of the rods on the tension side may be estimated from the expression

$$a_1 \geqq \frac{f_s A}{u' \Sigma o}, \quad \dots \dots \dots \quad (21)$$

and that of the rods on the compression side from the expression

$$a_2 \geqq \frac{f_s' A'}{u' \Sigma o}, \quad \dots \dots \dots \quad (22)$$

where u' = the allowable intensity of the bond stress on the rods.

In any case the overlap must be great enough to provide sufficient reinforcement throughout the portion of the beam in which negative bending may exist.

209. Web Reinforcement. — In each of the cases which have been considered tensile stresses in the concrete have been assumed to be negligible; and the only stress which has been considered as existing, on the portion of a cross section between the neutral axis and the tension reinforcement, has been a vertical shearing stress, of uniform intensity v. This stress must be accompanied by a shearing stress of equal intensity on any longitudinal layer, intersecting the cross section.

If the assumptions of the theory are adhered to, therefore, the state of stress at any point below the neutral layer should be considered similar to that existing at the neutral layer in any homogeneous beam (Art. 92); that is, at any point between the neutral layer and the tension reinforcement, the planes of principal stress would make angles of 45° with the vertical and the stress intensities on each of these planes would be equal to v, the stress on one plane being tension and on the other, compression.

If the concrete does not resist tension, it is evident that special reinforcement, in addition to the longitudinal bars, will be required to resist the tensile stresses on the diagonal planes. This secondary reinforcement may consist of small rods, bent into special forms, which for convenience are placed in vertical planes, at distances s apart, as indicated in Fig. (277). Sometimes the secondary reinforcement is inclined at 45° with the vertical, in the direction of the diagonal tensile stress; which would evidently be a more efficient arrangement, if this stress alone were to be considered.

When the secondary reinforcing units are vertical, the rods may be bent into stirrup shaped forms, or into forms with four vertical parts, as indicated in the cross section A–A (Fig. 277). The spaces between two adjacent units may be found on the assumption that each unit supports a stress

$$F = vb's\sqrt{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

equal to the total tension on a plane at 45° with the horizontal, of width b' and length equal to $s\sqrt{2}$, the diagonal distance between the units. The diagonal force F would produce a combination of tension and shear at any section through the vertical reinforcing unit, the total tensile stress component on the section of the unit being equal to

$$P = F\cos 45^\circ = vb's. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

If the allowable value of P is known, the proper spacing of the secondary reinforcing units in any portion of the span can be found from (2), when the average value of v for that portion is known, provided the assumption that the concrete does not resist tension is consistently adhered to.

The value of P must be determined more or less arbitrarily; but, since each reinforcing unit is subjected to a combination of equal shear and tension, we may assume

$$P = 0.6\,f_s n'a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

where n' = the number of vertical parts in the unit and a = the area of the cross section of a single part, provided there is a sufficient bond between the unit and the concrete. For rough, or deformed, rods with the ends bent over, as indicated (Fig. 277), there will probably be a sufficient bond if

$$d_0 \geqq \frac{d}{50}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

where d_0 = the diameter of the rod. If square rods are used d_0 will represent the side of the square.

It is evident from (2) that the required spacing of the reinforcing units will increase as v diminishes. In the tee beam the web reinforcement serves the double purpose of resisting the diagonal tension and strengthening the bond between the flange and the web; and hence it is customary in such a beam to limit the maximum value of s, a common rule being

$$s \leqq \frac{3}{4}d. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

On account of the approximations in the theory, all of the foregoing equations must be regarded as empirical, to a greater extent even than those in the preceding articles, and subject to modification to suit the conditions under which they are used. One modification may be mentioned; namely, that of replacing v in (2) with a value $v' = v - c$, where c is a small value for a tensile stress intensity which the concrete alone is assumed to resist.

Although the tee beam has been taken as an illustration, since it is in beams of this type that web reinforcement is nearly always required, it is evident that the foregoing method of computing the spacing of the web reinforcement could be applied with equal facility to a rectangular beam, if necessary.

210. Deflection of Reinforced Concrete Beams. — A general equation for the elastic curve of a reinforced concrete beam, which is of uniform dimensions and uniformly reinforced throughout, can be deduced in the following manner.

If we let r = the radius of curvature at any point in the elastic curve and follow the notation previously adopted, we shall have, as in the common beam theory (Art. 95),

$$\frac{1}{r} = \frac{e_s}{d - kd} = \frac{f_s}{E_s d\,(1 - k)}. \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

The equation for the stress intensity in the tension reinforcement, in each of the cases considered, has been found to be in the form

$$f_s = \frac{M}{Ajd};$$

and, substituting this value of f_s in (1), we have

$$\frac{1}{r} = \frac{M}{E_s Ajd^2\,(1 - k)}. \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

As in the ordinary beam theory, $\frac{1}{r} = \frac{d^2v}{dx^2}$; hence we may write

$$\frac{d^2v}{dx^2} = \frac{M}{E_sAjd^2\,(1-k)} = \frac{M}{B}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$B = E_sAjd^2\,(1-k) = \text{a constant.}$$

Therefore,

$$v = \frac{1}{B}\int\!\!\int M\,dx\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

the constant B evidently taking the place of the constant EI for the homogeneous beam of uniform section.

Hence, by substituting the value of B for EI in the formulas for the maximum deflection of simple homogeneous beams under various load systems (Art. 106), we may obtain the formulas for computing the maximum deflections in simple reinforced concrete beams, of any type of cross section, when subjected to similar load systems.

In a continuous beam, or a beam fixed at the supports, however, we have seen that the arrangement of the longitudinal reinforcement must be varied to provide for the tensile stresses throughout the span; and hence the values of B for the different sections of such a beam would vary in such a manner that an accurate solution for the deflection would be impracticable.

211. Columns, Axially Loaded. — The stress intensities in a reinforced concrete column can be readily determined under the assumptions made in developing the theory for reinforced concrete beams (Art. 203). When the resultant of the load coincides with the axis of the column, it is evident that the stress intensities in both the concrete and the reinforcing bars will be uniform. We will adopt the following notation:

A = the total area of the cross section of the column,
A_s = the total area of the cross section of the reinforcing bars,
A_c = the net area of the cross section of the concrete, allowing for the space occupied by the reinforcement,
p_0 = the ratio of the area A_s to the area A,
f_c = the stress intensity in the concrete,
f_s = the stress intensity in the reinforcing bars,
R = the resultant axial load,
E_s = the modulus of elasticity of the steel reinforcement,
E_c = the modulus of elasticity of the concrete,
n = the ratio of E_s to E_c.

Under the assumptions mentioned, the longitudinal strain in the concrete must be equal to that in the reinforcement; and hence

$$\frac{f_s}{E_s} = \frac{f_c}{E_c}$$

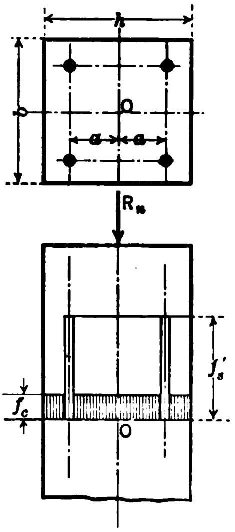

FIG. 278.

and, therefore,

$$f_s = nf_c. \quad . \quad . \quad . \quad . \quad (1)$$

For equilibrium

$$R = f_s A_s + f_c A_c; \quad . \quad . \quad . \quad (2)$$

and, substituting the value of f_s from (1) and solving for f_c, we have

$$f_c = \frac{R}{A_c + nA_s}. \quad . \quad . \quad . \quad (3)$$

If desired (3) may be expressed in terms of the total area A and the ratio $p_0 = \frac{A_s}{A}$; in which case, by substituting $A_c = A - A_s$, we have

$$f_c = \frac{R}{A\,[1 + p_0(n - 1)]}. \quad . \quad . \quad (4)$$

The sketch (Fig. 278) represents a square column, but the foregoing equations will evidently apply to an axially loaded column having a cross section of any shape.

The allowable value of the working stress intensity f_c will depend to a certain extent on the ratio of the unsupported length of the column to the width b of the cross section; but, for ratios of $\frac{l}{b}$ less than a certain quantity, say

$$\frac{l}{b} \leqq 16, \text{ or } 20,$$

we may assume the working stress intensity to be constant, as in the case of the homogeneous column.

212. Columns, Eccentrically Loaded. — Two cases will be considered, based on the conditions stated in Art. (203). In both cases the cross sections will be square, or rectangular, and the reinforcement will be assumed to be symmetrically arranged, so that the center of the reinforcement will coincide with the center of the concrete. If the resultant of the external forces, acting on the column, is not normal to the section it will be resolved into a normal component R_n and a shearing component, the latter evidently having no effect on the normal stress intensity at any point.

(a) *When the neutral axis NN lies within the boundary of the cross section* (Fig. 279). — We will adopt the following notation, the dimensions of the cross section being indicated in the sketch:

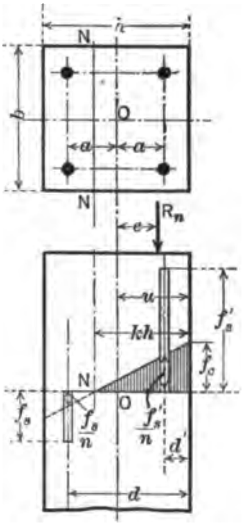

FIG. 279.

A = the area of the cross section of the reinforcing bars which are in tension,

A' = the area of the reinforcing bars which are in compression,

f_s = the intensity of stress in the tension reinforcement,

f_s' = the intensity of stress in the compression reinforcement,

f_c = the maximum compressive stress intensity in the concrete,

e = the eccentricity of the load, that is, the distance between the resultant normal force R_n and the center of the cross section,

k = the ratio of the distance of the neutral axis from the compression side to the total depth of the section h,

$p = \frac{A}{bh}$ and $p' = \frac{A'}{bh}$, which are the ratios of the areas of the tension and compression reinforcing bars, respectively, to the *total* area of the cross section.

For a symmetrical section $A = A'$ and hence $p = p'$. Since the stress intensity in any reinforcing bar would be n times the stress intensity which would exist in the surrounding concrete, provided the concrete were assumed to resist tension as well as compression and the ratio of the moduli of elasticity were the same for tension as for compression, we may write

$$\frac{\frac{f_s}{n}}{f_c} = \frac{d - kh}{kh} \quad \text{and} \quad \frac{\frac{f_s'}{n}}{f_c} = \frac{kh - d'}{kh};$$

and hence

$$f_s = nf_c\left(\frac{d}{kh} - 1\right), \quad \ldots\ldots\ldots \quad (1)$$

$$f_s' = nf_c\left(1 - \frac{d'}{kh}\right). \quad \ldots\ldots \quad (2)$$

Since R_n must equal the resultant normal stress on the section, we have

$$R_n = \frac{f_c bkh}{2} + f_s'A' - f_sA \text{ (very nearly)}, \quad \ldots \quad (3)$$

the stress on the concrete being calculated without deducting the area of the cross section of the reinforcing bars.

Since the bending moment will be equal to the moment of resistance, we shall have, making the same approximation as in (3),

$$M = R_ne = \frac{f_c bkh}{2}\left(\frac{h}{2} - \frac{kh}{3}\right) + f_s'A'a + f_sAa. \quad \ldots \quad (4)$$

Substituting the values of f_s and f_s' from (1) and (2), and putting $A' = A = pbh$ in (3) and (4) and reducing, we obtain

$$R_n = \frac{f_c bkh}{2} + nf_c\left(1 - \frac{d'}{kh}\right)pbh - nf_c\left(\frac{d}{kh} - 1\right)pbh$$

$$= f_c bh\left[\frac{k}{2} + pn\left(2 - \frac{1}{k}\right)\right] \quad \ldots\ldots\ldots \quad (5)$$

and

$$M = R_ne = \frac{f_c bkh}{2}\left(\frac{h}{2} - \frac{kh}{3}\right) + anf_c pbh\left[\left(1 - \frac{d'}{kh}\right) + \left(\frac{d}{kh} - 1\right)\right]$$

$$= \frac{f_c b}{k}\left[2pna^2 + \frac{k^2h^2}{12}(3 - 2k)\right]. \quad \ldots\ldots\ldots \quad (6)$$

Solving (6) for R_n and equating to (5), we have

$$\frac{f_c b}{ke}\left[2pna^2 + \frac{k^2h^2}{12}(3 - 2k)\right] = f_c bh\left[\frac{k}{2} + pn\left(2 - \frac{1}{k}\right)\right],$$

which reduces to

$$k^3 + 3\left(\frac{e}{h} - \frac{1}{2}\right)k^2 + \frac{12\,pne}{h}k = \frac{6\,pn}{h}\left(\frac{2\,a^2}{h} + e\right), \quad . \quad (7)$$

from which the value of k may be obtained.

Having the value of k, we can obtain from (5) or (6) the value of the maximum stress intensity in the concrete,

$$f_c = \frac{2\,R_nk}{bh\,[k^2 + 2\,pn\,(2\,k - 1)]} = \frac{12\,Mk}{b\,[24\,pna^2 + k^2h^2\,(3 - 2\,k)]}. \quad (8)$$

The stress intensities in the reinforcing bars can be determined from (1) and (2) after the value of f_c has been found.

(b) *When the neutral axis NN lies outside of the boundary of the cross section* (Fig. 280).

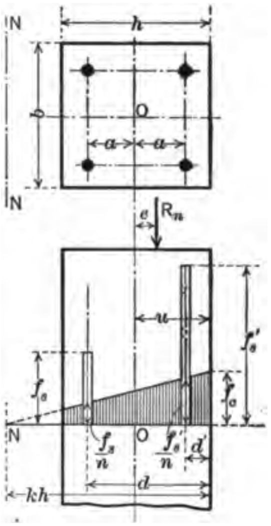

FIG. 280.

Using a notation similar to that in the preceding case and assuming $A = A'$,

$$\frac{\frac{f_s}{n}}{f_c} = \frac{kh - d}{kh} \quad \text{and} \quad \frac{\frac{f_s'}{n}}{f_c} = \frac{kh - d'}{kh},$$

the stress intensity f_s being compression instead of tension in this case.

Hence

$$f_s = nf_c\left(1 - \frac{d}{kh}\right), \quad . \quad . \quad (9)$$

$$f_s' = nf_c\left(1 - \frac{d'}{kh}\right) \quad . \quad . \quad (10)$$

and the stress intensity in the concrete, at the side of the cross section nearest the neutral axis, will be equal to

$$f_c' = f_c\left(\frac{kh - h}{kh}\right) = f_c\left(1 - \frac{1}{k}\right). \quad . \quad . \quad . \quad . \quad (11)$$

The resultant stress on the section will evidently be equal to

$$R_n = \frac{1}{2}(f_c + f_c')\,bh + f_s'A' + f_sA \text{ (very nearly)}, \quad . \quad (12)$$

the stress on the concrete being calculated without deducting the area of the reinforcement from the cross section of the concrete. The bending moment, making the same approximation, will be equal to

$$M = R_n e = \frac{1}{2}(f_c - f_c')\,bh \times \frac{h}{6} + f_s'A'a - f_sAa. \quad . \quad (13)$$

Substituting the values of f_s', f_s and f_c', from (9), (10) and (11), and putting $A' = A = pbh$ in (12) and (13) and reducing, we obtain

$$R_n = \frac{f_c}{2}\left(2 - \frac{1}{k}\right)bh + nf_c\left(1 - \frac{d'}{kh}\right)pbh + nf_c\left(1 - \frac{d}{kh}\right)pbh$$

$$= f_c bh\,(1 + 2pn)\left(1 - \frac{1}{2k}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$M = R_n e = \frac{f_c bh^2}{12\,k} + anf_c pbh\left[\left(1 - \frac{d'}{kh}\right) - \left(1 - \frac{d}{kh}\right)\right]$$

$$= \frac{f_c b}{12\,k}\left[24\,pna^2 + h^2\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Solving (15) for R_n and equating to (14), we have

$$\frac{f_c b}{12\,ke}\left[24\,pna^2 + h^2\right] = f_c bh\,(1 + 2\,pn)\left(1 - \frac{1}{2\,k}\right),$$

from which we obtain

$$k = \frac{24\,pna^2 + h^2}{12\,eh\,(1 + 2\,pn)} + \frac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Solving (14) or (15) we have, for the value of the maximum stress intensity in the concrete,

$$f_c = \frac{2\,R_n k}{bh\,[(1 + 2\,pn)(2\,k - 1)]} = \frac{12\,Mk}{b\,[24\,pna^2 + h^2]}. \quad . \quad . \quad (17)$$

The stress intensities in the steel bars can be determined from (9) and (10) after the value of f_c is known.

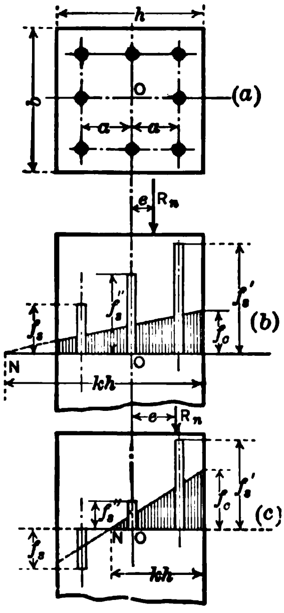

FIG. 281.

(c) *Columns with eight reinforcing bars, symmetrically arranged.* — The formulas for k and f_c in the two preceding cases may be readily modified to suit the arrangement of the reinforcing bars, shown in the cross section Fig. (281*a*), for the case where the neutral axis is within the boundary of the cross section and the distribution of stress is similar to that shown in Fig. (281*c*), or, for the case in which the neutral axis is outside of the section and the stress distribution is similar to that shown in Fig. (281*b*).

Assuming the bars to be of the same size, we will use the same notation as before and, in addition, let A'' = the total area of the two bars in the central plane through O and f_s'' = the intensity of the compression stress in these two bars.

Then in either of the cases mentioned

$$A'' = \frac{2}{3}A' = \frac{2}{3}A = \frac{2}{3}pbh \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and

$$f_s'' = nf_c \frac{kh - \frac{h}{2}}{kh} = nf_c\left(1 - \frac{1}{2\,k}\right). \quad . \quad . \quad . \quad (19)$$

Proceeding in the same manner as in Case (a), we shall have for the resultant of the normal stress on the cross section, when the stress distribution is similar to that indicated in Fig. (281*c*),

$$R_n = \frac{f_c bkh}{2} + f_s'A' + f_s''A'' - f_sA = f_c bh\left[\frac{k}{2} + \frac{4}{3}pn\left(2 - \frac{1}{k}\right)\right]; \quad (20)$$

the bending moment being represented by equation (6), as before.

Solving (6) for R_n, equating to (20) and reducing, we have

$$k^3 + 3\left(\frac{e}{h} - \frac{1}{2}\right)k^2 + \frac{16\,pne}{h}k = \frac{4\,pn}{h}\left(\frac{3\,a^2}{h} + 2\,e\right), \quad . \quad (21)$$

from which the value of k may be obtained. Having the value of k, the value of f_c may be obtained from (20).

Similarly, when the stress distribution is similar to that indicated in Fig. (281*b*), the resultant normal stress on the cross section will be equal to

$$R_n = \frac{1}{2}(f_c + f_c')\,bh + f_s'A' + f_s''A'' + f_sA = f_c bh\left(1 + \frac{8}{3}pn\right)\left(1 - \frac{1}{2\,k}\right); \, (22)$$

the bending moment being represented by (15), as before.

Solving (15) for R_n, equating to (22) and then solving for k, we have

$$k = \frac{24\,pna^2 + h^2}{12\,eh\left(1 + \frac{8}{3}pn\right)} + \frac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Having the value of k, the value of f_c may be obtained from (22).

(d) *Columns with more than eight reinforcing bars.* — If desired expressions for the values of k and f_c may be readily obtained for square, or rectangular, columns reinforced with more than eight bars, symmetrically arranged, by the same method as that employed in Case (c).

(e) *Unsymmetrical reinforcement.* — When the reinforcing bars are not symmetrically arranged, the true central axis of the column will not pass through the geometric center of the square, or rectangle, but through a point which may be called the center of gravity of the equivalent cross section; that is, the center of gravity of the cross section which would be formed if the cross section of the reinforcing bars were replaced with n times the area in concrete. For illustration; if in Case (a) the area A' were not equal to A (Fig. 279), the distance u to the true center of gravity of the cross section would not be equal to $\frac{h}{2}$, but could be found from the equation

$$u = \frac{bh \times \frac{h}{2} + nA'd' + nAd}{bh + nA' + nA} \quad \ldots \ldots \quad (24)$$

In such a case the fundamental equation for R_n (equation 3) would remain unchanged, but the equation for M (equation 4) would be written in the form

$$M = \frac{f_c bkh}{2}\left(u - \frac{kh}{3}\right) + f_s'A'(u - d') + f_s A(d - u). \quad (25)$$

By substituting the values of f_s and f_s', from (1) and (2), and the values of A' and A, in terms of b, h, p' and p, in (3) and (25), expressions for the values of k and f_c can be obtained in the same manner as in Case (a).

Expressions for the value of M, similar to (25), could evidently be written for each of the other cases considered; and expressions for the maximum stress intensity in the concrete obtained in the same manner.

(f) *Alternative method.* — In cases where the neutral axis of the resultant stress lies outside of the cross section, a solution can be made by adapting the formula for the maximum stress, due to combined axial compression and bending in a homogeneous column, (Art. 126) to the conditions in the concrete column as follows:

Let A_0 = the area of the equivalent cross section, mentioned under Case (e), obtained by replacing the cross section of the reinforcing steel with n times its area in concrete, and

I_0 = the moment of inertia of the equivalent section about an axis through its center of gravity, perpendicular to the plane of bending.

The formula

$$f_c = \frac{R_n}{A_0} + \frac{Mu}{I_0} \quad \ldots \ldots \ldots \quad (26)$$

would then give the correct value of the maximum stress intensity in the concrete, provided the neutral axis of the resultant stress were outside of the section, as shown in Figs. (280) or (281*b*).

For a cross section similar to that in Fig. (280) the value of u would be given by (24); the area of the equivalent cross section

$$A_0 = bh + nA' + nA \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

and the moment of inertia of the equivalent cross section

$$I_0 = \frac{bu^3}{3} + b\left(\frac{h-u}{3}\right)^3 + nA'(u-d')^2 + nA(d-u)^2. \quad (28)$$

When $A = A'$, $u = \frac{h}{2}$, and (28) would reduce to

$$I_0 = \frac{bh^3}{12} + 2na^2A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The minimum stress intensity on the cross section would be equal to

$$f_c' = \frac{R_n}{A_0} - \frac{M(h-u)}{I_0}. \quad . \quad . \quad . \quad . \quad . \quad (30)$$

A negative result, obtained by the solution of (30), would indicate that the neutral axis was inside of the boundary of the section, in which case the formulas for f_c and f_c' would not be correct. If, however, the negative value for f_c' were small, as compared to the value of f_c, given by (26), the latter could be considered sufficiently accurate for ordinary purposes.

This method has the advantage of being applicable to columns having circular, or other shaped, cross sections, where a solution by the method used in Cases (a), (b) and (c) would be much more complex.

213. Hooped Reinforcement. — Concrete columns of circular section are frequently reinforced with steel hoops, or bands, or a continuous rod wound into the form of a helix, embedded near the surface. This reinforcement would resist to a certain extent the lateral expansion, which accompanies the longitudinal compression in the concrete (Art. 5); in other words, the hoops would prevent a certain amount of lateral strain and reduce, by a corresponding amount, the compression strain in the direction of the axis of the column.

This reduction in strain is very small for loads within the working load; but experiments have shown that the ultimate strength of concrete columns has been considerably increased by the use of this reinforcement.

The effect of hoop reinforcement may be allowed for by using a higher value for the working stress intensity f_c than that used for columns with longitudinal reinforcement only. The allowable increase in the value of f_c should be determined from a comparison of the results of experiments on the strengths of the two types of columns, rather than from any theory built upon the relation of the lateral and longitudinal strains.

214. Floor Slabs Supported on Columns, without Beams or Girders. — A system of reinforced concrete floor construction, which does away with the necessity of floor beams, or girders, was originated by C. A. P. Turner. In this system, frequently called the *mushroom system*, the floor slab is supported directly on the tops of columns, which are specially formed for the purpose, and is reinforced by sets of rods running in four directions, as indicated in Fig. (282).

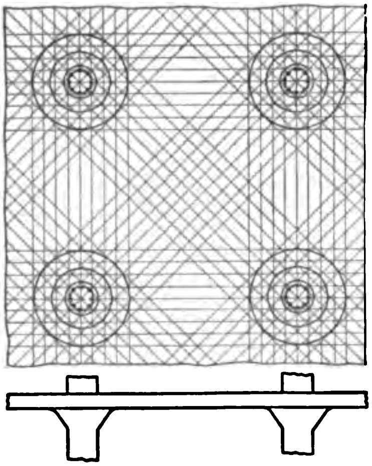

FIG. 282.

Methods of determining the stresses in such a slab have been proposed, which are based on the theory for determining the stresses in a homogeneous circular flat plate, supported at the center and subjected to uniform load, combined with a load at the circumference; the radius of the circle being taken equal to the mean distance of an assumed line of inflexion in the portion of the slab surrounding a column.

On account of the lack of homogeneity in the slab, such a method is at best very approximate and a better method in this case would appear to be a purely empirical one, based on the results of experiments made to determine the ultimate strength of slabs of this type under actual working conditions; supplemented by measurements of the deflections of the slabs under series of loads, both below and above the allowable working load.

215. Problems. — Reinforced Concrete Beams and Columns. — For the sake of brevity all quantities in the following problems will be expressed, as far as possible, by the symbols adopted in the different cases which have been considered.

Problem 1.

A rectangular concrete beam, having the dimensions, $b = 10''$, $d = 12''$, is reinforced on the tension side only with 2 steel bars, $\frac{3}{4}''$ square, and loaded with a uniformly distributed load of 500 lbs. per ft. If the span is 16 ft. and the beam is assumed to be freely supported at the ends, determine the values of f_s, f_c, v and u, assuming $n = 15$.

Problem 2.

Find the proper size and spacing of the reinforcement for a concrete floor slab, supported on two sides only and having a span of 15 ft., assuming $d = 5''$, $f_s = 16{,}000$ lbs. per sq. in., $f_c = 700$ lbs. per sq. in., $n = 15$.

Problem 3.

Determine the safe uniform load, in lbs. per sq. ft., including the weight of the slab, for the floor in Problem (2), assuming the slab to be continuous over the supports, with the reinforcement arranged as indicated in Fig. (274); and calculate the values of v and u under this load.

Problem 4.

A concrete floor, supported on concrete girders, as indicated in Fig. (274), which are spaced 15 ft. on centers and reinforced with 4 steel bars $1\frac{1}{8}''$ square, has the following dimensions: span of girders = 24 ft., $t = 6''$, $b' = 15''$, $d = 22''$. If the total load on the floor, including the weight of the floor and girders, averages 150 lbs. per sq. ft., calculate the values of f_s, f_c, v and u, assuming $n = 15$ and $M = \frac{wl^2}{10}$.

Problem 5.

Calculate the size and required spacing of the web reinforcing units in the girders in Problem (4), assuming the allowable values, $f_s = 16{,}000$ lbs. per sq. in., $f_c = 650$ lbs. per sq. in., $v = 120$ lbs. per sq. in.

Problem 6.

Calculate the values of f_c, f_s and f_s' at the sections at the ends of floor girders in Problem (4), assuming one-half of the reinforcing rods in each span to be bent up and carried over the supports, as indicated in Fig. (277), the distance from the centers of the rods to the top of the floor being $2\frac{1}{2}''$, and the distance of the centers of the rods on the lower side from the under side of the beam being assumed to be $2\frac{1}{2}''$.

Problem 7.

Determine the values of f_c and f_s due to an axial load of 200,000 lbs. on a concrete column, having a cross section $16'' \times 16''$, reinforced with eight $\frac{3}{4}''$ square bars, arranged as shown in Fig. (281*a*), the bars being placed with centers $2''$ inside of the column. Assume $n = 15$.

Problem 8.

Determine the values of f_c, f_s and f_s' for the column in Problem (7), assuming the column to be subjected to an eccentric load of 120,000 lbs., acting through a principal axis at a distance $e = 2''$ from the center. Assume $n = 15$.

Problem 9.

Solve Problem (8) assuming the column to be subjected to a load of 80,000 lbs., with an eccentricity $e = 6''$.

Problem 10.

Determine the allowable load on a concrete column 16″ square, reinforced with eight 1″ square bars, arranged as in Problem (7), with centers 2″ from the sides of the column; assuming $f_c = 650$ lbs. per sq. in. and $n = 15$:

(a) When the load is central;

(b) When the load has an eccentricity of 1″;

(c) When the load has an eccentricity of 8″.

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